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Some new integral inequalities for (s,m)-convex and (α,m)-convex functions

The paper considers several new integral inequalities for functions the second derivatives of which, with respect to the absolute value, are (s,m)-convex and (α,m)-convex functions. These results are related to well-known Hermite-Hadamard type integral inequality, Simpson type integral inequality, and Jensen type inequality. In other words, new upper bounds for these inequalities using the indicated classes of convex functions have been obtained. These estimates are obtained using a direct definition for a convex function, classical integral inequalities of Hölder and power mean types. Along with the new outcomes, the paper presents results confirming the existing in literature upper bound estimates for integral inequalities (in particular well known in literature results obtained by U. Kırmacı in [7] and M.Z. Sarıkaya and N. Aktan in [35]). The last section presents some applications of the obtained estimates for special computing facilities (arithmetic, logarithmic, generalized logarithmic average and harmonic average for various quantities).

Keywords: convex function, (s,m)-convex, (α,m)-convex, Hermite–Hadamard inequalitiy, Jensen inequality, Hölder inequality, power mean inequality.

Introduction

Convexity has become a very attractive topic for many authors over the past decades, since it has applications in many areas of pure and applied mathematics. The following basic two definitions are well known in the literature [1]:

Definition 1. The function $f:[a,b]\to\mathbb{R}$, is said to be convex, if we have

$$f(\lambda x+(1-\lambda)y)\le \lambda f(x)+(1-\lambda)f(y)$$

for all $x,y\in[a,b]$ and $\lambda\in[0,1]$.

Definition 2. A function $f:[a,b]\to R$ is called either midconvex or convex in the Jensen sence, or J–convex on $[a,b]$ if for all points $x,y\in[a,b]$ the inequality

$$f\left(\frac{x+y}{2}\right)\le\frac{f(x)+f(y)}{2} \tag{1}$$

is valid. Many important inequalities are established for the class of convex functions, but one of the most important is so called Hermite–Hadamard's inequality (or Hadamard's inequality). This double inequality is stated as follows in literature:

Let $f:I\subseteq\mathbb{R}\to\mathbb{R}$ be a convex function and let $a,b\in I$, with $a<b$. The following double inequality:

$$f\left(\frac{a+b}{2}\right)\le\frac{1}{b-a}\int_a^b f(x)dx\le\frac{f(a)+f(b)}{2}. \tag{2}$$

The above inequality is in the reversed direction if f is concave.

In [2] Toader defines the m-convexity:

Definition 3. Let real function f be defined on a nonempty interval I of real numbers $\mathbb{R}$. The function f is said to be m-convex on I if inequality

$$f(\lambda x+m(1-\lambda)y)\le\lambda f(x)+m(1-\lambda)f(y)$$

holds for all $x,y\in I$ and $m,\lambda\in[0,1]$.

In [3] Breckner defined a new class of functions that are s-convex in the second sense:

Definition 4. $f\colon [0,\infty)\to\mathbb{R}$ is said to be s-convex function in the second sense if

$$f\left(\lambda x+(1-\lambda)\,y\right)\le \lambda^{s} f\left(x\right)+\left(1-\lambda\right)^{s} f\left(y\right)$$

holds for all $x,y\in[0,\infty)$, $\lambda\in[0,1]$ and for some fixed $s\in(0,1]$. It is clear that the ordinary convexity of functions defined on $[0,\infty)$ for $s=1$.

In [4] Miheşan introduced the following class of functions:

Definition 5. $f:[0,\infty)\to\mathbb{R}$ is said to be (α,m)-convex function if

$$f\left(\lambda x+(1-\lambda)\,y\right)\le \lambda^{\alpha} f\left(x\right)+m\left(1-\lambda^{\alpha}\right) f\left(y\right)$$

holds for all $x,y\in[0,\infty)$, $\lambda\in[0,1]$; and for some fixed $\alpha,m\in(0,1]$.

A series of works ([5–38] and references therein) devoted to (α,m)-convex and (s,m)-convex functions and established some Hermite-Hadamard, Ostrovski, Jensen et al. type inequalities (1) and (2).

The following theorem was proved by Dragomir and Pearce, in [5]:

Theorem 1. Let $f:[0,\infty)\to\mathbb{R}$ be a m-convex function with $m\in(0,1]$. If $0\le a<b<\infty$ and $f\in L_1[a,b]$, then one has the inequality:

$$\frac{1}{b-a}\int_a^b f(x)dx\le min\left\{\frac{f(a)+mf(\frac{b}{m})}{2},\frac{f(b)+mf(\frac{a}{m})}{2}\right\}. \tag{3}$$

Some generalizations of this result can be found in [36–38].

In [6] Özdemir et al. the following lemma is proved.

Lemma 1. Let $f:I\subseteq\mathbb{R}\to\mathbb{R}$ be a twice differentiable mapping on I° (I° is interior of I), where $a,b\in I$ and $m\in(0,1]$. If $f''\in L[a,b]$, then the following equality holds

$$\frac{f(a)+f(mb)}{2}-\frac{1}{mb-a}\int_a^{mb} f(x)dx=$$

$$=\frac{(mb-a)^2}{2}\int_0^1(t-t^2)f''(ta+m(1-t)b)dt. \tag{4}$$

In [7], Kırmacı proved the following lemma

Lemma 2. Let $f:I\subset\mathbb{R}\to\mathbb{R}$ be twice differentiable function on I° with $f^{''}\in L[a,b]$. Then we have

$$\frac{(b-a)^2}{2}(I_1+I_2)=\frac{1}{b-a}\int_a^b f(x)dx-\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right], \tag{5}$$

where

$$I_1=\int_0^{1/2} t\left(t-0.5\right)f''(ta+(1-t)b)dt,\ I_2=\int_{1/2}^1\left(t-0.5\right)\left(t-1\right)f''(ta+(1-t)b)dt$$

and I° denotes the interior of I.

In [8] B. Bayraktar and M. Gürbüz the following lemma is proved.

Lemma 3. Let $f:I\subset\mathbb{R}\to\mathbb{R}$ be a twice differentiable function on I°($I^{\circ}$is interior of I), where $a,b\in I$. If $f^{''}\in L\left[a,b\right]$, then we have

$$\frac{f(a)+f(mb)}{2}-f\left(\frac{a+mb}{2}\right)=$$

$$=\frac{(mb-a)^2}{2}\left[\int_0^{1/2} tf''(at+m(1-t)b)dt+\int_{1/2}^1(1-t)f''(at+m(1-t)b)dt\right]. \tag{6}$$

In this paper we give some integral inequalities of Hadamard type and inequalities Jensen type for twice differentiable (s,m)-convex and (α,m)-convex functions and give some applications to the special means of real numbers.

1 Some new results for (s,m)-convex functions

We start with the definition [9] of a (s,m)-convex functions.

Definition 6. For some fixed $s\in(0,1]$ and $m\in[0,1]$ a mapping $f:I\subset[0,\infty)\to\mathbb{R}$ is said to be (s,m)-convex in the second sense on I if

$$f(tx+m(1-t)y)\le t^s f(x)+m(1-t)^s f(y)$$

holds for all $x,y\in I$ and $t\in[0,1]$.

It should be noted that the following proposition is true:

Proposition 1. Any m-convex function is (s,m)-convex function.

Proof. Indeed, for m-convex functions we have

$$f(tx+m(1-t)y)\le tf(x)+m(1-t)f(y),\ \forall\ m,\ t\in[0,1].$$

Since $t\le t^s$ and $1-t\le(1-t)^s$ for all $s\in(0,1]$ then we can write

$$tf(x)+m(1-t)f(y)\le t^s f(x)+m(1-t)^s f(y)$$

and then

$$f(tx+m(1-t)y)\le t^s f(x)+m(1-t)^s f(y).$$

The proof is completed.

Theorem 2. Let $f:[0,\infty)\to\mathbb{R}$ be an (s,m)-convex function with $s,m\in(0,1]$. If $0\le a<b<\infty$ and $f\in L[a,b]$, then has the inequality:

$$\frac{1}{b-a}\int_a^b f(x)dx\le\frac{1}{s+1}\left[\frac{f(a)+mf(\frac{b}{m})}{2}+\frac{f(b)+mf(\frac{a}{m})}{2}\right]. \tag{7}$$

Proof. It's obvious that

$$\int_0^1 f(ta+(1-t)b)dt=\int_0^1 f((1-t)a+tb)dt=\frac{1}{b-a}\int_a^b f(x)dx; \tag{8}$$

and

$$\int_0^1\left[f(ta+(1-t)b)+f((1-t)a+tb)\right]dt=\frac{2}{b-a}\int_a^b f(x)dx. \tag{9}$$

Since the function f is (s,m)-convex functions for all $t\in[0,1]$

$$f(ta+(1-t)b)=f\left(ta+m(1-t)\frac{b}{m}\right)\le t^s f(a)+m(1-t)^s f\left(\frac{b}{m}\right)$$

and

$$f(tb+(1-t)a)\le t^s f(b)+m(1-t)^s f\left(\frac{a}{m}\right)$$

then

$$\int_0^1\left[f(ta+(1-t)b)+f((1-t)a+tb)\right]dt\le$$

$$\le\int_0^1\left[t^s f(a)+m(1-t)^s f\left(\frac{b}{m}\right)\right]dt+\int_0^1\left[t^s f(b)+m(1-t)^s f\left(\frac{a}{m}\right)\right]dt=$$

$$=\frac{f(a)+mf(\frac{b}{m})}{s+1}+\frac{f(b)+mf(\frac{a}{m})}{s+1}.$$

Taking into account equality (9) completes the proof.

Remark 1. From (7) for $m=1$ and $s=1$ we have right hand inequality (2)

$$\frac{1}{b-a}\int_a^b f(x)dx\le\frac{f(a)+f(b)}{2}.$$

Corollary 1. It is obvious that for (s,m)-convex functions the inequalities

$$\frac{1}{b-a}\int_a^b f(x)dx \le \min\left\{\frac{f(a)+mf(\frac{b}{m})}{s+1},\ \frac{f(b)+mf(\frac{a}{m})}{s+1}\right\}. \tag{10}$$

Proof. Since the

$$\int_0^1 f(ta+(1-t)b)dt \le \frac{f(a)+mf(\frac{b}{m})}{s+1}$$

and

$$\int_0^1 f(tb+(1-t)a)dt \le \frac{f(b)+mf(\frac{a}{m})}{s+1}$$

and taking into account equalities (8) we have (10).

Remark 2. If we choose $s=1$ from (10) we have (3).

Theorem 3. Let $f: I\subset[0,b^*]\to\mathbb{R}$ be a twice differentiable mapping on I°such that $f''\in L[a,b]$ where $a,b\in I$. If $|f''|^q$ is (s,m)-convex on $[a,b]$ for $s,m\in(0,1]$, $q\ge 1$, then the following inequality holds

$$\frac{f(a)+f(mb)}{2}\le \tag{11}$$

$$\le\frac{(mb-a)^2}{2}\frac{(|f''(a)|^q+m|f''(b)|^q)^{1/q}}{6^{1-\frac{1}{q}}(s+2)(s+3)}.$$

Proof. Suppose that $q=1$. From (4) and using the (s,m)-convexity of $|f''|$, we have

$$\left|\frac{f(a)+f(mb)}{2}-\frac{1}{mb-a}\int_a^{mb}f(x)dx\right|\le\frac{(mb-a)^2}{2}\int_0^1(t-t^2)\left|f''(ta+m(1-t)b)\right|dt\le$$

$$\le\frac{(mb-a)^2}{2}\int_0^1(t-t^2)\left[t^s\left|f''(a)\right|+m(1-t)^s\left|f''(b)\right|\right]dt=$$

$$=\frac{(mb-a)^2}{2}\frac{1}{(s+2)(s+3)}(\left|f''(a)\right|+m\left|f''(b)\right|)$$

which completes the proof for $q=1$.

Suppose now that $q>1$. From (4) in Lemma 1 and using the Hölder's integral inequality for $q>1$, we have

$$\int_0^1(t-t^2)\left|f''(ta+m(1-t)b)\right|dt= \tag{12}$$

$$=\int_0^1(t-t^2)^{\frac{1}{p}}(t-t^2)^{\frac{1}{q}}\left|f''(ta+m(1-t)b)\right|dt\le$$

$$\le\left[\int_0^1\left((t-t^2)^{\frac{1}{p}}\right)^p dt\right]^{\frac{1}{p}}\left(\int_0^1\left[(t-t^2)^{\frac{1}{q}}\left|f''(ta+m(1-t)b)\right|\right]^q dt\right)^{\frac{1}{q}},$$

where $\frac{1}{q}+\frac{1}{p}=1$.

Since $|f''|^q$ is (s,m)-convex on$[a,b]$, we know that for all $t\in[0,1]$,

$$\left|f''(ta+m(1-t)b)\right|^q\le t^s\left|f''(a)\right|^q+m(1-t)^s\left|f''(b)\right|^q. \tag{13}$$

From (12) and (13) we have

$$\left|\frac{f(a)+f(mb)}{2}-\frac{1}{mb-a}\int_a^{mb}f(x)dx\right|\le\frac{(mb-a)^2}{2}\left(\frac{1}{6}\right)^{\frac{1}{p}}\times$$

$$\times\left[\left|f''(a)\right|^q\int_0^1(t-t^2)t^s dt+m\left|f''(b)\right|^q\int_0^1(t-t^2)(1-t)^s dt\right],$$

here

$$\int_0^1 (t-t^2)t^s dt = \frac{1}{(s+2)(s+3)} \text{ and } \int_0^1 (t-t^2)(1-t)^s dt = \frac{1}{(s+2)(s+3)},$$

we have (11). The proof is completed.

Corollary 2. From (11) for $s = m = q = 1$, we have estimates obtained by Sarıkaya and Aktan (see [35], Proposition 2):

$$\left| \frac{f(a)+f(mb)}{2} - \frac{1}{mb-a} \int_a^{mb} f(x)dx \right| \leq \frac{(b-a)^2}{24} \left[|f''(a)| + |f''(b)| \right].$$

Theorem 4. Let $f : I \subset \mathbb{R} \to \mathbb{R},\ I \subset [0, \infty)$, be twice differentiable function on I° such that $f^{''} \in L[a, b]$ with $0 \leq a < b < \infty$. If $\left| f^{''} \right|$ is (s, m)-convex function with $s, m \in (0, 1]$ then we have

$$\frac{1}{b-a} \int_a^b f(x)dx - \frac{1}{2} \left[\frac{f(a)+f(b)}{2} + f\left(\frac{a+b}{2} \right) \right] \leq$$

$$\leq \frac{(b-a)^2}{2} \xi (1+\tau) \left[|f''(a)| + m \left| f''\left(\frac{b}{m} \right) \right| \right], \tag{14}$$

where

$$\xi = \frac{1}{(s+2)(s+3)2^{s+3}} \text{ and } \tau = \frac{2^{s+2}(s-1)+s+5}{s+1}$$

Proof. Since f'' is a (s, m)-convex function

$$f''(ta+(1-t)b) = f''\left(ta + m(1-t)\frac{b}{m} \right) \leq t^s f(a) + m(1-t)^s f\left(\frac{b}{m} \right), \forall t \in [0, 1]$$

From equality (5) and using the triangle inequality, we can write

$$\left| \frac{1}{b-a} \int_a^b f(x)dx - \frac{1}{2} \left[\frac{f(a)+f(b)}{2} + f\left(\frac{a+b}{2} \right) \right] \right| \leq$$

$$\leq |f''(a)| \int_0^{1/2} t^{s+1} (0.5-t)\, dt + m \left| f''\left(\frac{b}{m} \right) \right| \int_0^{1/2} t\,(0.5-t)\,(1-t)^s\, dt \quad +$$

$$+ |f''(a)| \int_{1/2}^1 t^s\,(t-0.5)\,(1-t)dt + m \left| f''\left(\frac{b}{m} \right) \right| \int_{1/2}^1 (t-0.5)\,(1-t)^{s+1}\, dt. \tag{15}$$

Obviously, the first and third integrals are easy to calculate:

$$\int_0^{1/2} t^{s+1}\,(0.5-t)\, dt = \xi, \quad \int_{1/2}^1 t^s\,(t-0.5)\,(1-t)dt = \xi\tau.$$

If we do $1-t=z$ transformations in second and fourth integrals, we get:

$$\int_{1/2}^1 (t-0.5)\,(1-t)^{s+1}\, dt = \xi \text{ and } \int_0^{1/2} t\,(0.5-t)\,(1-t)^s\, dt = \xi\tau.$$

Substituting the values of the integrals in inequality (15) and completing the grouping, we complete the proof.

Corollary 3. Let $f : I \to \mathbb{R},\ I \subset [0, \infty)$ be twice differentable function on I° such as $f'' \in L[a, b]$, $0 \leq a < b < \infty$. If $|f''|$ is m-convex with $m \in (0, 1]$ then we have

$$\left| \frac{1}{b-a} \int_a^b f(x)dx - \frac{1}{2} \left[\frac{f(a)+f(b)}{2} + f\left(\frac{a+b}{2} \right) \right] \right| \leq$$

$$\leq \frac{(b-a)^2}{96} \left[|f''(a)| + m \left| f''\left(\frac{b}{m} \right) \right| \right]. \tag{16}$$

Proof. In inequality (14) if we choose $s = 1$ we have:

$$\xi = \frac{1}{(s+2)(s+3)2^{s+3}} = \frac{1}{3\cdot 2^6};\ \tau = \frac{2^{s+2}(s-1)+s+5}{s+1} = 3, \xi(1+\tau) = \frac{1}{48}$$

and from (14) we get (16). The proof is completed. This inequality were obtained by Kırmacı (see [7], Corollary 1).

Corollary 4. If $\|f''\|_\infty = \sup\limits_{x\in[a,b]} |f''(x)| < \infty$ and $m \in (0,1]$, we have

$$\left| \frac{1}{b-a}\int_a^b f(x)dx - \frac{1}{2}\left[\frac{f(a)+f(b)}{2} + f\left(\frac{a+b}{2}\right)\right]\right| \le \frac{(b-a)^2}{96}(1+m)\,\|f''\|_\infty .$$

Also putting $m = 1$ we get inequality

$$\left| \frac{1}{b-a}\int_a^b f(x)dx - \frac{1}{2}\left[\frac{f(a)+f(b)}{2} + f\left(\frac{a+b}{2}\right)\right]\right| \le \frac{(b-a)^2}{48}\,\|f''\|_\infty .$$

The same estimates were obtained by U. Kırmacı (see [7], Remark 1).

2 Some new results for (α, m)-convex functions

The following theorem gives an upper estimate the value of the inequality (1) for a (α, m)-convex function.

Theorem 5. Let $f : [0,\infty) \to \mathbb{R}$ be twice differentiable function on I° such as $f^{''} \in L\,[a,b]$ with $0 \le a < b < \infty$. If $\frac{b}{m} \in I^\circ$ and $\left|f^{''}\right|$ is (α, m)-convex function with $\alpha, m \in (0,1]$ then we have

$$\left|\frac{f(a)+f(mb)}{2} - f\left(\frac{a+mb}{2}\right)\right| \le \tag{17}$$

$$\le \frac{(mb-a)^2}{2}\left[(\zeta+\eta)\,|f''(a)| + m\,|f''(b)|\left|\frac{1}{4} - (\zeta+\eta)\right|\right],$$

where

$$\zeta = \frac{1}{(\alpha+2)\,2^{\alpha+2}} \quad \text{and} \quad \eta = \frac{2^{\alpha+2}-\alpha-3}{(\alpha+1)\,(\alpha+2)\,2^{\alpha+2}}.$$

Proof. Using the triangle inequality for the equality (6) in Lemma 3, we can write

$$\frac{f(a)+f(mb)}{2} - f\left(\frac{a+mb}{2}\right) \le \frac{(mb-a)^2}{2}\times$$

$$\times\left[\left|\int_0^{1/2} tf''(at+m(1-t)b)dt\right| + \left|\int_{1/2}^1 (1-t)f''(at+m(1-t)b)dt\right|\right] =$$

$$= \frac{(mb-a)^2}{2}\left(|I_1| + |I_2|\right). \tag{18}$$

Since f'' is a (α, m)-convex function

$$|I_1| \le \int_0^{1/2} t|f''(at+m(1-t)b)|dt \le |f''(a)|\int_0^{1/2} t^{\alpha+1}dt + m|f''(b)|\int_0^{1/2} t\,(1-t^\alpha)\,dt =$$

$$= \frac{1}{(\alpha+2)2^{\alpha+2}}|f''(a)| + m|f''(b)|\left|\frac{1}{8} - \frac{1}{(\alpha+2t)2^{\alpha+2}}\right| =$$

$$= \zeta|f''(a)| + m|f''(b)|\left|\frac{1}{8} - \zeta\right|.$$

For the second integral $|I_2|$ we can write

$$|I_2| \leq |f''(a)| \int_{1/2}^{1} t^{\alpha}(1-t)\,dt + m\,|f''(b)| \int_{1/2}^{1} (1-t)(1-t^{\alpha})\,dt =$$

$$= \frac{2^{\alpha+2}-\alpha-3}{(\alpha+1)(\alpha+2)\,2^{\alpha+2}}\,|f''(a)| + m\,|f''(b)| \left|\frac{1}{8} - \frac{2^{\alpha+2}-\alpha-3}{(\alpha+1)(\alpha+2)\,2^{\alpha+2}}\right| =$$

$$= \eta\,|f''(a)| + m\,|f''(b)| \left|\frac{1}{8}-\eta\right|.$$

Substituting these inequalities for $|I_1|$ and $|I_2|$ into inequality (18), we complete the proof.

Corollary 5. In inequality (17) if we choose $\alpha = 1$ and $m = 1$ we have:

$$\left|\frac{f(a)+f(b)}{2} - f\left(\frac{a+b}{2}\right)\right| \leq \frac{(b-a)^2}{16}\left[|f''(a)| + |f''(b)|\right].$$

The same estimates were obtained by Bayraktar and Gürbüz (see [8], Corollary 2.2).

Theorem 6. Let $f : I = [0, b^*] \to \mathbb{R}$ be a twice differentiable function on I° such as $f'' \in L[a,b]$ where $a, b \in I^\circ$. If $\frac{b}{m} \in I^\circ$ and $|f''|^q$ is (α,m)-convex on I, for $\alpha, m \in (0,1]$ and $q \geq 1$, the following inequality holds

$$\left|\frac{f(a)+f(mb)}{2} - f\left(\frac{a+mb}{2}\right)\right| \leq \frac{(mb-a)^2}{2^{4-\frac{3}{q}}} \times F,$$

where

$$F = \left[\eta\,|f''(a)|^q + m\left|\frac{1}{8}-\eta\right|\,|f''(b)|^q\right]^{\frac{1}{q}} + \left[\zeta\,|f''(a)|^q + m\left|\frac{1}{8}-\zeta\right|\,|f''(b)|^q\right]^{\frac{1}{q}};$$

$$\eta = \frac{1}{(\alpha+2)\,2^{\alpha+2}} \text{ and } \zeta = \frac{2^{\alpha+2}-1}{(\alpha+1)(\alpha+2)\,2^{\alpha+2}}.$$

Proof. Using the triangle inequality for the equality (6) in Lemma , we can write

$$\left|\frac{f(a)+f(mb)}{2} - f\left(\frac{a+mb}{2}\right)\right| \leq$$

$$\leq \frac{(mb-a)^2}{2}\left[\int_0^{1/2} t\,|f''(ta+m(1-t)b)|\,dt + \int_{1/2}^{1}(1-t)\,|f''(ta+m(1-t)b)|\,dt\right] =$$

$$= \frac{(mb-a)^2}{2}\,(I_1+I_2)\,. \tag{19}$$

Using the power mean inequality and (α,m)-convexity of $|f''|^q$ on $[a,b]$ we get

$$I_1 \leq \left(\int_0^{1/2} t dt\right)^{1-\frac{1}{q}} \left[\int_0^{1/2} t\,|f''(at+m(1-t)b)|^q\,dt\right]^{\frac{1}{q}} \leq$$

$$\leq 2^{\frac{3(1-q)}{q}}\left[|f''(a)|^q \int_0^{1/2} t^{\alpha+1}dt + m\,|f''(b)|^q \left|\int_0^{1/2} t(1-t^{\alpha})dt\right|\right]^{\frac{1}{q}}.$$

And calculating these integrals, we have

$$I_1 \leq 2^{\frac{3(1-q)}{q}}\left[\eta\,|f''(a)|^q + \left|\frac{1}{8}-\eta\right| m\,|f''(b)|^q\right]^{\frac{1}{q}}. \tag{20}$$

Similarly for I_2 we can write

$$I_2 \leq \left(\int_{1/2}^{1}(1-t)\,dt\right)^{1-\frac{1}{q}} \left[\int_{1/2}^{1}(1-t)\,|f''(at+m(1-t)b)|^q\,dt\right]^{\frac{1}{q}} \leq$$

$$\leq 2^{\frac{3(1-q)}{q}}\left[|f''(a)|^q\left|\int_{1/2}^{1} t^{\alpha}\,(1-t)\,dt\right| + m\,|f''(b)|^q\left|\int_{1/2}^{1}(1-t)\,(1-t^{\alpha})dt\right|\right]^{\frac{1}{q}}.$$

And calculating these integrals, we have

$$I_2 \leq 2^{2^{\frac{3(1-q)}{q}}}\left[\zeta\,|f''(a)|^q + m|\frac{1}{8}-\zeta|\,|f''(b)|^q\right]^{\frac{1}{q}}. \tag{21}$$

Substituting these inequalities for (20) and (21) into inequality (19) and rearranging we complete the proof.

Corollary 6. In Theorem 6 if we choose $\alpha = m = q = 1$, we have

$$\left|\frac{f(a)+f(b)}{2} - f\left(\frac{a+b}{2}\right)\right| \leq \frac{(b-a)^2}{32}\left[3\,|f''(a)| + |f''(b)|\right].$$

3 Applications to special means

We now consider the means for arbitrary real numbers α, β $(\alpha \neq \beta)$. We take

1 *Arithmetic mean*: $A(\alpha,\beta) = \frac{\alpha+\beta}{2}$.

2 *Logarithmic mean*: $L(\alpha,\beta) = \frac{\alpha-\beta}{\ln|\alpha|-\ln|\beta|}$, $|\alpha| \neq |\beta|$, $\alpha,\beta \neq 0$.

3 *Generalized log-mean*: $L_n(\alpha,\beta) = \left[\frac{\beta^{n+1}-\alpha^{n+1}}{(n+1)(\beta-\alpha)}\right]^{\frac{1}{n}}, n \in Z\backslash\{-1,0\},\ \alpha,\beta \in \mathbb{R}^+$.

4 *Harmonic mean*: $H = H(a,b) = \frac{2\alpha\beta}{\alpha+\beta}$, $\alpha+\beta \neq 0$.

Now, using some results, we give some applications to special means of real numbers.

Proposition 2. Let $a,b \in \mathbb{R}^+$, $a<b$ and $n \in Z\backslash\{-1\}$. Then we have

$$|L_n^n(a,b) - A\left[A(a^n,b^n), A^n(a,b)\right]| \leq \frac{(b-a)^2}{48}n(n-1)A(a^{n-2},b^{n-2}).$$

Proof. The assertion follows from Corollary 3 for $m=1$ applied to the (s,m)-convex function $f(x)=x^n$, $x \in \mathbb{R}$.

Proposition 3. Let $a,b \in \mathbb{R}^+$, $a<b$ then we have

$$\left|L^{-1}(a,b) - A\left[H^{-1}(a,b),\ A^{-1}(a,b)\right]\right| \leq \frac{(b-a)^2}{24}H^{-1}(a^3,b^3).$$

Proof. The assertion follows from Corollary 3 for $m=1$ applied to the (s,m)-convex function $f(x)=\frac{1}{x}$, $x \in \mathbb{R}^+$.

Proposition 4. Let $a,b \in \mathbb{R}^+$, $a<b$ and $n \in Z\backslash\{-1\}$. Then, we have

$$|A(a^n,b^n) - A^n(a,b)| \leq \frac{(b-a)^2}{8}n(n-1)A(a^{n-2},b^{n-2}).$$

Proof. The assertion follows from Corollary 5 applied to the (s,m)-convex function $f(x)=x^n$, $x \in \mathbb{R}$.

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(S, m)-дөңес және (α, m)-дөңес функциялар үшін кейбір жаңа интегралдық теңсіздіктер

Мақалада (s, m)-дөңес және (α, m)-дөңес функциялар үшін бірнеше жаңа интегралдық теңсіздіктер ұсынылған. Бұл нәтижелер жақсы белгілі Эрмит-Адамар типті интегралдық теңсіздікпен, Симпсон типті интегралдық теңсіздікпен және Йенсен типті теңсіздікпен байланысты. Басқаша айтқанда, дөңес функциялардың көрсетілген кластар арқылы осы теңсіздіктер үшін жоғарыдан жаңа бағалар алынды. Мақалада келтірілген нәтижелер дөңес функциялардың анықтамаларын тікелей пайдалануы мен Гельдер типті және дәрежелік орташа типті классикалық интегралдық теңсіздіктерді қолдану арқылы алынды. Жаңа нәтижелермен бірге авторлар әдебиеттегі интегралдық теңсіздіктерге арналған жоғары шекара бағаларын растайтын нәтижелерге қолжеткізді (дербес жағдайда M.Z. Sarıkaya және N. Aktan [35] және U. Kırmacı [7] алынған әдебиеттердегі жақсы белгілі нәтижелер). Мақаланың соңғы бөлімінде арнайы есептеу құралдары үшін алынған бағаларды кейбір қосымшалары келтірілген, яғни әртүрлі шамалар үшін арифметикалық, логарифмдік, жалпыланған логарифмдік орташа және гармоникалық орташа.

Кілт сөздер: дөңес функция, (s, m)-дөңес, (α, m)-дөңес, Эрмит-Адамар теңсіздігі, Йенсен теңсіздігі, Гёльдер теңсіздігі, орташа дәрежелі үшін теңсіздік.

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Некоторые новые интегральные неравенства для (s, m)-выпуклых и (α, m)-выпуклых функций

В статье представлено несколько новых интегральных неравенств для (s, m)-выпуклых и (α, m)-выпуклых функций. Эти результаты связаны с хорошо известным интегральным неравенством типа Эрмита-Адамара, интегральным неравенством типа Симпсона и с неравенством типа Йенсена. Другими словами, получены новые оценки сверху для этих неравенств с использованием указанных классов выпуклых функций. Представленные результаты получены с помощью непосредственно определения выпуклых функций, а также классических интегральных неравенств типа Гельдера и типа степенного среднего. Наряду с новыми результатами авторами получены результаты, подтверждающие существующие в литературе оценки верхних границ для интегральных неравенств (в частности, хорошо известные в литературе результаты M.Z. Sarıkaya и N. Aktan в [35] и U. Kırmacı в [7]). В последнем разделе статьи приведены некоторые приложения полученных оценок для специальных вычислительных средств, а именно: арифметическое, логарифмическое, обобщенное логарифмическое, среднее и среднее гармоническое для различных величин.

Ключевые слова: выпуклая функция, (s, m)-выпуклая, (α, m)-выпуклая, неравенство Эрмита-Адамара, неравенство Йенсена, неравенство Гёльдера, неравенство для среднестепенного.