UDC 517.958

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A solvability conditions of mixed problems for equations of parabolic type with involution

In this work the partial differential equations with involutions are considered. The mixed problems for the parabolic type equation, with constant and variable constants, corresponding to the Dirichlet type boundary conditions is investigated. The involution is contained by the second derivative with respect to the variable x, which is the difficult case for investigations. One-dimensional differential operators with involution have an infinite number of positive and negative eigenvalues. This means that the operator on the right-hand side of the equation under study is not semi-bounded. In the case of classical problems, ordinary differential operators usually appear on the right-hand side of the equations, which are semi-bounded. Therefore, the incorrectness of mixed problems for a parabolic equation with an involution is discussed in this paper. Examples are given. Sufficient conditions for the initial data are found when the problem under study has a unique solution. The representation of the solution in the form of partial sums of the Fourier series in eigenfunctions is found. The density in the space $L_2(-1,1)$ of the set of initial functions is proved everywhere, when the problem has a unique solution.

Keywords: Fourier method, mixed problem, involution, eigenfunctions, basis.

Introduction

We study the solvability of the following problem:

$$u_t(x,t) = u_{xx}(-x,t), \quad -1 \le x \le 1, \quad t \ge 0; \tag{1}$$

$$u_x(-1,t) = u_x(1,t) = 0, \quad u(x,0) = \varphi(x), \tag{2}$$

A transformation S of a function $f(x)$ from the class $L_2(-1,1)$ is said to be an involution, if $(S^2 f)(x) = f(x)$. In particular, a transformation of the form $(Sf)(x) = f(-x)$ is involution. Equation (1) is said to be an equation of parabolic type with involution. This name has nothing to do with the well-known classification of equations of mathematical physics.

A necessary condition of the existence of a solution of problem (1), (2) is the consistency of the initial data with equation (1) and boundary conditions (2). Therefore, we will require that

$$\varphi(x) \in C^2[-1,1] \,\text{and} \varphi'(-1) = \varphi'(1) = 0.$$

We say, that problem (1), (2) is well-posed, if 1) the solution of the problem exists, 2) the solution of the problem is unique, (3) the solution of the problem depends continuously on the initial data (is stable).

The application of the Fourier method to problem (1), (2) leads to a spectral problem with involution

$$-X''(-x) = \lambda X(x), \quad X'(-1) = X'(1) = 0. \tag{3}$$

Questions of the well-posedness of mixed problems for differential equations with involution are considered in [1–3]. In works [4, 5], inverse problems for equations with involution are considered. Spectral problems with involution were investigated in [6–15]. Mixed problems for equations of the form (1), apparently, are considered first in this paper.

The incorrectness of the mixed problem (1), (2)

It is well known [13, 14] that the spectral problem (3) is self-adjoint and has two series of eigenvalues $\lambda_{k1} = k^2\pi^2$, $\lambda_{k2} = -\left(k + \frac{1}{2}\right)^2\pi^2$. The corresponding eigenfunctions have the form $X_{k1}(x) = \cos k\pi x$, $k = 0, 1, 2, , , , ,;$ $X_{k2}(x) = \sin\left(k + \frac{1}{2}\right)\pi x$, $k = 0, 1, 2, , ,;$ which form a complete orthonormal system in the class $L_2(-1,1)$.

We note, that the spectral problem (3) has an infinite number of negative eigenvalues, the double differentiation operator on the left-hand side of the differential equation (3) (or the operator of double differentiation with respect to x on the right-hand side of the differential equation (1) is not semi-bounded. That is the fundamental difference of the equation has studied from many different equations.

The standard method consists of the idea of representing the formal solution of the mixed problem (1), (2) in the following form of infinite series

$$u(x,t)=\sum_{\lambda_{k1}} A_k e^{-\lambda_{k1}t}\cos k\pi x+\sum_{\lambda_{k2}} B_k e^{-\lambda_{k2}t}\sin\left(k+\frac{1}{2}\right)\pi x, \tag{4}$$

where

$$A_k=\int_{-1}^{1}\varphi(x)\cos k\pi x dx,\quad B_k=\int_{-1}^{1}\varphi(x)\sin\left(k+\frac{1}{2}\right)\pi x dx. \tag{5}$$

If function $\varphi(x)$ is not infinite times differentiable, i.e. if the Fourier coefficients B_k of the function $\varphi(x)$ do not decrease with sufficient rapidity, then the second term in (4) diverges, since $\lambda_{k2}<0$. Therefore, in the case of general initial value, the mixed problem (1), (2) may not have a solution. In the case when the solution exists but it does not have the property of stability i.e. does not depend continuously on the initial value. For example, perturbation

$$u_\delta(x,t)=\varepsilon e^{-\lambda_{k2}t}\sin\left(k+\frac{1}{2}\right)\pi x$$

does not exceed the number ε for $t=0$, but will be greater than any preassigned number C_0 for $t=\delta$, at sufficiently small ε and δ and sufficiently large k. Thus, the mixed problem (2) in the case of parabolic type with involution (1) is not well-posed. Nevertheless, we show that the solution of the mixed problem under study exists and is unique.

The solvability classes of the mixed problem (1), (2)

First of all, let us show the uniqueness of the solution of the mixed problem.

Theorem 1. If a solution of the mixed problem (1), (2) exists, then it is unique.

Proof. Assume that the mixed problem (1), (2) exists. Any solution $u(x,t)$ of problem (1), (2), as a function of x, can be represented as a Fourier series.

$$u(x,t)=\sum_{k=0}^{\infty}T_{k1}(t)\cos k\pi x+\sum_{k=0}^{\infty}T_{k2}(t)\sin\left(k+\frac{1}{2}\right)\pi x$$

by orthonormal basis $\{X_k(x)\}=\left\{X_{k1}=cosk\pi x,\ X_{k2}=\sin\left(k+\frac{1}{2}\right)\pi x\right\}$. Since this series converges in the sense of the norm of the space $L_2(-1,1)$, then it also converges in the sense of the scalar product.
Therefore

$$T_{k1}(t)=(u(x,t),\cos k\pi x),\quad T_{k2}(t)=\left(u(x,t),\sin\left(k+\frac{1}{2}\right)\pi x\right).$$

We write these two equations in brief in the form

$$T_k(t)=(u(x,t),X_k(x)). \tag{6}$$

We multiply both sides of equation (1) by scalar product to $X_k(x)$, which gives

$$(u_t,\ X_k)=(u_{xx}(-x,t),X_k).$$

The right-hand side of the equality obtained is twice integrable by parts, and on the left side we use the rule of differentiation with respect to the parameter t under the integral sign. Taking into account the equation (3), we obtain relation $\frac{\partial}{\partial t}(u,X_k)=\lambda_{k1}(u(x,t),X_k)$. By substitution the equality (6), we obtain the Cauchy problem for an ordinary first-order differential equation

$$T'_k(t)=-\lambda_k T_k(t),\quad T_k(0)=(\varphi,\ X_k).$$

The initial condition is obtained from (6) for $t=0$. By the uniqueness of the solution of the Cauchy problem, $T_k(t)$ is uniquely determined. This proves the uniqueness of the solution of problem (1), (2). Theorem 1 is proved.

We show classes of admissible initial functions $\varphi(x)$, for which problem (1), (2) has a solution. First we show that the series (4) is a solution of the problem (1), (2) if all the coefficients of B_k are zero.

Theorem 2. If initial function $\varphi(x)$ is even, belongs to the class $C^2[-1,1]$ and satisfies the conditions $\varphi'(-1) = \varphi'(1) = 0$, then the solution of problem (1), (2) exists, is unique and can be represented as a series (4).

Proof. If $\varphi(x)$ - even function, then all the Fourier coefficients B_k of the form (5) are equal to zero. Therefore, the series (4) takes the form

$$u(x,t) = \sum_{k=0}^{\infty} A_k e^{-k^2\pi^2 t} \cos k\pi x. \tag{7}$$

In order to prove the theorem we have to show that the series (7) converges for any $t > 0$, and it can be term-by-term differentiated once with respect to the variable t and twice with respect to the variable x. The last two operations are possible under the condition of uniform convergence of the series

$$-\sum_{k=0}^{\infty} k^2\pi^2 A_k e^{-k^2\pi^2 t} \cos k\pi x \tag{8}$$

for all $t > 0$. Uniform convergence of the series (8) is proved in the same way as in the case of a classical equation of parabolic type (see, for example, [16; 203]).
The convergence of the series (7) follows from the convergence of the majorant series

$$\sum_{k=0}^{\infty} |A_k \cos k\pi x|. \tag{9}$$

The convergence of the series (9) is proved in exactly the same way as the absolute and uniform convergence of the classical Fourier series in the trigonometric system is proved [16; 203]. Thus, the solution of problem (1), (2) exists, unique and can be represented as a series (7). The proof of the theorem is completed.

Next, let consider the problem (1), (2), where the initial function $\varphi(x)$ is a trigonometric polynomial

$$\varphi(x) = \sum_{k=0}^{N_1} a_k \cos k\pi x + \sum_{k=0}^{N_2} b_k \sin\left(k+\frac{1}{2}\right)\pi x. \tag{10}$$

We note that functions in the form of series or polynomials in eigenfunctions are used in the study of various problems. For example, in [17] (see also references in it) functions of the type (10) are used in the study of the spectral properties of loaded differential operators.

Theorem 3. If initial function $\varphi(x)$ is a trigonometric polynomial of the form (10), then the solution of problem (1), (2) exists, is unique and can be represented in the form

$$u(x,t) = \sum_{k=0}^{N_1} A_k \cos k\pi x\, e^{-k^2\pi^2 t} + \sum_{k=0}^{N_2} B_k sin\left(k+\frac{1}{2}\right)\pi x\, e^{\left(k+\frac{1}{2}\right)^2\pi^2 t},$$

where

$$A_k = \int_{-1}^{1} \varphi(x)\cos k\pi x dx, \quad k = 0,1,2,...,N_1;$$

$$B_k = \int_{-1}^{1} \varphi(x)\sin\left(k+\frac{1}{2}\right)\pi x dx, \quad k = 0,1,2,...,N_2.$$

Proof. The validity of the theorem follows from the fact that the coefficients

$$A_k = 0, \quad k = N_1+1, ..., \quad B_k = 0, \quad k = N_2+1, ...$$

and from the statement of the Theorem 1.

Since the set of trigonometric polynomials in the complete orthonormal system $\{X_{k1}, X_{k2}\}$ is everywhere dense in $L_2(-1,1)$, then from theorem 3 implies

Theorem 4. The set M of admissible initial functions is everywhere dense in $L_2(-1,1)$, if the mixed problem (1), (2) is solvable for all function from M.

A mixed problem for an equation with a variable coefficient

We consider the mixed problem (2) for an equation with a variable coefficient

$$u_t(x,t) = u_{xx}(-x,t) + q(x)\,u(x,t), \quad -1 \le x \le 1, \quad t \ge 0. \tag{11}$$

The application of the Fourier method to problem (11), (2) leads to a spectral problem with involution

$$-X''(-x) + q(x)\,X(x) = \lambda X(x), \quad X'(-1) = X'(1) = 0. \tag{12}$$

In the paper [15] it is shown that the baseness eigenfunction $\{X_k(x)\}$ of spectral problem (12) in the space $L_2(-1,1)$. If coefficient $q(x)$ is a real continuous function in the interval under consideration, then this basis is an orthonormal basis by virtue of the self-adjointness of the spectral problem (11). Therefore initial function $\varphi(x)$ can be decomposed into convergent in norm of the space $L_2(-1,1)$ Fourier series by orthonormal basis $\{X_k(x)\}$.

We have following

Theorem 5. If in equation (11) the coefficient $q(x)$ is a real continuous function and an initial function $\varphi(x)$ is a polynomial of the following form

$$u(x,t) = \sum_{\substack{\lambda_{k1}>0,\\ k=\overline{1,N}}} A_k X_{k1}(x) + \sum_{\substack{\lambda_{k2}<0,\\ k=\overline{1,M}}} B_k X_{k2}(x),$$

then the solution of problem (11), (2) exists, unique and can be represented by the following

$$u(x,t) = \sum_{\substack{\lambda_{k1}>0,\\ k=\overline{1,N}}} A_k e^{-\lambda_{k1}t} X_{k1}(x) + \sum_{\substack{\lambda_{k2}<0,\\ k=\overline{1,M}}} B_k e^{-\lambda_{k2}t} X_{k2}(x),$$

where

$$A_k = \int\limits_{-1}^{1} \varphi(x)\,X_{k1}(x)\,dx, \quad k = 1,2,...,N;$$

$$B_k = \int\limits_{-1}^{1} \varphi(x)\,X_{k2}(x)\,dx, \quad k = 1,2,...,M,$$

$X_{k1}(x)$, $X_{k2}(x)$ — eigenfunction, corresponding to eigenvalues $\lambda_{k1} > 0$ and $\lambda_{k2} < 0$ respectively.

By virtue of the density of the set of polynomials in the complete orthonormal system $\{X_k(x)\}$ in class $L_2(-1,1)$, for the mixed problem (11), (2) the assertion of Theorem 4 is satisfies.

In conclusion, we note that all the results formulated remain valid in the case of conditions $u(-1,t) = u(1,t) = 0$, $u(x,0) = \varphi(x)$,

This work was supported by the Committee of Science of the Ministry of Education and Science of the Republic Kazakhstan, project no. AP0531225.

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Ә.Ә. Сәрсенбі

Инволюциясы бар параболалық түрдегі теңдеулер үшін аралас есептердің шешімділік шарттары

Мақалада инволюциясы бар параболалық түрдегі теңдеу үшін шеттік шарттары Дирихле түрінде болатын аралас есептер қарастырылды. Коэффициенттері тұрақты және айнымалы болатын теңдеулер зерттелген. Теңдеудің x айнымалысы бойынша екінші туындысында инволюция бар. Мұндай жағдайда есептерді зерттеудің өз қиындықтары бар. Инволюциясы бар бірөлшемді дифференциалды операторлардың оң және теріс таңбалы меншікті мәндері шексіз көп болады. Бұл теңдеудің оң жағындағы оператор оң анықталған емес дегенді білдіреді. Классикалық жағдайларда әдетте теңдеудің оң жағындағы операторлар оң анықталған болып келеді. Сондықтан автор инволюциясы бар параболалық түрдегі теңдеу үшін аралас есептердің қойылымы коректілі емес болатындығын талқылаған. Мысалдар келтірген. Қарастырылып отырған есептердің жалғыз шешімі бар болуын қамтамасыз ететін бастапқы функциялар үшін жеткілікті шарттар алған. Мұндай бастапқы функциялар жиыны

$L_2(-1,1)$ кеңістігінде тығыз орналасқан жиын болатындығы есептің жалғыз шешімі бар болған жағдайда көрсетілген. Шешімнің меншікті функциялар бойынша Фурье қатарының дербес қосындылары түрінде кескінделетіндігі анықталған.

Кілт сөздер: Фурье тәсілі, аралас есеп, инволюция, меншікті функциялар, базис.

А.А. Сарсенби

Условия разрешимости смешанных задач для уравнений параболического вида с инволюцией

Работа посвящена изучению смешанных задач для уравнения параболического вида с инволюцией с краевыми условиями типа Дирихле. Рассмотрены уравнения с постоянными и переменными коэффициентами. Инволюцию содержит вторая производная по переменной x. Этот случай является трудным для изучения. Одномерные дифференциальные операторы с инволюцией имеют бесконечное число положительных и отрицательных собственных значений. Это означает, что оператор в правой части изучаемого уравнения не является полуограниченным. В случае классических задач в правой части уравнений обычно стоят обыкновенные дифференциальные операторы, которые являются полуограниченными. Поэтому в статье автором показана некорректность смешанных задач для уравнения параболического вида с инволюцией. Приведены примеры. Найдены достаточные условия на начальные данные, когда изучаемая задача имеет единственное решение. Найдено представление решения в виде частичных сумм ряда Фурье по собственным функциям. Доказана всюду плотность в пространстве $L_2(-1,1)$ множества начальных функций, когда задача имеет единственное решение.

Ключевые слова: метод Фурье, смешанная задача, инволюция, собственные функции, базис.

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