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Families of theories of abelian groups and their closures

In studying the structural properties of elementary theories, a relationship between theories with respect to a series of natural operators plays an important role. This relationship can be determined by placing models of given theories in various formula definable sets. Such sets include, for example, sets defined by unary predicates or equivalence relations. In this way, P-operators and E-operators arise, as well as their closures, and e-spectra, i.e. the numbers of new theories that may be generated by these operators. For E-operators, applicable to the families of theories of abelian groups, closures and generating sets, as well as their e-spectra are described. Szmielew invariants are used as a tool for the established characterization of a theory belonging to the E-closure of a family of theories of abelian groups. A series of families of theories corresponding to the sets of Szmielew invariants, properties of these families, and values of e-spectra are also described.

Keywords: family of theories, abelian group, E-operator, generating set, closure, e-spectrum.

0.1 Introduction

In a series of papers topological properties for families of theories are studied [1–6]. The notions of P-operator and E-operator were introduced allowing to study links between theories with respect to appropriate closure operators. These operators give possibilities to generate new theories by given families of ones, and, in special cases, to find minimal/least generating sets. Counting e-spectra for families of theories we get characteristics for maximal variations of theories in closed sets of theories with respect to generating sets.

We continue this investigation applying for families of theories of abelian groups and describing closures for families of theories of abelian groups with respect to the E-operator. In Sections 2 and 3 we consider basic notions and known results for families of theories and theories of abelian groups. In Section 4 we characterize the property when a theory of abelian groups belongs to E-closure of a given family of theories of abelian groups. This characterization allows to describe closed sets of theories of abelian groups with(out) least generating sets. Examples of these descriptions, for natural families of theories, are presented in Section 5.

0.2 Preliminaries

Throughout the paper we use the following terminology in [1, 2].

Let $P = (P_i)_{i\in I}$, be a family of nonempty unary predicates, $(\mathcal{A}_i)_{i\in I}$ be a family of structures such that P_i is the universe of $\mathcal{A}_i$, $i \in I$, and the symbols P_i are disjoint with languages for the structures $\mathcal{A}_j$, $j \in I$. The structure $\mathcal{A}_P \rightleftharpoons \bigcup\limits_{i\in I} \mathcal{A}_i$ expanded by the predicates P_i is the *P-union* of the structures $\mathcal{A}_i$, and the operator mapping $(\mathcal{A}_i)_{i\in I}$ to $\mathcal{A}_P$ is the *P-operator*. The structure $\mathcal{A}_P$ is called the *P-combination* of the structures $\mathcal{A}_i$ and denoted by $\mathrm{Comb}_P(\mathcal{A}_i)_{i\in I}$ if $\mathcal{A}_i = (\mathcal{A}_P|_{A_i})|_{\Sigma(\mathcal{A}_i)}$, $i \in I$. Structures $\mathcal{A}'$, which are elementary equivalent to $\mathrm{Comb}_P(\mathcal{A}_i)_{i\in I}$, will be also considered as P-combinations.

Clearly, all structures $\mathcal{A}' \equiv \mathrm{Comb}_P(\mathcal{A}_i)_{i\in I}$ are represented as unions of their restrictions $\mathcal{A}'_i = (\mathcal{A}'|_{P_i})|_{\Sigma(\mathcal{A}_i)}$ if and only if the set $p_\infty(x) = \{\neg P_i(x) \mid i \in I\}$ is inconsistent. If $\mathcal{A}' \neq \mathrm{Comb}_P(\mathcal{A}'_i)_{i\in I}$, we write $\mathcal{A}' = \mathrm{Comb}_P(\mathcal{A}'_i)_{i\in I\cup\{\infty\}}$, where $\mathcal{A}'_\infty = \mathcal{A}'|_{\bigcap\limits_{i\in I} \overline{P_i}}$, maybe applying Morleyzation. Moreover, we write $\mathrm{Comb}_P(\mathcal{A}_i)_{i\in I\cup\{\infty\}}$ for $\mathrm{Comb}_P(\mathcal{A}_i)_{i\in I}$ with the empty structure $\mathcal{A}_\infty$.

Note that if all predicates P_i are disjoint, a structure $\mathcal{A}_P$ is a P-combination and a disjoint union of structures $\mathcal{A}_i$. In this case the P-combination $\mathcal{A}_P$ is called *disjoint*. Clearly, for any disjoint P-combination $\mathcal{A}_P$, $\mathrm{Th}(\mathcal{A}_P) = \mathrm{Th}(\mathcal{A}'_P)$, where $\mathcal{A}'_P$ is obtained from $\mathcal{A}_P$ replacing $\mathcal{A}_i$ by pairwise disjoint $\mathcal{A}'_i \equiv \mathcal{A}_i$, $i \in I$. Thus, in this case, similar to structures the P-operator works for the theories $T_i = \mathrm{Th}(\mathcal{A}_i)$ producing the theory

$T_P = \mathrm{Th}(\mathcal{A}_P)$, being *P-combination* of T_i, which is denoted by $\mathrm{Comb}_P(T_i)_{i\in I}$. In general, for non-disjoint case, the theory T_P will be also called a *P-combination* of the theories T_i, but in such a case we will keep in mind that this P-combination is constructed with respect (and depending) to the structure $\mathcal{A}_P$, or, equivalently, with respect to any/some $\mathcal{A}' \equiv \mathcal{A}_P$.

For an equivalence relation E replacing disjoint predicates P_i by E-classes we get the structure $\mathcal{A}_E$ being the *E-union* of the structures $\mathcal{A}_i$. In this case the operator mapping $(\mathcal{A}_i)_{i\in I}$ to $\mathcal{A}_E$ is the *E-operator*. The structure $\mathcal{A}_E$ is also called the *E-combination* of the structures $\mathcal{A}_i$ and denoted by $\mathrm{Comb}_E(\mathcal{A}_i)_{i\in I}$; here $\mathcal{A}_i = (\mathcal{A}_E|_{A_i})|_{\Sigma(\mathcal{A}_i)}$, $i \in I$. Similar above, structures $\mathcal{A}'$, which are elementary equivalent to $\mathcal{A}_E$, are denoted by $\mathrm{Comb}_E(\mathcal{A}'_j)_{j\in J}$, where $\mathcal{A}'_j$ are restrictions of $\mathcal{A}'$ to its E-classes. The E-operator works for the theories $T_i = \mathrm{Th}(\mathcal{A}_i)$ producing the theory $T_E = \mathrm{Th}(\mathcal{A}_E)$, being *E-combination* of T_i, which is denoted by $\mathrm{Comb}_E(T_i)_{i\in I}$ or by $\mathrm{Comb}_E(\mathcal{T})$, where $\mathcal{T} = \{T_i \mid i \in I\}$.

Clearly, $\mathcal{A}' \equiv \mathcal{A}_P$ realizing $p_\infty(x)$ is not elementary embeddable into $\mathcal{A}_P$ and can not be represented as a disjoint P-combination of $\mathcal{A}'_i \equiv \mathcal{A}_i$, $i \in I$. At the same time, there are E-combinations such that all $\mathcal{A}' \equiv \mathcal{A}_E$ can be represented as E-combinations of some $\mathcal{A}'_j \equiv \mathcal{A}_i$. We call this representability of $\mathcal{A}'$ to be the *E-representability*.

If there is $\mathcal{A}' \equiv \mathcal{A}_E$ which is not E-representable, we have the E'-representability replacing E by E' such that E' is obtained from E adding equivalence classes with models for all theories T, where T is a theory of a restriction $\mathcal{B}$ of a structure $\mathcal{A}' \equiv \mathcal{A}_E$ to some E-class and $\mathcal{B}$ is not elementary equivalent to the structures $\mathcal{A}_i$. The resulting structure $\mathcal{A}_{E'}$ (with the E'-representability) is a *e-completion*, or a *e-saturation*, of $\mathcal{A}_E$. The structure $\mathcal{A}_{E'}$ itself is called *e-complete*, or *e-saturated*, or *e-universal*, or *e-largest*.

For a structure $\mathcal{A}_E$ the number of *new* structures with respect to the structures $\mathcal{A}_i$, i. e., of the structures $\mathcal{B}$ which are pairwise elementary non-equivalent and elementary non-equivalent to the structures $\mathcal{A}_i$, is called the *e-spectrum* of $\mathcal{A}_E$ and denoted by $e\text{-}\mathrm{Sp}(\mathcal{A}_E)$. The value $\sup\{e\text{-}\mathrm{Sp}(\mathcal{A}') \mid \mathcal{A}' \equiv \mathcal{A}_E\}$ is called the *e-spectrum* of the theory $\mathrm{Th}(\mathcal{A}_E)$ and denoted by $e\text{-}\mathrm{Sp}(\mathrm{Th}(\mathcal{A}_E))$.

If $\mathcal{A}_E$ does not have E-classes $\mathcal{A}_i$, which can be removed, with all E-classes $\mathcal{A}_j \equiv \mathcal{A}_i$, preserving the theory $\mathrm{Th}(\mathcal{A}_E)$, then $\mathcal{A}_E$ is called *e-prime*, or *e-minimal*.

For a structure $\mathcal{A}' \equiv \mathcal{A}_E$ we denote by $\mathrm{TH}(\mathcal{A}')$ the set of all theories $\mathrm{Th}(\mathcal{A}_i)$ of E-classes $\mathcal{A}_i$ in $\mathcal{A}'$.

By the definition, an e-minimal structure $\mathcal{A}'$ consists of E-classes with a minimal set $\mathrm{TH}(\mathcal{A}')$. If $\mathrm{TH}(\mathcal{A}')$ is the least for models of $\mathrm{Th}(\mathcal{A}')$ then $\mathcal{A}'$ is called *e-least*.

Definition [2]. Let $\overline{\mathcal{T}}$ be the class of all complete elementary theories of relational languages. For a set $\mathcal{T} \subset \overline{\mathcal{T}}$ we denote by $\mathrm{Cl}_E(\mathcal{T})$ the set of all theories $\mathrm{Th}(\mathcal{A})$, where $\mathcal{A}$ is a structure of some E-class in $\mathcal{A}' \equiv \mathcal{A}_E$, $\mathcal{A}_E = \mathrm{Comb}_E(\mathcal{A}_i)_{i\in I}$, $\mathrm{Th}(\mathcal{A}_i) \in \mathcal{T}$. As usual, if $\mathcal{T} = \mathrm{Cl}_E(\mathcal{T})$ then $\mathcal{T}$ is said to be *E-closed*.

The operator Cl_E of E-closure can be naturally extended to the classes $\mathcal{T} \subset \overline{\mathcal{T}}$ as follows: $\mathrm{Cl}_E(\mathcal{T})$ is the union of all $\mathrm{Cl}_E(\mathcal{T}_0)$ for subsets $\mathcal{T}_0 \subseteq \mathcal{T}$.

For a set $\mathcal{T} \subset \overline{\mathcal{T}}$ of theories in a language Σ and for a sentence φ with $\Sigma(\varphi) \subseteq \Sigma$ we denote by $\mathcal{T}_\varphi$ the set $\{T \in \mathcal{T} \mid \varphi \in T\}$.

Proposition 2.1 [2]. *If $\mathcal{T} \subset \overline{\mathcal{T}}$ is an infinite set and $T \in \overline{\mathcal{T}} \setminus \mathcal{T}$ then $T \in \mathrm{Cl}_E(\mathcal{T})$ (i.e., T is an accumulation point for $\mathcal{T}$ with respect to E-closure Cl_E) if and only if for any formula $\varphi \in T$ the set $\mathcal{T}_\varphi$ is infinite.*

Theorem 2.2 [2]. *If $\mathcal{T}'_0$ is a generating set for a E-closed set $\mathcal{T}_0$ then the following conditions are equivalent:*
(1) *$\mathcal{T}'_0$ is the least generating set for $\mathcal{T}_0$;*
(2) *$\mathcal{T}'_0$ is a minimal generating set for $\mathcal{T}_0$;*
(3) *any theory in $\mathcal{T}'_0$ is isolated by some set $(\mathcal{T}'_0)_\varphi$, i.e., for any $T \in \mathcal{T}'_0$ there is $\varphi \in T$ such that $(\mathcal{T}'_0)_\varphi = \{T\}$;*
(4) *any theory in $\mathcal{T}'_0$ is isolated by some set $(\mathcal{T}_0)_\varphi$, i.e., for any $T \in \mathcal{T}'_0$ there is $\varphi \in T$ such that $(\mathcal{T}_0)_\varphi = \{T\}$.*

0.3 Theories of abelian groups

Let A be an abelian group. Then kA denotes its subgroup $\{ka \mid a \in A\}$ and $A[k]$ denotes the subgroup $\{a \in A \mid ka = 0\}$. It p is a prime number and $pA = \{0\}$ then $\dim A$ denotes the dimension of the group A, considered as a vector space over a field with p elements. The following numbers, for arbitrary p and n (p is prime and n is natural) are called the *Szmielew invariants* for the group A [7]:

$$\alpha_{p,n}(A) = \min\{\dim((p^nA)[p]/(p^{n+1}A)[p]), \omega\};$$

$$\beta_p(A) = \min\{\inf\{\dim((p^nA)[p] \mid n \in \omega\}, \omega\};$$

$$\gamma_p(A) = \min\{\inf\{\dim((A/A[p^n])/p(A/A[p^n])) \mid n \in \omega\}, \omega\};$$

$$\varepsilon(A) \in \{0,1\} \,\text{and}\, \varepsilon(A) = 0 \Leftrightarrow (nA = \{0\} \,\text{for some}\, n \in \omega, n \neq 0).$$

It is known [7, Theorem 8.4.10] that two abelian groups are elementary equivalent if and only if they have same Szmielew invariants. Besides, the following proposition holds.

Proposition 3.1 [7, Proposition 8.4.12]. *Let for any p and n the cardinals $\alpha_{p,n}$, β_p, $\gamma_p \leq \omega$, and $\varepsilon \in \{0,1\}$ are given. Then there is an abelian group A such that the Szmielew invariants $\alpha_{p,n}(A)$, $\beta_p(A)$,$\gamma_p(A)$, and $\varepsilon(A)$ are equal to $\alpha_{p,n}$, β_p, γ_p, and ε, respectively, if and only if the following conditions hold:*

(1) *if for prime p the set $\{n \mid \alpha_{p,n} \neq 0\}$ is infinite then $\beta_p = \gamma_p = \omega$;*

(2) *if $\varepsilon = 0$ then for any prime p, $\beta_p = \gamma_p = 0$ and the set $\{\langle p, n\rangle \mid \alpha_{p,n} \neq 0\}$ is finite.*

We denote by $\mathbf{Q}$ the additive group of rational numbers, $\mathbf{Z}_{p^n}$ — the cyclic group of the order p^n, $\mathbf{Z}_{p^\infty}$ — the quasi-cyclic group of all complex roots of 1 of degrees p^n for all $n \geq 1$, R_p — the group of irreducible fractions with denominators which are mutually prime with p. The groups $\mathbf{Q}$, $\mathbf{Z}_{p^n}$, R_p, $\mathbf{Z}_{p^\infty}$ are called *basic*. Below the notations of these groups will be identified with their universes.

Since abelian groups with same Szmielew invariants have same theories, any abelian group A is elementary equivalent to a group

$$\oplus_{p,n}\mathbf{Z}_{p^n}^{(\alpha_{p,n})} \oplus \oplus_p \mathbf{Z}_{p^\infty}^{(\beta_p)} \oplus \oplus_p R_p^{(\gamma_p)} \oplus \mathbf{Q}^{(\varepsilon)},$$

where $B^{(k)}$ denotes the direct sum of k subgroups isomorphic to a group B. Thus, any theory of an abelian group has a model being a direct sum of based groups.

Recall that any complete theory of an abelian group is based by the set of positive primitive formulas [7, Lemma 8.4.5], reduced to the set of the following formulas:

$$\exists y(m_1x_1 + \ldots + m_nx_n \approx p^k y); \tag{11}$$

$$m_1x_1 + \ldots + m_nx_n \approx 0, \tag{12}$$

where $m_i \in \mathbf{Z}$, $k \in \omega$, p is a prime number [8], [7, Lemma 8.4.7]. Formulas (11) and (12) allow to witness that Szmielew invariants defines theories of abelian groups modulo Proposition 3.1.

0.4 Families of theories of abelian groups and their closures

Denote by $\overline{\mathcal{TA}}$ the set of all theories of abelian groups. Below we consider families $\mathcal{T} \subseteq \overline{\mathcal{TA}}$ and corresponding families $\mathcal{T}'$, where constants 0 are replaced by unary predicates P_0 with unique realizations 0, and operations + are replaced by ternary predicates S, where $\models S(a,b,c) \Leftrightarrow a+b=c$. Clearly, each theory $T \in \mathcal{T}$ can be reconstructed by the correspondent theory $T' \in \mathcal{T}'$ and vice versa. So we can freely replace the closure $\mathrm{Cl}_E(\mathcal{T}')$ (and its elements) by the correspondent set of theories of abelian groups, denoted by $\mathrm{Cl}_E(\mathcal{T})$ (as well as by correspondent theories).

Now we fix a family $\mathcal{T}$. In view of Proposition 2.1 and the basedness by the set of formulas (11) and (12) we have the following lemmas.

Lemma 4.1. A family $\mathrm{Cl}_E(\mathcal{T})$ does not contain theories with new finite invariants $\alpha_{p,n}$, β_p, γ_p as well as invariants with new p and n.

Lemma 4.2. A family $\mathrm{Cl}_E(\mathcal{T})$ contains a theory with infinite invariant $\alpha_{p,n}$ if and only if $\mathcal{T}$ contains a theory with that infinite invariant or $\mathcal{T}$ has theories with infinitely many distinct finite invariants $\alpha_{p,n}$.

Using Proposition 3.1 and Lemma 4.2 we have.

Lemma 4.3. A family $\mathrm{Cl}_E(\mathcal{T})$ contains a theory with an infinite invariant β_p (respectively, γ_p) if and only if $\mathcal{T}$ contains a theory with that infinite invariant, or $\mathcal{T}$ has theories with infinitely many distinct finite invariants $\alpha_{p,n}$, or $\mathcal{T}$ has theories with infinitely many distinct finite invariants β_p (γ_p).

Lemma 4.4. A family $\mathrm{Cl}_E(\mathcal{T})$ contains a theory with $\varepsilon = 1$ if and only if $\mathcal{T}$ contains a theory with $\varepsilon = 1$, or $\mathcal{T}$ has theories forming infinite set $\{\langle p, n\rangle \mid \alpha_{p,n} \neq 0\}$, or $\mathcal{T}$ has theories with positive invariants β_p or γ_p.

Lemmas 4.2 – 4.4 describe approximations of new infinite Szmielew invariants by finite ones.

Applying Proposition 2.1, the basedness of theories in Section 3, and Lemmas 4.1 – 4.4 we can describe E-closures for families of theories of abelian groups.

For this aim we remember the following fact for a family $\mathcal{T}$ of language uniform theories T_I defined by sets I of nonempty predicates.

Recall [3] that a theory T in a predicate language Σ is said to be *language uniform*, or a LU-*theory* if for each arity n any substitution on the set of non-empty n-ary predicates preserves T.

Proposition 4.5 [3, Proposition 6]. *If $T_J \notin \mathcal{T}$ then $T_J \in \mathrm{Cl}_E(\mathcal{T})$ if and only if for any finite set $J_0 \subset I_0$ there are infinitely many $T_I\mathcal{T}$ with*

$$J \cap J_0 = I \cap J_0. \tag{13}$$

The equations (13) for indexes mean the *local correspondence* between $\mathcal{T}$ and T_J. Using replacements of these index sets by sequences of Szmielew invariants for theories of abelian groups we get the local correspondence for families of theories in $\overline{\mathcal{TA}}$: for a family $\mathcal{T} \subseteq \overline{\mathcal{TA}}$, a theory $T \in \overline{\mathcal{TA}} \setminus \mathcal{T}$ *locally corresponds* to $\mathcal{T}$ if replaced (13) holds modulo infinite Szmielew invariants and, besides, simultaneously the sequence of infinite Szmielew invariants for T, which are not represented in infinitely many theories in $\mathcal{T}$, is approximated by sequences of corresponding finite Szmielew invariants for theories in $\mathcal{T}$ used for replaced (13).

Theorem 4.6. If $\mathcal{T}$ is an infinite family of theories of abelian groups and $T \notin \mathcal{T}$ is a theory of an abelian group then $T \in \mathrm{Cl}_E(\mathcal{T})$ if and only if T has infinite models (i. e., T has some infinite $\alpha_{p,n}$ or some positive β_p, γ_p, ε) and locally corresponds to $\mathcal{T}$.

Proof. If $T \in \mathrm{Cl}_E(\mathcal{T})$ then T has infinite models since finite models define only finitely many positive Szmielew invariants, these invariants are exhausted by finite $\alpha_{p,n}$, and theories with these invariants are isolated. If T locally does not correspond to $\mathcal{T}$ then $T \notin \mathrm{Cl}_E(\mathcal{T})$ in view of Proposition 2.1.

Conversely if T has infinite models and locally corresponds to $\mathcal{T}$ then $\mathrm{Cl}_E(\mathcal{T})$ contains a theory with same Szmielew invariants as for T and thus $T \in \mathrm{Cl}_E(\mathcal{T})$.

0.5 Generating sets and e-spectra

Theorem 2.2, Proposition 3.1, and Theorem 4.6 allow to characterize families of theories of abelian groups with(out) least generating sets as well as to describe e-spectra for E-combinations of theories in $\overline{\mathcal{TA}}$.

Following the series of Szmielew invariants, for a theory $T \in \overline{\mathcal{TA}}$ we consider the *support* $\mathrm{Supp}(T)$ of T being the set of positive Szmielew invariants for T. Now we denote by $\mathcal{FS}$ (respectively, $\mathcal{CFS}$) the set of all theories in $\overline{\mathcal{TA}}$ having (co)finite supports. By $\mathcal{ICIS}$ we denote the set of all theories in $\overline{\mathcal{TA}}$ having infinite and co-infinite supports. By $\mathcal{F}$ we denote the set of all theories in $\overline{\mathcal{TA}}$ with finite Szmielew invariants, and by $\mathcal{INF}$ — with infinite Szmielew invariants $\alpha_{p,n} > 0$, $\beta_p > 0$, $\gamma_p > 0$.

Clearly, $\mathrm{Cl}_E(\mathcal{FS}) = \overline{\mathcal{TA}}$ and $\mathrm{Cl}_E(\mathcal{CFS}) = \overline{\mathcal{TA}}$ implying $\mathrm{Cl}_E(\mathcal{ICIS}) = \overline{\mathcal{TA}}$. Note also that $\mathrm{Cl}_E(\mathcal{F}) = \overline{\mathcal{TA}}$ whereas $\mathcal{INF}$ is E-closed.

By $\mathbf{A}$, $\mathbf{B}$, $\mathbf{\Gamma}$, $\mathbf{E}$ we denote the classes of all theories in $\overline{\mathcal{TA}}$ whose positive Szmielew invariants are exhausted by $\alpha_{p,n}$, β_p, γ_p, ε, respectively. For $\mathbf{X}, \mathbf{Y}, \mathbf{Z}, \mathbf{U} \in \{\mathbf{A}, \mathbf{B}, \mathbf{\Gamma}, \mathbf{E}\}$ we denote by $\mathbf{XY}$, $\mathbf{XYZ}$, $\mathbf{XYZU}$, respectively, the set of all theories in $\overline{\mathcal{TA}}$ whose positive Szmielew invariants are exhausted by corresponding $\alpha_{p,n}$, β_p, γ_p, ε for $\mathbf{X}, \mathbf{Y}, \mathbf{Z}, \mathbf{U}$.

For $\mathbf{X}$ as above and for a sequence S of some Szmielew invariants, we write $\mathbf{X}_S$ for the set of of all theories T in $\mathbf{X}$ such that Szmielew invariants for T equal to corresponding values in S. If the sequences S do not have finite positive values we denote by $\mathbf{X}^\infty$ the union of these $\mathbf{X}_S$. If for a subset P_0 of the set P of all prime numbers the sequences S do not have positive values for $p \in P \setminus P_0$ we denote by $\mathbf{X}_{P_0}$ the union of these $\mathbf{X}_S$. We write $\mathbf{X}_p$ instead X_{P_0} if P_0 is a singleton $\{p\}$.

As above we denote by $\mathbf{X}_{P_0}\mathbf{Y}_{P'_0}$, $\mathbf{X}_{P_0}\mathbf{Y}_{P'_0}\mathbf{Z}_{P''_0}$, $\mathbf{X}_{P_0}\mathbf{Y}_{P'_0}\mathbf{Z}_{P''_0}\mathbf{U}_{P'''_0}$, respectively, the set of all theories in $\overline{\mathcal{TA}}$ whose positive Szmielew invariants are exhausted by corresponding $\alpha_{p,n}$, β_p, γ_p, ε for $\mathbf{X}_{P_0}, \mathbf{Y}_{P'_0}, \mathbf{Z}_{P''_0}, \mathbf{U}_{P'''_0}$.

Remark 5.1. If S_T is a sequence of all Szmielew invariants for a theory T then $\mathbf{A}_{S_T}\mathbf{B}_{S_T}\mathbf{\Gamma}_{S_T}\mathbf{E}_{S_T} = \{T\}$. As Proposition 3.1 asserts, some Szmielew invariants can be reconstructed automatically using the rest. Therefore, for instance, if Szmielew invariants $\alpha_{p,n}$, β_p, γ_p for T imply $\varepsilon = 1$ then we have $\mathbf{A}_{S'_T}\mathbf{B}_{S'_T}\mathbf{\Gamma}_{S'_T} = \{T\}$ for the subsequence S'_T of S_T which is obtained removing ε. We have a similar effect for $\beta_p = \omega$ and $\gamma_p = \omega$ with $|\{n \mid \alpha_{p,n} \neq 0\}| = \omega$. In such a case S_T can be reconstructed both from S'_T and from S''_T which is obtained from S'_T removing considered β_p and γ_p.

In general case, subsequences S'''_T of S_T define, for combinations of $\mathbf{A}_{S'''_T}$, $\mathbf{B}_{S'''_T}$, $\mathbf{\Gamma}_{S'''_T}$, $\mathbf{E}_{S'''_T}$, the following possibilities for cardinalities: 1 (if T is uniquely defined), 2 (having, for instance, positive $\alpha_{p,n}$ only and varying ε), ω (having finitely many free positions for Szmielew invariants which can vary independently from 0 to ω), 2^ω (having countably many free positions for Szmielew invariants which can vary independently from 0 to ω).

Recall [9] that a group $\mathcal{A}$ is *divisible* if for any natural $n > 0$ and any element $a \in A$ the equation $nx = a$ has a solution in $\mathcal{A}$.

Theorem 5.2. [9]. *Any divisible subgroup $\mathcal{A}$ of abelian group $\mathcal{B}$ is a direct summand in $\mathcal{B}$.*

Theorem 5.3. [9]. *Any nonzero divisible abelian group $\mathcal{A}$ is represented a direct sum of groups isomorphic to $\mathbf{Q}$ or $\mathbf{Z}_{p^\infty}$.*

Recall [9, 10] that a group $\mathcal{A}$ is *bounded* if there is a positive number n such that $n\mathcal{A} = \{0\}$. Otherwise the group $\mathcal{A}$ is called *unbounded*. A group $\mathcal{A}$ is *torsion free* if all nonunit elements have infinite order.

The following proposition is implied by Proposition 3.1 and summarizes possibilities for combinations of $\mathbf{A}$, $\mathbf{B}$, $\mathbf{\Gamma}$, $\mathbf{E}$.

Proposition 5.4. 1. $\mathbf{AB} = \mathbf{A\Gamma} = \mathbf{AB\Gamma} = \mathbf{A} \subset \mathcal{FS}$, $\mathbf{A}$ is divided into $\mathbf{A} \cap \mathcal{F}$, consisting of theories with finite models, and $\mathbf{A} \setminus \mathcal{F}$, consisting of theories with infinite bounded models.

2. $\mathbf{B} = \mathbf{\Gamma} = \mathbf{B\Gamma} = \mathbf{O}$, where $\mathbf{O}$ is a singleton consisting of the one-element group.

3. $\mathbf{AE}$ consists of theories T without β_p and γ_p in $\mathrm{Supp}(T)$ and such that sets $\{n \mid \alpha_{p,n} \neq 0\}$ are finite for each p, i.e., with bounded quotients with respect to maximal divisible subgroups.

4. $\mathbf{BE}$ consists of all theories of divisible abelian groups.

5. $\mathbf{\Gamma E}$ consists of all theories of torsion free abelian groups.

6. $\mathbf{ABE}$ consists of all theories of abelian groups $\mathcal{A}$ with bounded quotients relative to maximal divisible subgroups $\mathcal{B}$, i. e., the theories $\mathrm{Th}(\mathcal{A}/\mathcal{B})$ form the set $\mathbf{A}$.

7. $\mathbf{A\Gamma E}$ consists of all theories of abelian groups without β_p in $\mathrm{Supp}(T)$ and such that sets $\{n \mid \alpha_{p,n} \neq 0\}$ are finite for each p.

8. $\mathbf{B\Gamma E}$ consists of all theories of abelian groups such that quotients with respect to maximal divisible subgroups are torsion free.

9. $\mathbf{AB\Gamma E} = \overline{\mathcal{TA}}$.

Since theories of finite abelian groups with unbounded $\alpha_{p,n}$ are isolated and force theories with infinite $\alpha_{p,n}$, β_p, γ_p, and positive ε, as well as since these values for distinct p are independent, we have the following proposition.

Proposition 5.5. 1. *For any* $P_0 \subseteq P$, $\mathrm{Cl}_E\left(\bigcup\limits_{p\in P_0}(\mathbf{A}_p \cap \mathcal{F})\right) = \mathrm{Cl}_E\left(\bigcup\limits_{p\in P_0}\mathbf{A}_p\right) = \bigcup\limits_{p\in P_0}\mathbf{A}_p\mathbf{B}_p^\infty\mathbf{\Gamma}_p^\infty\mathbf{E}$ *with the least generating set* $\bigcup\limits_{p\in P_0}(\mathbf{A}_p \cap \mathcal{F})$.

2. *For any* $P_0 \subseteq P$, $\mathrm{Cl}_E(\mathbf{A}_{P_0} \cap \mathcal{F}) = \mathrm{Cl}_E(\mathbf{A}_{P_0}) = \mathbf{A}_{P_0}\mathbf{B}_{P_0}^\infty\mathbf{\Gamma}_{P_0}^\infty\mathbf{E}$ *with the least generating set* $\mathbf{A}_{P_0} \cap \mathcal{F}$; $\mathrm{Cl}_E\left(\bigcup\limits_{p\in P_0}(\mathbf{A}_p \cap \mathcal{F})\right)$ *is a subset of* $\mathrm{Cl}_E(\mathbf{A}_{P_0} \cap \mathcal{F})$ *which is proper if and only if* $|P_0| \geq 2$.

Taking $P_0 = P$ we have the following

Corollary 5.6. $\mathrm{Cl}_E(\mathbf{A} \cap \mathcal{F}) = \mathrm{Cl}_E(\mathbf{A}) = \mathbf{AB}^\infty\mathbf{\Gamma}^\infty\mathbf{E}$ *with the least generating set* $\mathbf{A} \cap \mathcal{F}$.

Clearly, e-$\mathrm{Sp}(T) = 0$ for any theory T being a E-combination with unique finite structure, in particular, for a finite abelian group. Now we divide $\mathbf{A}_p$ into singletons $\mathbf{A}_{\alpha_{p,n}}$ consisting of theories of abelian groups with unique positive Szmielew invariant $\alpha_{p,n}$. For a fixed p and n and an infinite union $\bigcup\limits_{\alpha_{p,n}}\mathbf{A}_{\alpha_{p,n}}$ produces a family of theories whose E-combination $T_{p,n}$ has e-$\mathrm{Sp}(T_{p,n}) = 1$ witnessed by $\mathbf{A}_{\alpha_{p,n}}$ with $\alpha_{p,n} = \omega$. Uniting the families $\bigcup\limits_{\alpha_{p,n}}\mathbf{A}_{\alpha_{p,n}}$ for $p \in P_0$ we get E-combinations T with e-$\mathrm{Sp}(T) = |P_0|$ which is obtained by additivity as in [4]. Taking finite direct sums $\oplus_{p,n}\mathbf{Z}_{p^n}^{(\alpha_{p,n})}$ we again can produce infinite $\alpha_{p,n}$ for E-closures such that these $\alpha_{p,n}$ can be independently achieved or not achieved. Thus we get 2^ω possibilities for variations of infinite $\alpha_{p,n}$ which is witnessed by some E-combinations T with e-$\mathrm{Sp}(T) = 2^\omega$. Since there are 2^ω distinct theories of abelian groups this value is maximal. Summarizing the arguments we have arbitrary admissible values of e-spectra and obtain the following

Theorem 5.7. For any $\lambda \in \omega \cup \{\omega, 2^\omega\}$ *there is an* E*-combination* T *of theories of finite abelian groups* (*in* $\mathbf{A} \cap \mathcal{F}$ *and with least generating set*) *such that* e-$\mathrm{Sp}(T) = \lambda$.

Now we define a subfamily of $\mathrm{Cl}_E(\mathbf{A})$ producing an E-combination without the least generating set. Choose an infinite set $P_0 \subseteq P$ and take a countable set $D \subset \mathcal{P}(P_0)$ such that $\langle D, \subseteq\rangle$ is a dense linearly ordered set isomorphic to $\langle \mathbf{Q}, \leq\rangle$ and without cuts (A, A') having $\bigcup A \neq \bigcap A'$. Denote by $\mathrm{Cl}_E(\mathbf{A})_D$ the family

$$\left\{\mathrm{Th}\left(\oplus_{p\in X}\mathbf{Z}_p^{(\omega)}\right) \mid X \in D\right\}.$$

Clearly, $\mathrm{Cl}_E(\mathbf{A})_D$ does not have isolated points and has 2^ω cuts producing $|\mathrm{Cl}_E(\mathrm{Cl}_E(\mathbf{A})_D)| = 2^\omega$. Moreover, for any $P_0 \subseteq P$ with $|P \setminus P_0| = \omega$ we can take continuum many infinite $P_0' \subset P$ which are disjoint from P_0 and produce continuum many theories in corresponding sets $\mathrm{Cl}_E(\mathbf{A})_{D'}$, for $D' \subset \mathcal{P}(P_0')$, and separated from $\mathrm{Cl}_E(\mathbf{A})_D$ with respect to Hausdorff topology.

Similarly we can add theories in $\mathbf{A} \cap \mathcal{F}$ with positive invariants for $P_0' \subset P$ which is disjoint from P_0 and produce the value 2^ω for e-spectrum. Again the E-closure of that extended family does not have the least generating set.

Thus, by Theorem 2.2, the following theorem holds.

Theorem 5.8. There are 2^ω families $\mathrm{Cl}_E(\mathbf{A})_D$ whose E-closures do not have least generating sets and whose E-combinations T satisfy $e\text{-}\mathrm{Sp}(T)=2^\omega$.

Addind/replacing the arguments above for $\alpha_{p,n}$ with β_p and/or γ_p we get the following theorems.

Theorem 5.9. For any $\lambda\in\omega\cup\{\omega,2^\omega\}$ there is an E-combination T of theories in **BE** (*respectively,* **ΓE**, **AΓE**, **BΓE**, $\overline{\mathcal{TA}}$) *and with least generating set*) *such that $e\text{-}\mathrm{Sp}(T)=\lambda$.*

Theorem 5.10. There are 2^ω families $\mathrm{Cl}_E(\mathbf{BE})_D$ (respectively, $\mathrm{Cl}_E(\mathbf{\Gamma E})_D$, $\mathrm{Cl}_E(\mathbf{A\Gamma E})_D$, $\mathrm{Cl}_E(\mathbf{B\Gamma E})_D$, $\mathrm{Cl}_E(\overline{\mathcal{TA}})_D$) whose E-closures do not have least generating sets and whose E-combinations T satisfy $e\text{-}\mathrm{Sp}(T)=2^\omega$.

Clearly, Theorems 5.8 and 5.10 are witnessed by subfamilies of $\mathcal{ICIS}$.

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Абельдік группалар теориясының жиынтықтары және олардың тұйықтамасы

Элементарлы теориялардың қасиеттерінің құрылымдарын оқу барысында маңызды рөлді кәдімгі операторлар қатарына қатысты теориялардың арасындағы өзара байланыстары атқарады. Бұл өзара байланысты осы теориялардың модельдерін әртүрлі формульді анықталған жиындарға орналастыру арқылы анықтауға болады. Осындай теорияларға, мысалы, бірорынды предикатпен немесе эквивалентті қатынаспен берілген жиындар жатады. Сол себепті, P-операторлар және E-операторлар және олардың тұйықтамалары e-спектрлар, яғни осы теориялармен туындалатын жаңа теориялар саны, пайда болады. E-операторлар үшін абельдік группалар теориясының жиынтықтарына сәйкес тұйықтамалар және туындалған жиындар, сонымен бірге e-спектрлар сипатталады. Құрал ретінде қойылған сипаттамада теорияның E-тұйықтамасында осы абельдік группалар теориясының жиынтығы үшін шмелевтік инварианттары қолданылады. Бұл мақалада теориялар жиындықтарының сериялары және сәйкесінше шмелевтік инварианттарының жиынтықтары анықталады, осы жиынтықтардың қасиеттері зерттеледі, сонымен қатар e-спектрлардың мағыналары сипатталады.

Кілт сөздер: теориялар жиынтығы, абельдік группа, E-оператор, туындалған жиын, тұйықтама, e-спектр.

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Семейства теорий абелевых групп и их замыкания

При изучении структурных свойств элементарных теорий важную роль играет взаимосвязь между теориями относительно ряда естественных операторов. Эту взаимосвязь можно определять, помещая модели данных теорий в различные формульно определимые множества. К таким множествам относятся, например, множества, задаваемые одноместными предикатами или отношениями эквивалентности. Таким образом, возникают P-операторы и E-операторы, их замыкания, а также e-спектры, т.е. новые теории, которые могут порождаться данными операторами. Для E-операторов, применительно к семействам теорий абелевых групп, описываются замыкания и порождающие множества, а также их e-спектры. В качестве инструмента для установленной характеризации попадания теории в E-замыкание данного семейства теорий абелевых групп используются шмелевские инварианты. Определяются серии семейств теорий, соответствующих совокупностям шмелевских инвариантов, исследуются свойства этих семейств, а также описываются значения e-спектров.

Ключевые слова: семейство теорий, абелева группа, E-оператор, порождающее множество, замыкание, e-спектр.

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