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The tri-harmonic Neumann problem

In this article investigated the tri-harmonic Neumann function for the unit dics. For harmonics functions the Neumann's boundary problem is well studied and solved under certain conditions through Neumann's function, sometimes it is also called Green's function of the second order. Any case of finding of Green function of the corresponding boundary value problem is very important for this or that area D as it contains extensive information, allowing to write out a large number of analytical solutions in the form of integrated ratios. At the same time the specified procedure makes the main difficulty at the solution Dirichlet and Neumann problems and in an explicit form Green function is known only for a small number of simple areas. The harmonics Green function with itself consistently leads to the subsequent polyharmonic Green function which can be used to solve the subsequent Dirichlet problem for higher order of the Poisson equation. Methods of integrated transformation have received tri-harmonic Neumann function in explicit form for the unit disc of the complex plane with biharmonic Neumann function. With Neumann's function an integrated idea is given by development for the tri-harmonic operator. Above-mentioned polyharmonic Green function for the unit disc gives rise to the solution some specific polyharmonic objective of Dirichlet problem. In the same way harmonic Neumann function with itself consistently leads to the subsequent polyharmonic Neumann function. Received in the present article result allows to expect interesting prospects in further development of the analytical theory of boundary valua problems in complex analysis for the equations of elliptic type.

Keywords: Neumann function, Green function, harmonic function, potential, field, the Dirichlet problem.

The Neumann function for the Laplacian of the unit disc is given as

$$N_1(z,\zeta) = -\log|(\zeta - z)(1 - z\bar{\zeta})|^2. \tag{1}$$

This function are related to the fundamental solution of the Laplacian. The Neumann function on the boundary satisfies

$$\partial_{\nu_z} N_1(z,\zeta) = (z\partial_z + \bar{z}\partial_{\bar{z}})N_1(z,\zeta) = -2.$$

Neumann boundary conditions are given via outer normal derivatives ∂_ν. For the unit disc this is

$$\partial_\nu = z\partial_z + \bar{z}\partial_{\bar{z}}.$$

Typical for Neumann problems is that they are in general not well-posed. They are neither always solvable nor uniquely solvable. As well solvability conditions have to be determined as normalization conditions to be posed.

The bi-harmonic Neumann function has the form [1], [2].

$$-N_2(z,\zeta) = |\zeta - z|^2\left[\log|(\zeta - z)(1 - z\bar{\zeta})|^2 - 4\right] + 4\sum_{k=2}^{+\infty}\frac{(z\bar{\zeta})^k + (\bar{z}\zeta)^k}{k^2} +$$

$$+2\left[z\bar{\zeta} + \bar{z}\zeta\right]\log|1 - z\bar{\zeta}|^2 - (1 + |z|^2)(1 + |\zeta|^2)\left[\frac{\log(1 - z\bar{\zeta})}{z\bar{\zeta}} + \frac{\log(1 - \bar{z}\zeta)}{\bar{z}\zeta}\right] \tag{2}$$

and satisfies the Neumann problem

$$\partial_z\partial_{\bar{z}}N_2(z,\zeta) = N_1(z,\zeta) \text{ in D for fixed } \zeta \in \overline{\mathrm{D}};$$

$$\partial_{\nu_z}N_2(z,\zeta) = 2(1 - |\zeta|^2) \text{ on } \partial\mathrm{D} \text{ for fixed } \zeta \in \overline{\mathrm{D}}$$

and the normalization condition

$$\frac{1}{2\pi i}\int_{\partial\mathbb{D}} N_2(z,\zeta)\frac{dz}{z}=0.$$

Moreover, N_2 is symmetric in z and ζ, $N_2(z,\zeta)=N_2(\zeta,z)$.

Theorem 1. The Neumann problem

$$(\partial_z\partial_{\bar z})^2 w=f \text{ in } \mathbb{D},\ \partial_\nu w=\gamma_0,\ \partial_\nu\partial_z\partial_{\bar z}w=\gamma_1 \text{ on } \partial\mathbb{D};$$

$$\frac{1}{2\pi i}\int_{|\zeta|=1} w(\zeta)\frac{d\zeta}{\zeta}=c_0,\ \frac{1}{2\pi i}\int_{|\zeta|=1} w_{\zeta\bar\zeta}(\zeta)\frac{d\zeta}{\zeta}=c_1$$

for $f\in L_p(\mathbb{D},\mathbb{C})$, $2<p$, $\gamma_0,\gamma_1\in C(\partial\mathbb{D};\mathbb{C}), c_0, c_1\in\mathbb{C}$ is uniquely solvable if and only if

$$\frac{1}{2\pi i}\int_{|\zeta|=1}\gamma_0(\zeta)\frac{d\zeta}{\zeta}=2c_1-\frac{2}{\pi}\int_{|\zeta|<1}(1-|\zeta|^2)f(\zeta)d\xi d\eta$$

and

$$\frac{1}{2\pi i}\int_{|\zeta|=1}\gamma_1(\zeta)\frac{d\zeta}{\zeta}=\frac{2}{\pi}\int_{|\zeta|<1}f(\zeta)d\xi d\eta.$$

The solution is given as

$$w(z)=c_0-(1-|z|^2)c_1+\frac{1}{4\pi i}\int_{|\zeta|=1}\{N_1(z,\zeta)\gamma_0(\zeta)+N_2(z,\zeta)\gamma_1(\zeta)\}\frac{d\zeta}{\zeta}-$$
$$-\frac{1}{\pi}\int_{|\zeta|<1}N_2(z,\zeta)f(\zeta)d\xi d\eta. \tag{3}$$

Definition. The Neumann-3 function for the unit disc $\mathbb{D}$ is

$$N_3(z,\zeta)=-\frac{1}{4}|\zeta-z|^4\log|(\zeta-z)(1-z\bar\zeta)|^2+n_3(z,\zeta), \tag{4}$$

where $n_3(z,\zeta)$ is tri-harmonic in both variables with proper boundary behavior. The properties of the third Neumann function are

$$\partial_z\partial_{\bar z}N_3(z,\zeta)=N_2(z,\zeta) \text{ in } \mathbb{D}\backslash\{\zeta\} \text{ for } \zeta\in\mathbb{D};$$
$$\partial_\nu N_3(z,\zeta)=-\frac{1}{2}(1-|\zeta|^2)^2-\frac{1}{2}\partial_\nu N_2(z,\zeta) \text{ on } \partial\mathbb{D} \text{ for } \zeta\in\mathbb{D},$$

where

$$\partial_{\nu_z}N_2(z,\zeta)=2(1-|\zeta|^2) \text{ on } \partial\mathbb{D} \text{ for } \zeta\in\mathbb{D},$$

so that

$$\partial_{\nu_z}N_3(z,\zeta)=-\left[\frac{1}{2}(1-|\zeta|^2)^2+(1-|\zeta|^2)\right];$$
$$\frac{1}{2\pi i}\int_{|z|=1}N_3(z,\zeta)\frac{dz}{z}=0 \text{ for } \zeta\in\mathbb{D};$$
$$N_3(z,\zeta)=N_3(\zeta,z) \text{ for } z,\zeta\in\mathbb{D}.$$

It is important that the normal derivative of $N_3(z,\zeta)$ with respect to z does depend on ζ but not on z. In order to find $N_3(z,\zeta)$ in a proper way some particular Neumann problems are investigated. We must calculate function $n_3(z,\zeta)$. From a formula (3) we will express function:

$$n_3(z,\zeta)=N_3(z,\zeta)+\frac{1}{4}|\zeta-z|^4\log|(\zeta-z)(1-z\bar\zeta)|^2. \tag{5}$$

We will prove some properties to which function (5) satisfies:

$$\partial_z n_3(z,\zeta) = \partial_z N_3(z,\zeta) - \frac{1}{2}(\zeta - z)(\overline{\zeta - z})^2 \log|(\zeta - z)(1 - z\bar{\zeta})|^2 -$$

$$-\frac{1}{4}|\zeta - z|^4 \left(\frac{1}{\zeta - z} + \frac{\bar{\zeta}}{1 - z\bar{\zeta}}\right);$$

$$\partial_z\partial_{\bar{z}} n_3(z,\zeta) = N_2(z,\zeta) + |\zeta - z|^2 \log|(\zeta - z)(1 - z\bar{\zeta})|^2 + 2|\zeta - z|^2 -$$

$$-\frac{1}{2}|\zeta - z|^2(1 - |\zeta|^2)\left(\frac{1}{1 - z\bar{\zeta}} + \frac{1}{1 - \bar{z}\zeta}\right);$$

$$(\partial_z\partial_{\bar{z}})^2 n_3(z,\zeta) = 6 - (1 - |\zeta|^2)\left(\frac{1}{1 - z\bar{\zeta}} + \frac{1}{1 - \bar{z}\zeta}\right) - \frac{1}{2}(1 - |\zeta|^2)^2\left(\frac{1}{(1 - \bar{z}\zeta)^2} + \frac{1}{(1 - \bar{z}\zeta)^2}\right),$$

for $|z| = 1$ then

$$\partial_\nu n_3(z,\zeta) = -\frac{1}{2}(1 - |\zeta|^2)^2 - (1 - |\zeta|^2) + \frac{1}{2}|\zeta - z|^2(2 - z\bar{\zeta} - \bar{z}\zeta)\log|(\zeta - z)(1 - z\bar{\zeta})|^2 + \frac{1}{2}|\zeta - z|^4;$$

$$\partial_\nu\partial_z\partial_{\bar{z}} n_3(z,\zeta) = 4(2 - z\bar{\zeta} - \bar{z}\zeta) + 2(2 - z\bar{\zeta} - \bar{z}\zeta)\log|1 - z\bar{\zeta}|^2 - (1 - |\zeta|^2) +$$

$$+\frac{1}{2}(1 - |\zeta|^2)(2 - z\bar{\zeta} - \bar{z}\zeta) - (1 - |\zeta|^2)\left(\frac{1 - \bar{z}\zeta}{1 - z\bar{\zeta}} + \frac{1 - z\bar{\zeta}}{1 - \bar{z}\zeta}\right)$$

follows.

Next the first solvability conditions of Theorem is verified.

$$\frac{1}{2\pi i}\int_{|z|=1} \partial_\nu n_3(z,\zeta)\frac{dz}{z} = 2c_1 - \frac{2}{\pi}\int_{|z|<1}(1 - |z|^2)\partial_z\partial_{\bar{z}} n_3(z,\zeta)dxdy.$$

At the beginning consider the left-hand side

$$\frac{1}{2\pi i}\int_{|z|=1} \partial_\nu n_3(z,\zeta)\frac{dz}{z} = \frac{1}{2\pi i}\int_{|z|=1}\left\{-\frac{1}{2}(1 - |\zeta|^2)^2 - (1 - |\zeta|^2) +\right.$$

$$\left.+\frac{1}{2}|\zeta - z|^2(2 - z\bar{\zeta} - \bar{z}\zeta)\log|(\zeta - z)(1 - z\bar{\zeta})|^2 + \frac{1}{2}|\zeta - z|^4\right\}\frac{dz}{z} =$$

$$= -\frac{1}{2}(1 - |\zeta|^2)^2 - (1 - |\zeta|^2) + 3|\zeta|^2 + \frac{1}{2}|\zeta|^4 + 3|\zeta|^2 +$$

$$+\frac{1}{2}|\zeta|^4 + 1 + 4|\zeta|^2 + |\zeta|^4 = 10|\zeta|^2 + |\zeta|^4 - 1.$$

Also to solve the right-side of the condition, namely:

$$2c_1 - \frac{2}{\pi}\int_{|z|<1}(1 - |z|^2)\partial_z\partial_{\bar{z}} n_3(z,\zeta)dxdy,$$

where

$$c_1 = \frac{1}{2\pi i}\int_{|z|=1}\partial_z\partial_{\bar{z}} n_3(z,\zeta)\frac{dz}{z} =$$

$$= \frac{1}{2\pi i}\int_{|\zeta|=1}\left\{N_2(z,\zeta) + |\zeta - z|^2\log|(\zeta - z)(1 - z\bar{\zeta})|^2 +\right.$$

$$\left.+2|\zeta - z|^2 - \frac{1}{2}|\zeta - z|^2(1 - |\zeta|^2)\left[\frac{1}{1 - z\bar{\zeta}} + \frac{1}{1 - \bar{z}\zeta}\right]\right\}\frac{dz}{z} =$$

$$= 4|\zeta|^2 + 2 + 2|\zeta|^2 - \frac{1}{2}(1-|\zeta|^2) - \frac{1}{2}(1-|\zeta|^2) = 7|\zeta|^2 + 1$$

and solving separately:

$$\frac{2}{\pi}\int_{|z|<1}(1-|z|^2)(\partial_z\partial_{\bar z})^2 n_3(z,\zeta)dxdy =$$

$$= \frac{2}{\pi}\int_{|z|<1}(1-|z|^2)\left\{6-(1-|\zeta|^2)\left(\frac{1}{1-z\bar\zeta}+\frac{1}{1-\bar z\zeta}\right)-\right.$$

$$\left.-\frac{1}{2}(1-|\zeta|^2)^2\left(\frac{1}{(1-\bar z\zeta)^2}+\frac{1}{(1-z\bar\zeta)^2}\right)\right\}dxdy =$$

$$= 6 - 2(1-|\zeta|^2) - (1-|\zeta|^2)^2 = 3 + 4|\zeta|^2 - |\zeta|^4$$

it follows that

$$2c_1 - \frac{2}{\pi}\int_{|\zeta|<1}(1-|z|^2)(\partial_z\partial_{\bar z})^2 n_3(z,\zeta)d\xi d\eta = 10|\zeta|^2 + |\zeta|^4 - 1.$$

We have proved the validity of the first condition:

$$10|\zeta|^2 + |\zeta|^4 - 1 = 10|\zeta|^2 + |\zeta|^4 - 1.$$

In the next step we verify the second solvability condition of Theorem 1:

$$\frac{1}{2\pi i}\int_{|z|=1}\partial_\nu\partial_z\partial_{\bar z} n_3(z,\zeta)\frac{dz}{z} = \frac{2}{\pi}\int_{|z|<1}(\partial_z\partial_{\bar z})^2 n_3(z,\zeta)dxdy.$$

The left-hand side is

$$\frac{1}{2\pi i}\int_{|z|=1}\partial_\nu\partial_z\partial_{\bar z} n_3(z,\zeta)\frac{dz}{z} =$$

$$= \frac{1}{2\pi i}\int_{|z|=1}\left\{4(2-z\bar\zeta-\bar z\zeta) + 2(2-z\bar\zeta-\bar z\zeta)\log|1-z\bar\zeta|^2 - (1-|\zeta|^2)+\right.$$

$$\left.+\frac{1}{2}(1-|\zeta|^2)(2-z\bar\zeta-\bar z\zeta) - (1-|\zeta|^2)\left(\frac{1-\bar z\zeta}{1-z\bar\zeta}+\frac{1-z\bar\zeta}{1-\bar z\zeta}\right)\right\}\frac{dz}{z} =$$

$$= 8 + 4|\zeta|^2 - (1-|\zeta|^2)^2 - (1-|\zeta|^2)(1-|\zeta|^2) = 6 + 8|\zeta|^2 - 2|\zeta|^4.$$

Then evaluating the right-hand side of the condition shows

$$\frac{2}{\pi}\int_{|\zeta|<1}(\partial_z\partial_{\bar z})^2 n_3(z,\zeta)dxdy =$$

$$= \frac{2}{\pi}\int_{|\zeta|<1}\left\{6-(1-|\zeta|^2)\left(\frac{1}{1-z\bar\zeta}+\frac{1}{1-\bar z\zeta}\right)-\right.$$

$$\left.-\frac{1}{2}(1-|\zeta|^2)^2\left(\frac{1}{(1-z\bar\zeta)^2}+\frac{1}{(1-\bar z\zeta)^2}\right)\right\}dxdy =$$

$$= 12 - 4(1-|\zeta|^2) - 2(1-|\zeta|^2)^2 = 8 + 4|\zeta|^2 - 2 + 4|\zeta|^2 - 2|\zeta|^4 = 6 + 8|\zeta|^2 - 2|\zeta|^4.$$

Hence the second conditions is satisfied, i.e.

$$6 + 8|\zeta|^2 - 2|\zeta|^4 = 6 + 8|\zeta|^2 - 2|\zeta|^4.$$

In order to find the solution of Theorem 1, we also must calculate

$$c_0 = \frac{1}{2\pi i}\int_{|\zeta|=1} n_3(z,\zeta)\frac{d\zeta}{\zeta} =$$

$$= \frac{1}{2\pi i}\int_{|\zeta|=1}\left\{N_3(z,\zeta) + \frac{1}{4}|\zeta - z|^4 \log|(\zeta - z)(1 - z\bar{\zeta})|^2\right\}\frac{d\zeta}{\zeta}.$$

Evaluating this integral shows $c_0 = 2|\zeta|^2 - \frac{3}{2}|\zeta|^4$.

Thus we have verified all the necessary and sufficient conditions of solvability of Theorem 1. According to Theorem 1 the function $n_3(z,\zeta)$ is given as [3]:

$$n_3(z,\zeta) = c_0 + (1 - |z|^2)c_1 - \frac{1}{4\pi i}\int_{|\tilde{\zeta}|=1}\Big\{N_1(z,\tilde{\zeta})\partial_\nu n_3(\tilde{\zeta},\zeta) +$$

$$+ N_2(z,\tilde{\zeta})\partial_\nu\partial_{\tilde{\zeta}}\partial_{\bar{\tilde{\zeta}}} n_3(\tilde{\zeta},\zeta)\Big\}\frac{d\tilde{\zeta}}{\tilde{\zeta}} - \frac{1}{\pi}\int_{|\zeta|<1} N_2(z,\tilde{\zeta})f(\tilde{\zeta})d\tilde{\xi}d\tilde{\eta} =$$

$$= 2|\zeta|^2 - \frac{3}{2}|\zeta|^4 + (1 - |z|^2)(7|\zeta|^2 + 1) + \frac{1}{4\pi i}\int_{|\tilde{\zeta}|=1}\left\{\left(\log|(\tilde{\zeta} - z)(1 - z\bar{\tilde{\zeta}})^2|\right)\times\right.$$

$$\times\left(-\frac{1}{2}(1 - |\zeta|^2)^2 - (1 - |\zeta|^2) + \frac{1}{2}|\zeta - \tilde{\zeta}|^2(2 - \zeta\bar{\tilde{\zeta}} - \bar{\zeta}\tilde{\zeta})\log|(\zeta - \tilde{\zeta})(1 - \tilde{\zeta}\bar{\zeta})|^2 +\right.$$

$$\left.\left. + \frac{1}{2}|\zeta - \tilde{\zeta}|^4\right)\right\}\frac{d\tilde{\zeta}}{\tilde{\zeta}} + \frac{1}{4\pi i}\int_{|\zeta|=1}\left\{\left(|\tilde{\zeta} - z|^2[\log|(\tilde{\zeta} - z)(1 - z\bar{\tilde{\zeta}})|^2 - 4] +\right.\right.$$

$$+ 4\sum_{k=2}^{+\infty}\frac{(z\bar{\tilde{\zeta}})^k + (\bar{z}\tilde{\zeta})^k}{k^2} + 2(z\bar{\tilde{\zeta}} + \bar{z}\tilde{\zeta})\log|1 - z\bar{\tilde{\zeta}}|^2 - (1 + |z|^2)(1 + |\zeta|^2)\left[\frac{\log(1 - z\bar{\tilde{\zeta}})}{z\bar{\tilde{\zeta}}} +\right.$$

$$\left.\left. + \frac{\log(1 - \bar{z}\tilde{\zeta})}{\bar{z}\tilde{\zeta}}\right]\right)\left(4(2 - \tilde{\zeta}\bar{\zeta} - \bar{\tilde{\zeta}}\zeta) + 2(2 - \tilde{\zeta}\bar{\zeta} - \bar{\tilde{\zeta}})\log|1 - \tilde{\zeta}\bar{\zeta}|^2 - (1 - |\zeta|^2) +\right.$$

$$\left.\left. + \frac{1}{2}(1 - |\zeta|^2)(2 - \tilde{\zeta}\bar{\zeta} - \bar{\tilde{\zeta}}\zeta) - (1 - |\zeta|^2)\left(\frac{1 - \bar{\tilde{\zeta}}\zeta}{1 - \tilde{\zeta}\bar{\zeta}} + \frac{1 - \tilde{\zeta}\bar{\zeta}}{1 - \bar{\tilde{\zeta}}\zeta}\right)\right)\right\}\frac{d\tilde{\zeta}}{\tilde{\zeta}} -$$

$$- \frac{1}{\pi}\int_{|\zeta|<1}\left\{\left(\left(|\tilde{\zeta} - z|^2[\log|(\tilde{\zeta} - z)(1 - z\bar{\tilde{\zeta}})|^2 - 4] + 4\sum_{k=2}^{+\infty}\frac{(z\bar{\tilde{\zeta}})^k + (\bar{z}\tilde{\zeta})^k}{k^2} +\right.\right.\right.$$

$$+ 2(z\bar{\tilde{\zeta}} + \bar{z}\tilde{\zeta})\log|1 - z\bar{\tilde{\zeta}}|^2 - (1 + |z|^2)(1 + |\zeta|^2)\left[\frac{\log(1 - z\bar{\tilde{\zeta}})}{z\bar{\tilde{\zeta}}} +\right.$$

$$\left.\left. + \frac{\log(1 - \bar{z}\tilde{\zeta})}{\bar{z}\tilde{\zeta}}\right]\right)(6 - (1 - |\zeta|^2)\left(\frac{1}{1 - \tilde{\zeta}\bar{\zeta}} + \frac{1}{1 - \bar{\tilde{\zeta}}\zeta}\right) -$$

$$\left.\left.\left. - \frac{1}{2}(1 - |\zeta|^2)^2\left(\frac{1}{(1 - \bar{\tilde{\zeta}}\zeta)^2} + \frac{1}{(1 - \tilde{\zeta}\bar{\zeta})^2}\right)\right)\right)\right\}d\tilde{\xi}d\tilde{\eta}. \tag{6}$$

Then (6) we will insert in (4), from here we will receive [4]:

$$N_3(z,\zeta) = -\frac{1}{4}|\zeta - z|^4\log|(\zeta - z)(1 - z\bar{\zeta})|^2 + n_3(z,\zeta) = \frac{3}{2}\left(|\zeta|^4 + |z|^4\right) + 5\left(|\zeta|^2 + 1\right)\left(|z|^2 + 1\right) +$$

$$+ 2\left(|\zeta|^2 + |z|^2 + 6\right) + \frac{1}{4}\left(|\zeta|^2 - 1\right)\left(|z|^2 - 1\right)\left(\bar{z}\zeta + z\bar{\zeta}\right) + \frac{1}{4}|\zeta - z|^4\log\left|\frac{1 - z\bar{\zeta}}{\zeta - z}\right|^2 -$$

$$- \left[2\left(|z|^2 + 2\right)\left(|\zeta|^2 + 2\right) + \frac{1}{2}\left(|\zeta|^4 + |z|^4\right)\right]\log|1 - z\bar{\zeta}|^2 +$$

$$+\left[\frac{1}{2}\left(|\zeta|^2+|z|^2\right)\left(|z|^2+1\right)\left(|\zeta|^2+1\right)+4\left(|\zeta|^2+|z|^2+2\right)\right]\times$$

$$\times\left[\frac{\log\left(1-\bar{z}\zeta\right)}{\bar{z}\zeta}+\frac{\log\left(1-z\bar{\zeta}\right)}{z\bar{\zeta}}\right]-$$

$$-\frac{1}{4}\left(|z|^4+1\right)\left(|\zeta|^4+1\right)\left[\frac{\log\left(1-\bar{z}\zeta\right)}{\left(\bar{z}\zeta\right)^2}+\frac{\log\left(1-z\bar{\zeta}\right)}{\left(z\bar{\zeta}\right)^2}+\frac{1}{\bar{z}\zeta}+\frac{1}{z\bar{\zeta}}\right]+$$

$$+\sum_{l=0}^{\infty}\left\{\left[\frac{8}{(l+1)^3}-\frac{\left(|\zeta|^2+1\right)\left(|z|^2+1\right)}{(l+2)^2}-\frac{4\,|z|^2+4\,|\zeta|^2+6}{(l+1)^6}\right]\left[\left(\bar{z}\zeta\right)^{l+1}+\left(z\bar{\zeta}\right)^{l+1}\right]\right\},\qquad z,\zeta\in D. \tag{7}$$

Theorem 2. The tri-harmonic Neumann problem

$$(\partial_z\partial_{\bar{z}})^3 w=f \text{ in } \mathbb{D},\ f\in L_p(\mathbb{D};\mathbb{C}),\ 2<p<+\infty;$$

$$\partial_\nu w=\gamma_0,\ \partial_\nu\partial_z\partial_{\bar{z}}w=\gamma_1\ \partial_\nu(\partial_z\partial_{\bar{z}})^2 w=\gamma_2 \text{ on } \partial\mathbb{D},\ \gamma_0,\gamma_1,\gamma_2\in C(\partial\mathbb{D};\mathbb{C}),$$

satisfying

$$\frac{1}{2\pi i}\int_{|\zeta|=1}w(\zeta)\frac{d\zeta}{\zeta}=c_0,\ \frac{1}{2\pi i}\int_{|\zeta|=1}\partial_\zeta\partial_{\bar{\zeta}}w(\zeta)\frac{d\zeta}{\zeta}=c_1,\ \frac{1}{2\pi i}\int_{|\zeta|=1}(\partial_\zeta\partial_{\bar{\zeta}})^2 w(\zeta)\frac{d\zeta}{\zeta}=c_2$$

is uniquely solvable if and only if

$$\frac{1}{2\pi i}\int_{\partial\mathbb{D}}\gamma_0(\zeta)\frac{d\zeta}{\zeta}=2c_1-c_2-\frac{1}{16\pi i}\int_{\partial\mathbb{D}}\gamma_2(\zeta)\frac{d\zeta}{\zeta}+\frac{1}{\pi}\int_{\mathbb{D}}\left((1-|\zeta|^2)^2-\frac{1}{2}\right)f(\zeta)d\xi d\eta;$$

$$\frac{1}{2\pi i}\int_{\partial\mathbb{D}}\gamma_1(\zeta)\frac{d\zeta}{\zeta}=c_1-2c_2-\frac{2}{\pi}\int_{\mathbb{D}}(1-|\zeta|^2)f(\zeta)d\xi d\eta$$

and

$$\frac{1}{2\pi i}\int_{\partial\mathbb{D}}\gamma_2(\zeta)\frac{d\zeta}{\zeta}=\frac{2}{\pi}\int_{\mathbb{D}}f(\zeta)d\xi d\eta.$$

The solution is given as

$$w(z)=c_0-c_1(1-|z|^2)-c_2\left(\frac{1}{4}(1-|z|^2)^2+\frac{1}{2}(1-|z|^2)\right)+\frac{1}{4\pi i}\int_{\partial\mathbb{D}}\{N_1(z,\zeta)\gamma_0(\zeta)+$$

$$+N_2(z,\zeta)\gamma_1(\zeta)+N_3(z,\zeta)\gamma_2(\zeta)\}\frac{d\zeta}{\zeta}-\frac{1}{\pi}\int_{\mathbb{D}}f(\zeta)N_3(z,\zeta)d\xi d\eta.$$

Proof. Rewriting the Neumann-3 problem as the system

$$(\partial_z\partial_{\bar{z}})^2 w=\omega \text{ in } \mathbb{D},\ \partial_\nu w=\gamma_0,\ \partial_\nu\partial_z\partial_{\bar{z}}w=\gamma_1 \text{ on } \partial\mathbb{D};$$

$$\frac{1}{2\pi i}\int_{\partial\mathbb{D}}w(\zeta)\frac{d\zeta}{\zeta}=c_0,\ \frac{1}{2\pi i}\int_{\partial\mathbb{D}}\partial_\zeta\partial_{\bar{\zeta}}w(\zeta)\frac{d\zeta}{\zeta}=c_1$$

and

$$\partial_z\partial_{\bar{z}}\omega=f \text{ in } \mathbb{D},\ \partial_\nu\omega=\gamma_2 \text{ on } \partial\mathbb{D},\ \frac{1}{2\pi i}\int_{\partial\mathbb{D}}\omega(\zeta)\frac{d\zeta}{\zeta}=c_2$$

leads to the solvability conditions

$$\frac{1}{2\pi i}\int_{\partial\mathbb{D}}\gamma_0(\zeta)\frac{d\zeta}{\zeta} = 2c_1 - \frac{2}{\pi}\int_{\mathbb{D}}(1-|\zeta|^2)\omega(\zeta)d\xi d\eta \tag{8}$$

and

$$\frac{1}{2\pi i}\int_{\partial\mathbb{D}}\gamma_1(\zeta)\frac{d\zeta}{\zeta} = \frac{2}{\pi}\int_{\mathbb{D}}\omega(\zeta)d\xi d\eta. \tag{9}$$

The solution then is

$$w(z) = c_0 - (1-|z|^2)c_1 + \frac{1}{4\pi i}\int_{\partial\mathbb{D}}\left\{\gamma_0(\zeta)N_1(z,\zeta)+\gamma_1(\zeta)N_2(z,\zeta)\right\}\frac{d\zeta}{\zeta} - $$

$$-\frac{1}{\pi}\int_{\mathbb{D}}\omega(\zeta)N_2(z,\zeta)d\xi d\eta; \tag{10}$$

$$\omega(\zeta) = c_2 + \frac{1}{4\pi i}\int_{\partial\mathbb{D}}\gamma_2(\tilde\zeta)N_1(\zeta,\tilde\zeta)\frac{d\tilde\zeta}{\tilde\zeta} - \frac{1}{\pi}\int_{\mathbb{D}}N_1(\zeta,\tilde\zeta)f(\tilde\zeta)d\tilde\xi d\tilde\eta.$$

Inserting ω into the (8)-(9) conditions gives

$$\frac{1}{2\pi i}\int_{\partial\mathbb{D}}\gamma_0(\zeta)\frac{d\zeta}{\zeta} = 2c_1 - \frac{1}{\pi}\int_{\mathbb{D}}(1-|\zeta|^2)\left\{c_2 + \frac{1}{4\pi i}\int_{\partial\mathbb{D}}\gamma_2(\tilde\zeta)N_1(\zeta,\tilde\zeta)\frac{d\tilde\zeta}{\tilde\zeta} - \right.$$

$$\left.-\frac{1}{\pi}\int_{\mathbb{D}}N_1(\zeta,\tilde\zeta)f(\tilde\zeta)d\tilde\xi d\tilde\eta\right\}d\xi d\eta$$

with

$$\frac{1}{\pi}\int_{\mathbb{D}}(1-|\zeta|^2)N_1(\zeta,\tilde\zeta)d\xi d\eta = \frac{1}{2}\left(1-\frac{1}{2}|\zeta|^2\right)^2 - \frac{1}{4}.$$

Then inserting ω into (10) shows

$$w(z) = c_0 - (1-|z|^2)c_1 - c_2\left(\frac{1}{\pi}\int_{\mathbb{D}}N_2(z,\zeta)d\xi d\eta\right) + \frac{1}{4\pi i}\int_{\partial\mathbb{D}}\{\gamma_0(\zeta)N_1(z,\zeta)+$$

$$+\gamma_1(\zeta)N_2(z,\zeta)\}\frac{d\zeta}{\zeta} - \frac{1}{4\pi i}\int_{\partial\mathbb{D}}\gamma_2(\tilde\zeta)\frac{1}{\pi}\int_{\mathbb{D}}N_1(\zeta,\tilde\zeta)N_2(z,\tilde\zeta)d\xi d\eta\frac{d\zeta}{\zeta} +$$

$$+\frac{1}{\pi}\int_{\mathbb{D}}f(\tilde\zeta)\frac{1}{\pi}\int_{\mathbb{D}}N_1(\zeta,\tilde\zeta)N_2(z,\zeta)d\xi d\eta d\tilde\xi d\tilde\eta.$$

So,we get

$$w(z) = c_0 - c_1(1-|z|^2) - c_2\left(\frac{1}{4}(1-|z|^2)^2 + \frac{1}{2}(1-|z|^2)\right) + \frac{1}{4\pi i}\int_{\partial\mathbb{D}}\{N_1(z,\zeta)\gamma_0+$$

$$+N_2(z,\zeta)\gamma_1(\zeta) + N_3(z,\zeta)\gamma_2(\zeta)\}\frac{d\zeta}{\zeta} - \frac{1}{\pi}\int_{\mathrm{D}}f(\zeta)N_3(z,\zeta)d\xi d\eta,$$

where $N_1(z,\zeta)$, $N_2(z,\zeta)$, $N_3(z,\zeta)$ are given (1), (2) and (7) respectively.

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С.К. Бургумбаева

Тригармоникалық Нейман есебі

Лаплас теңдеуі тиесілі элиптикалық типтес теңдеулері физикалық қолданылулары маңызды рөл атқарады. Оларға сығылмайтын сұйықтықтың ықтимал қозғалысы, электрстатикалық өрістің потенциалы, стационар жылу, диффузиялық үдерістер, өрістің потенциал өрісі, аэромеханиканың мәселелері жатады. Екінші ретті сызықтық эллиптикалық теңдеулері үшін, оның ішінде Лаплас теңдеуі үшін Дирихле және Нейман есептері негізгі шектік есептер болып табылады. Бұл шектік есептердің нақты шешімдерін табу үшін әртүрлі аналитикалық әдістер бар, мысалға, интегралдық теңдеулер әдістері, интегралдық түрлендіру әдістері, бейнелеу әдісі және т.с.с. Берілген D облысы үшін тиісті шектік есептің Грин функциясын табуда кез келген жағдайы өте маңызды, себебі ол көптеген аналитикалық шешімдерді интегралдық түрлендіру түрінде жазуға мүмкіндік беретін ауқымды ақпарат сақтайды. Айтылған әдістер Дирихле және Нейман есептерін шешуде негізгі қиындықтар туғызады және Грин функциясының айқын түрдегі шешімі қарапайым облыстар үшін белгілі. Бірлік шеңбері үшін Грин функциясы алғаш рет Алманзи еңбектерінде шешімін тапқан, бұл нәтиже Дирихле есебін шешуде алғашқы қадамдарының бірі болып саналады. Сонымен қоса гибрид бигармоникалық Грин функциясы туралы Г. Бегер еңбектерінде кездеседі және Грин функциясының гибрид полигармоникалық түрлерінің әртүрлі болуы еңбектерінде кездеседі. Мақала авторы тригармоникалық Нейман есебін бірлік шеңберде зерттеген. Нейман есебі гармоникалық функциялар үшін жақсы зерттелген және Нейман функциясы арқылы нақты шарттарда шешілген, кейбір жағдайларда бұл функцияны екінші ретті Грин функциясы деп атайды. Тригармоникалық Нейман функциясы бигармоникалық Нейман функциясымен кешен жазықтықтың бірлік шеңберінде айқын түрде табылған. Нейман функциясымен интегралдық көрінісі тригармоникалық оператордың дамуына жол береді. Мақалада алынған нәтиже эллиптикалық теңдеулер үшін шекаралық есептердің аналитикалық теориясын әрі қарай дамытудың қызықты болашағын болжауға мүмкіндік береді.

Кілт сөздер: Нейман функциясы, Грин функциясы, гармоникалық функция, потенциал, өріс, Дирихле есебі.

С.К. Бургумбаева

Тригармоническая задача Неймана

В статье исследована тригармоническая функция Неймана на единичной окружности. Для гармонических функций задача Неймана хорошо изучена и решена при определенных условиях через функцию Неймана, иногда ее также называют функцией Грина второго порядка. Всякий случай нахождения функции Грина соответствующей краевой задачи для той или иной области D весьма важен, так как содержит обширную информацию, позволяя выписать большое число аналитических решений в виде интегральных соотношений. В то же время указанная процедура составляет основную трудность при решении задач Дирихле и Неймана, и в явном виде функция Грина известна только для небольшого числа простых областей. Гармоническая функция Грина с собой последовательно приводит к последующей полигармонической функции Грина, которая может использоваться для решения последующей проблемы Дирихле для более высокого порядка уравнения Пуассона. Методами интегрального преобразования получена тригармоническая функция Неймана в явном виде на единичной окружности комплексной плоскости с бигармонической функцией Неймана. С функцией Неймана интегральное представление дает развитие для тригармонического оператора. Упомянутая выше полигармоническая функция Грина на единичной окружности дает начало решению некоторой конкретной полигармонической задачи Дирихле. Точно так же гармоническая функция Неймана с собой последовательно приводит к последующей полигармонической функции Неймана. Полученный автором результат позволяет предвидеть интересные перспективы в дальнейшем развитии аналитической теории краевых задач в комплексном анализе для уравнений эллиптического типа.

Ключевые слова: функция Неймана, функция Грина, гармоническая функция, потенциал, поле, задача Дирихле.