МАТЕМАТИКА
MATHEMATICS

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The natural solvability of the Navier-Stokes equations

It is known that the three-dimensional Navier-Stokes equations (ENS) the existence theorem of smooth solutions in the presence of smooth data for the whole with respect to time has not proved and the uniqueness theorem is violated in the class of generalized solutions. In a number of works by the author of this article, the results of search studies on the justification of the maximum principle for three-dimensional ENS are given. Over time, these studies have improved and later the justice of the simplest principle for maximum was shown for three-dimensional ENS. A further continuation of the search led to the determination of the relationship between pressure and the square of the velocity vector modulus from the properties of the ENS solutions. On the basis of this the answers to many problematic issues related to the solvability of the ENS were found. And in particular, in the selected spaces, the uniqueness of the weak and the existence of strong solutions of the problem for the three-dimensional Navier-Stokes equations for the whole of time are proved.

Keywords: Navier-Stokes equations, pressure in the Navier-Stokes equations, the uniqueness of weak generalized solutions, the existence of strong solutions.

Some introductory information

Unsolved problems in the theory of Navier-Stokes equations homogeneous fluid are given in [1, 2], etc. The initial-boundary value problem for Navier-Stokes equations [1] with respect to the velocity vector $\mathbf{U} = (U_1, U_2, U_3)$ and the pressure P in the domain $Q = (0, T] \times \Omega$:

$$\frac{\partial \mathbf{U}}{\partial t} - \mu\Delta\mathbf{U} + (\mathbf{U}, \nabla)\mathbf{U} + \nabla P = \mathbf{f}(t, \mathbf{x}),\ \operatorname{div}\mathbf{U} = 0; \tag{1a}$$

$$\mathbf{U}(0, \mathbf{x}) = \mathbf{\Phi}(\mathbf{x}),\ \mathbf{U}(t, \mathbf{x})\big|_{\mathbf{x}\in\partial\Omega} = 0, \tag{1b}$$

where $\mathbf{x} \in \Omega \subset R_3$; Ω — is a convex domain and $\partial\Omega$ is the boundary of Ω, $t \in [0, T]$, $T < \infty$; $\mathring{\mathbf{J}}(\Omega)$ — space solenoidal vectors; $\mathbf{L}_\infty(Q)-$ is the subspace of $\mathbf{C}(\bar{Q})$. $W^k_{p,0}(\Omega)$ is the Sobolev space functions equal to zero on $\partial\Omega$; The input data $\mathbf{f}$ and $\mathbf{\Phi}$ of the problem (1) meet the requirements:

i) $\mathbf{f}(t, \mathbf{x}) \in \mathbf{L}_\infty(0, T; \mathbf{L}_p(\Omega)) \cap \mathring{\mathbf{J}}(Q)$;

ii) $\mathbf{\Phi}(\mathbf{x}) \in \mathbf{L}_p(\Omega) \cap \mathbf{W}^1_{2,0}(\Omega) \cap \mathring{\mathbf{J}}(\Omega)$, $\forall p$.

Further, we use the Holder inequalities

$$\Big|\int_\Omega UV\,\mathbf{dx}\Big| \leq \Big(\int_\Omega |U|^p\,\mathbf{dx}\Big)^{\frac{1}{p}} \Big(\int_\Omega |V|^q\,\mathbf{dx}\Big)^{\frac{1}{q}} \tag{2}$$

and Jung for pair products

$$UV \leq \frac{1}{\epsilon p}|U|^p + \frac{\epsilon}{q}|V|^q, \quad \epsilon > 0, \quad \frac{1}{p} + \frac{1}{q} = 1, \tag{3}$$

in addition, the integration by parts formula

$$\int\limits_{\Omega} V \Delta U \, \mathbf{dx} = -\int\limits_{\Omega} \nabla V \, \nabla U \, \mathbf{dx} + \int\limits_{\partial\Omega} V \frac{\partial U}{\partial \mathbf{n}} \, \mathbf{dx}. \tag{4}$$

1 Explicit relation between pressure and square of the velocity vector module

From the properties of the solutions of the problem (1) a quadratic form connecting the pressure $P(t, \mathbf{x})$ from the components of the vector of the speed $\mathbf{U}(t, \mathbf{x})$:

$$\big(\mathbf{v}, \ \mathbf{B}\mathbf{v}'\big) = 0, \tag{5}$$

where $\mathbf{B} = \big\|U_\alpha U_\beta + \delta_\alpha^\beta P\big\|_{\alpha, \beta=1,2,3}$ — is the symmetric matrix; $\mathbf{v} = (v_1, v_2, v_3)$ — an arbitrary vector; the components $\{v_\alpha\}$ consist of arbitrary numbers such that $\sum\limits_{\alpha=1}^{3} v_\alpha^2 \neq 0$; $\mathbf{v}' = \big(v_1, \, v_2, \, v_3\big)'$ — vector column; δ_α^β — the Kronecker symbol.

Using the orthogonal matrix $\mathbf{T}$ from eigenvectors matrix $\mathbf{B}$ the quadratic form (5) is reduced to the sum of squares

$$\big(z_1, z_2, z_3\big)\mathbf{\Lambda}\big(z_1, z_2, z_3\big)' \equiv \big(\mathbf{z}, \mathbf{\Lambda z}\big) \equiv \sum_{\alpha=1}^{3} \lambda_\alpha z_\alpha^2 = 0, \tag{6}$$

where $\mathbf{\Lambda} = \operatorname{diag}\{\lambda_1, \lambda_2, \lambda_3\}$, The columns $\mathbf{T}$ consist of the eigenvectors of the corresponding to the eigenvalues of the matrix $\mathbf{B}$, $\mathbf{z} = \mathbf{T}'\mathbf{v}$. Notice, that $\mathbf{z} = \big(z_1, z_2, z_3\big)-$ is also an arbitrary vector.

Elements of the matrix $\mathbf{\Lambda}$ are determined from of the characteristic equation of the matrix $\mathbf{B}$, that is $\big|\mathbf{B} - \mathbf{\Lambda}\big| = 0$. Whence we obtain cubic equation $\lambda^3 + a\lambda^2 + b\lambda + c = 0$, where $a = -(\big|\mathbf{U}\big|^2 + 3P)$; $b = 2\big|\mathbf{U}\big|^2 P + 3P^2$; $c = -(\big|\mathbf{U}\big|^2 + P)P^2$. The solution of this equation we find using the Cardano formula. For this, substitution $\lambda = y - a/3$ we arrive at the «incomplete» type $y^3 + ry + d = 0$, $r = -1/3|\mathbf{U}|^4$, $d = -2/(27)|\mathbf{U}|^6$. Whence follows, that the discriminant of this equation is equal to zero, that is $D = (r/3)^3 + (d/2)^2 = 0$. Which means all the roots of the «incomplete» the equations are real, and two of them are equal to each other. In fact, $A = \sqrt[3]{-d/2} = 1/3\big|\mathbf{U}\big|^2$; $B = \sqrt[3]{-d/2} = 1/3|\mathbf{U}|^2$; $y_1 = A + B = 2/3|\mathbf{U}|^2$; $y_{2/3} = -(A+B)/2 = -1/3|\mathbf{U}|^2$. From here we find the roots λ_α, $\alpha = \overline{1,3}$ of the original cubic equations: $\lambda_1 = \big|\mathbf{U}\big|^2 + P$; $\lambda_2 = P$; $\lambda_3 = P$.

Now we rewrite the quadratic form reduced to the sum of squares (6)

$$\big(\mathbf{z}, \mathbf{\Lambda z}\big) \equiv \sum_{\alpha=1}^{3} \lambda_\alpha z_\alpha^2 = 0, \ \forall \mathbf{z}.$$

This relation is zero if and only then when $\lambda_\alpha = 0$, $\alpha = \overline{1,3}$. Where does it follow that $\lambda_1 = \big|\mathbf{U}\big|^2 + P = 0$, $\lambda_2 \equiv P_2 = 0$, $\lambda_3 \equiv P_3 = 0$. From here

$$P_1(t, \mathbf{x}) = -\big|\mathbf{U}\big|^2 \equiv -2E; \quad P_2(t, \mathbf{x}) = 0; \quad P_3(t, \mathbf{x}) = 0. \tag{7}$$

2 Estimations of the solution of the problem (1)

Theorem 1. If the input data of the problem (1) satisfy the requirements **i**), **ii**)*, then for the solutions of the problem (1) the following estimate holds:*

$$\|\mathbf{U}\|_{\mathbf{L}_\infty(Q)} \leq \|\mathbf{\Phi}\|_{\mathbf{L}_\infty(\Omega)} + T\|\mathbf{f}\|_{\mathbf{L}_\infty(Q)} \equiv A, \ \forall T < \infty. \tag{8}$$

Proof. We multiply the scalar equation (1a) by the vector function $pE^{p-1}\mathbf{U}$, the product is integrated over the domain Ω and use the identity $E^p = \frac{1}{2^p}|\mathbf{U}|^{2p}$, then

$$\frac{1}{2^p}\frac{d}{dt}\int\limits_{\Omega} |\mathbf{U}|^{2p} \, \mathbf{dx} - p\mu \int\limits_{\Omega} \Delta \mathbf{U} \mathbf{U} E^{p-1} \mathbf{dx} + p\int\limits_{\Omega} (\mathbf{U}, \nabla)\mathbf{U} E^{p-1} \mathbf{U} \mathbf{dx} +$$

$$+p\int\limits_{\Omega} E^{p-1}\nabla P\mathbf{U}\mathbf{dx} = p\int\limits_{\Omega} E^{p-1}\mathbf{U}\mathbf{f}\mathbf{dx}, \quad t\in(0,T]. \tag{9}$$

Each term (9) is transformed by integration by parts (4). In estimating the fourth term on the left-hand side of we take into account (7). We estimate the right-hand side by Holder's inequality (2) and, as a result, we get:

$$p\int\limits_{\Omega}(\frac{\partial\mathbf{U}}{\partial t},\mathbf{U})E^{p-1}d\mathbf{x} = \frac{1}{2^p}\frac{d}{dt}\int\limits_{\Omega}\left|\mathbf{U}\right|^{2p}d\mathbf{x}; \tag{10}$$

$$-p\mu\int\limits_{\Omega}(\Delta\mathbf{U},\mathbf{U})E^{p-1}d\mathbf{x} = p\mu\int\limits_{\Omega}E^{p-1}\sum_{\alpha=1}^{3}(\nabla U_\alpha)^2 d\mathbf{x} + p(p-1)\mu\int\limits_{\Omega}E^{p-2}(\nabla E)^2 d\mathbf{x} \geq 0'; \tag{11}$$

$$p\int\limits_{\Omega}(\mathbf{U},\nabla)\mathbf{U}E^{p-1}\mathbf{U}d\mathbf{x} = \int\limits_{\Omega}\mathbf{U}\nabla E^p d\mathbf{x} = -\int\limits_{\Omega}div\mathbf{U}E^p d\mathbf{x} + \int\limits_{\partial\Omega}(\mathbf{U},\mathbf{n})E^p d\mathbf{x} = 0'; \tag{12}$$

$$p\int\limits_{\Omega}E^{p-1}\mathbf{U}\nabla P d\mathbf{x} = -2p\int\limits_{\Omega}E^{p-1}\mathbf{U}\nabla E d\mathbf{x} = -2\int\limits_{\Omega}\mathbf{U}\nabla E^p d\mathbf{x} = 0; \tag{13}$$

$$p\int\limits_{\Omega}E^{p-1}\mathbf{U}\mathbf{f}d\mathbf{x} \leq \frac{p}{2^{p-1}}\left(\int\limits_{\Omega}\left|\mathbf{U}\right|^{2p}d\mathbf{x}\right)^{\frac{2p-1}{2p}}\left(\int\limits_{\Omega}\left|\mathbf{f}\right|^{2p}d\mathbf{x}\right)^{\frac{1}{2p}}. \tag{14}$$

From the identity (9), taking into account the relation (10)–(14), we have the estimate

$$\frac{1}{2^p}\frac{d}{dt}\int\limits_{\Omega}|\mathbf{U}|^{2p}\,\mathbf{dx} + p\mu\int\limits_{\Omega}E^{p-1}\sum_{\alpha=1}^{3}(\nabla U_\alpha)^2 d\mathbf{x} + p(p-1)\mu\int\limits_{\Omega}E^{p-2}(\nabla E)^2 d\mathbf{x} \leq$$

$$\leq \frac{p}{2^{p-1}}\left(\int\limits_{\Omega}\left|\mathbf{U}\right|^{2p}d\mathbf{x}\right)^{\frac{2p-1}{2p}}\left(\int\limits_{\Omega}\left|\mathbf{f}\right|^{2p}d\mathbf{x}\right)^{\frac{1}{2p}}, \quad t\in(0,T]. \tag{15}$$

Because of the nonnegativity of the second and third terms on the left-hand side of (15), from which we proceed to the strengthened inequality.

$$\frac{1}{2^p}\frac{d}{dt}\int\limits_{\Omega}\left|\mathbf{U}\right|^{2p}d\mathbf{x} \leq \frac{p}{2^{p-1}}\left(\int\limits_{\Omega}\left|\mathbf{U}\right|^{2p}d\mathbf{x}\right)^{\frac{2p-1}{2p}}\left(\int\limits_{\Omega}\left|\mathbf{f}\right|^{2p}d\mathbf{x}\right)^{\frac{1}{2p}}. \tag{16}$$

Both parts (16), dividing by a positive integral $\frac{p}{2^{p-1}}\left(\int\limits_{\Omega}\left|\mathbf{U}\right|^{2p}d\mathbf{x}\right)^{\frac{2p-1}{2p}}$, we write

$$\frac{d}{dt}\left(\int\limits_{\Omega}\left|\mathbf{U}\right|^{2p}d\mathbf{x}\right)^{\frac{1}{2p}} \leq \left(\int\limits_{\Omega}\left|\mathbf{f}\right|^{2p}d\mathbf{x}\right)^{\frac{1}{2p}}.$$

We integrate over t in the range from 0 to t and taking parity of exponents, leaving p behind it, we obtain

$$\left(\int\limits_{\Omega}\left|\mathbf{U}(t,\mathbf{x})\right|^{p}d\mathbf{x}\right)^{\frac{1}{p}} \leq \left(\int\limits_{\Omega}\left|\mathbf{\Phi}(\mathbf{x})\right|^{p}d\mathbf{x}\right)^{\frac{1}{p}} + \int\limits_{0}^{t}\left(\int\limits_{\Omega}\left|\mathbf{f}(\tau,\mathbf{x})\right|^{p}d\mathbf{x}\right)^{\frac{1}{p}}d\tau, \forall p=2m, m\in N.$$

Hence we have

$$\left\|\mathbf{U}\right\|_{L_\infty(0,T;L_p(\Omega))} \leq \left\|\mathbf{\Phi}(\mathbf{x})\right\|_{L_p(\Omega)} + T\left|\mathbf{f}\right\|_{L_\infty(0,T;L_p(\Omega))}, \ \forall p=2m. \tag{17}$$

Whence for $p=\infty$ we arrive at the proof of the theorem 1.

Corollary 1. For the solutions of the problem (1) the following estimates hold:

$$\|\mathbf{U}\|_{\mathbf{L}_\infty(0,T;L_2(\Omega))} \le \|\mathbf{\Phi}\|_{\mathbf{L}_2(\Omega)} + T\|\mathbf{f}\|_{\mathbf{L}_\infty(0,T;\mathbf{L}_2(\Omega))} \equiv A_1,\ \forall T < \infty; \tag{18}$$

$$\int_0^t \sum_{\alpha=1}^3 \big\|\nabla U_\alpha(\tau)\big\|^2_{L_2(\Omega)} d\tau \le \frac{1}{\mu}\Big(\big\|\mathbf{\Phi}\big\|^2_{L_2(\Omega)} + T/2(1+T)\big\|\mathbf{f}\big\|^2_{L_\infty(0,T;L_2(\Omega))}\Big) = A_2; \tag{19}$$

$$\big\|\mathbf{U}\big\|^3_{L_\infty(0,T;L_4(\Omega))} \le 3\Big(\big\|\mathbf{\Phi}\big\|^3_{L_4(\Omega)} + T^3\big\|\mathbf{f}\big\|^3_{L_\infty(0,T;L_4(\Omega))}\Big) = A_3; \tag{20}$$

$$\int_0^t \big\|\nabla E(\tau)\big\|^2_{L_2(\Omega)} d\tau \le \frac{1}{2\mu}\Big(\big\|\mathbf{\Phi}\big\|_{L_{4(\Omega)}} + TA_3\big\|\mathbf{f}\big\|_{L_\infty(0,T;L_4(\Omega))}\Big) = A_4, \quad t \in (0,T]. \tag{21}$$

Proof. Estimates (18), (20) follow from (17) respectively for $p = 1$ and $p = 4$. To prove (19) from (15) for $p = 1$, we have

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\mathbf{U}|^2\,\mathbf{dx} + \mu\int_\Omega \sum_{\alpha=1}^3 (\nabla U_\alpha)^2 d\mathbf{x} \le \Big(\int_\Omega \big|\mathbf{U}\big|^2 d\mathbf{x}\Big)^{\frac{1}{2}}\Big(\int_\Omega \big|\mathbf{f}\big|^2 d\mathbf{x}\Big)^{\frac{1}{2}}.$$

We integrate over t in the range from 0 to t,

$$\frac{1}{2}\int_\Omega |\mathbf{U}(t)|^2\,\mathbf{dx} + \mu\int_0^t \sum_{\alpha=1}^3 \int_\Omega (\nabla U_\alpha)^2 d\mathbf{x}d\tau \le \frac{1}{2}\int_\Omega |\Phi|^2\,\mathbf{dx} + \int_0^t (\int_\Omega \big|\mathbf{U}\big|^2 d\mathbf{x})^{\frac{1}{2}}(\int_\Omega \big|\mathbf{f}\big|^2 d\mathbf{x})^{\frac{1}{2}} d\tau, \quad t \in (0,T].$$

Hence, since the first integral on the left-hand side is nonnegative, we get

$$\mu\int_0^t \sum_{\alpha=1}^3 \int_\Omega \big\|\nabla U_\alpha(\tau)\big\|^2_{L_2(\Omega)} d\tau \le \frac{1}{2}\big\|\mathbf{\Phi}\big\|^2_{L_2(\Omega)} + \big\|\mathbf{U}\big\|_{L_\infty(0,T;L_2(\Omega))}\int_0^t \big\|\mathbf{f}(\tau)\big\|_{L_2(\Omega)} d\tau, \quad t \in (0,T].$$

From this, using the inequalities $2ab \le (a^2 + b^2)$, (18), we arrive at (19).

To prove (20), the estimate (15) is written for $p = 2$

$$\frac{1}{4}\frac{d}{dt}\int_\Omega |\mathbf{U}|^4\,\mathbf{dx} + \mu\int_\Omega E\sum_{\alpha=1}^3 (\nabla U_\alpha)^2 d\mathbf{x} + 2\mu\int_\Omega (\nabla E)^2 d\mathbf{x} \le \Big(\int_\Omega \big|\mathbf{U}\big|^4 d\mathbf{x}\Big)^{\frac{3}{4}}\Big(\int_\Omega \big|\mathbf{f}\big|^4 d\mathbf{x}\Big)^{\frac{1}{4}}, \quad t \in (0,T].$$

We integrate over t in the range from 0 to t, and then, as in the previous case, we find

$$2\mu\int_0^t\int_\Omega (\nabla E)^2 d\mathbf{x}d\tau \le \big\|\mathbf{\Phi}\big\|_{L_{4(\Omega)}} + T\,A_3\big\|\mathbf{f}\big\|_{L_\infty(0,T;L_4(\Omega))}, \quad t \in (0,T].$$

Hence we come to (21).

3 Weak generalized solutions

We multiply the equation (1a) by an arbitrary vector-valued function

$$\mathbf{Z}(t,\mathbf{x}) \in \mathbf{L}_\infty(Q) \cap W_2^1(Q) \cap \mathring{\mathbf{J}}(Q),$$

equal to zero for $\big(t = T\big) \wedge \big(\mathbf{x} \in \partial\Omega\big)$. The product is integrable over the domain $Q = [0,T] \times \Omega$ and with by integrating by parts, taking into account the conditions (1) from We transfer the first two terms from $\mathbf{U}$ to $\mathbf{Z}$. As a result, we get

$$\int_Q \Big(-\mathbf{U}\,\frac{\partial \mathbf{Z}}{\partial t} + \mu\sum_{k=1}^3 \nabla U_k \nabla Z_k + \big((\mathbf{U},\nabla)\mathbf{U} + \nabla P\big)\mathbf{Z}\Big)\,\mathbf{dx}\,dt =$$

$$= \int\limits_{\Omega} \mathbf{\Phi Z}(0,\mathbf{x})\mathbf{dx} + \int\limits_{Q} \mathbf{fZdx}\, dt. \tag{22}$$

Definition 1. We call the vector-function $\mathbf{U}$ *of spaces weakly generalized solution of the initial-boundary value problem for the Navier-Stokes equations (1a) from the spaces*

$$\mathbf{U} \in \mathbf{L}_\infty(Q) \cap \mathbf{L}_\infty\big(0,T;\mathbf{W}^1_{2,0}(\Omega)\big) \cap \mathring{\mathbf{J}}(Q);\ \forall t \in [0,T]\big) \tag{23}$$

and satisfying the identity (22) for any

$$\mathbf{Z}(t,\mathbf{x}) \in \mathbf{L}_\infty(Q) \cap W^1_2(Q) \cap \mathring{\mathbf{J}}(Q) \wedge \left(\mathbf{Z}\,\Big|_{(t=T)\wedge(\mathbf{x}\in\partial\Omega)} = 0\right).$$

The validity of the definition 1 follows from the fact that all the integrals occurring in (22) are finite for any $\mathbf{Z}$, from the class indicated.

From Theorem 1 and Corollary 1, the uniqueness of weak generalized solutions of the problem (1).

Theorem 2. If the input data $\mathbf{f}$ and $\mathbf{\Phi}$ satisfy the requirements **i**) and **ii**), then the problem (1) has the unique weak generalized solution $\mathbf{U}$ satisfying the identity (22) for any $\mathbf{Z}$ from the definition 1.

Proof. Let the functions $\mathbf{U}$ and $\mathbf{U}^*$ be two solutions of the problem (1). We set $\mathbf{V} = \mathbf{U} - \mathbf{U}^*$; $\nabla R = 2\nabla(E^* - E)$, then we have:

$$\frac{\partial \mathbf{V}}{\partial t} - \mu\Delta\mathbf{V} + (\mathbf{V},\nabla)\mathbf{U} + (\mathbf{U}^*,\nabla)\mathbf{V} + \nabla R = 0; \tag{24a}$$

$$\mathbf{V}(0,\mathbf{x}) = 0, \quad \mathbf{V}(t,\mathbf{x})\big|_{\partial\Omega} = 0, \quad \mathbf{x} \in \partial\Omega. \tag{24b}$$

From the equations (24a) we pass to the identity

$$\int\limits_{Q_t} \left(\frac{\partial \mathbf{V}}{\partial t}\mathbf{V} - \mu\Delta\mathbf{V}\mathbf{V} + (\mathbf{V},\nabla)\mathbf{U}\mathbf{V} + (\mathbf{U}^*,\nabla)\mathbf{V}\mathbf{V} + \nabla R\mathbf{V}\right)\mathbf{dx}d\tau = 0, \quad \forall t \in (0,T]. \tag{25}$$

We transform all terms by integration by parts. As $\mathbf{U},\mathbf{U}^* \in \mathring{\mathbf{J}}(Q)$, thereby $\mathbf{V} \in \mathring{\mathbf{J}}(Q)$, then

$$\int\limits_{Q_t} (\mathbf{U}^*,\nabla)\mathbf{V}\mathbf{V}\mathbf{dx} = 0, \quad \int\limits_{Q_t} \nabla R\mathbf{V}\mathbf{dx} = 0.$$

From (25) we find

$$\frac{1}{2}\big\|\mathbf{V}(t)\big\|^2_{\mathbf{L}_2(\Omega)} + \mu\sum_{k=1}^{3}\int\limits_0^t \|\nabla V_k(\tau)\|^2_{\mathbf{L}_2(\Omega)}d\tau = -\int\limits_{Q_t}\sum_{k,\beta=1}^{3} V_\beta\frac{\partial V_k}{\partial x_\beta}U_k\mathbf{dx}d\tau. \tag{26}$$

The integral on the right-hand side is estimated successively by the Holder inequality for $p = \infty$ and $q = 1$, as well as Young's (3) for $p = 2$ as a result we obtain the chain of inequalities

$$\Big|\int\limits_{Q_t}\sum_{k,\beta=1}^{3} V_\beta\frac{\partial V_k}{\partial x_\beta}U_k\mathbf{dx}d\tau\Big| \le \max_k \|U_k\|_{L_\infty(Q)}\sum_{k,\beta=1}^{3}\int\limits_{Q_t}\Big|\frac{\partial V_k}{\partial x_\beta}\Big|\big|V_\beta\big|\mathbf{dx}d\tau \le$$

$$\le A\epsilon/2\sum_{k,\beta=1}^{3}\int\limits_0^t\Big\|\frac{\partial V_k}{\partial x_\beta}\Big\|^2_{L_2(\Omega)}d\tau + A_5\int\limits_0^t\sum_{\beta=1}^{3}\big\|V_\beta\big\|^2_{L_2(\Omega)}d\tau \le$$

$$\le A\epsilon/2\sum_{k=1}^{3}\int\limits_0^t\big\|\nabla V_k(\tau)\big\|^2_{\mathbf{L}_2(\Omega)}d\tau + A_5\int\limits_0^t\big\|\mathbf{V}(\tau)\big\|^2_{\mathbf{L}_2(\Omega)}d\tau, \quad A_5 = 3A/(2\epsilon).$$

Taking into account the estimates (8), (19) and, using the latter for $\epsilon = 2\mu/A$ from (26), we find

$$\|\mathbf{V}(t)\|^2_{\mathbf{L}_2(\Omega)} \leq A_5 \int_0^t \|\mathbf{V}(\tau)\|^2_{\mathbf{L}_2(\Omega)} d\tau,\ A_5 = 3A^2/(4\mu), \quad \forall t \in (0,T].$$

Whence we have $\frac{d}{dt}\Big(\exp(-A_5 t)\|\mathbf{V}(t)\|^2_{\mathbf{L}_2(\Omega)}\Big) \leq 0$. From this inequality we conclude that $\mathbf{V} \equiv 0$, $\forall t \in (0,T]$, that is, that the solutions $\mathbf{U}$ and $\mathbf{U}^*$ match. The theorem 2 is proved.

4 Strong solutions

Definition 2. If in a domain Q a weak generalized solution of the initial-boundary value problem for Navier-Stokes equations has all possible generalized derivatives of the same order as the equations themselves, then this solution is called strong.

Theorem 3. If the input data of the problem (1) satisfies the requirements **i**), **ii**) and $\partial\Omega \in C^2$, then the problem (1) has a unique strong generalized solution $\mathbf{U}$ from spaces

$$\mathbf{U} \in \mathbf{W}^{2,1}_{2,0}(Q) \cap \mathring{\mathbf{J}}_\infty(Q), \forall t \in [0,T]\big),$$

satisfying the equations (1a) almost everywhere in Q, and for them the following estimates hold:

$$\|\mathbf{U}_t\|^2_{\mathbf{L}_2(Q)} \leq \mu \sum_{k=1}^3 \|\nabla\Phi_k\|^2_{\mathbf{L}_2(\Omega)} + 3T\|\mathbf{f}\|^2_{\mathbf{L}_\infty(0,T;\mathbf{L}_2(\Omega))} + 3(AA_2 + 4A_4) \equiv A_6; \tag{27}$$

$$\|\Delta\mathbf{U}\|^2_{\mathbf{L}_2(Q)} \leq A_6/\mu^2 \equiv A_7; \tag{28}$$

$$\|\nabla U_k\|^2_{\mathbf{L}_\infty(0,T;\mathbf{L}_2(\Omega))} \leq A_6/\mu \equiv A_8,\ k = \overline{1,3}; \tag{29}$$

$$\|\mathbf{U}\|_{\mathbf{L}_2(0,T;\mathbf{W}^2_2(\Omega))} \leq A_9\|\Delta\mathbf{U}\|_{\mathbf{L}_2(Q)}; \quad A_9 - const. \tag{30}$$

Proof. In order to establish the inequalities (27) from the equation (1a), we pass to the identity

$$\int_{Q_t} \big(\mathbf{U}_t - \mu\Delta\mathbf{U}\big)^2 \mathbf{dx}\, d\tau = \int_{Q_t} \big(\mathbf{f} - (\mathbf{U},\nabla)\mathbf{U} + 2\nabla E\big)^2 \mathbf{dx} d\tau. \tag{31}$$

We will square the integrands. After that the pair product on the left side is transformed by integration by parts. On the right side, Young's inequality for $\epsilon = 1$ and $p = 2$. Then from (31) we pass to the inequality

From the last inequality, taking into account estimates

$$3\int_0^t \int_\Omega |(\mathbf{U},\nabla)\mathbf{U}|^2 \mathbf{dx} d\tau \leq 3\|\mathbf{U}\|_{L_\infty(Q)} \sum_{k=1}^3 \int_0^t \|\nabla U_k(\tau)\|^2_{L_2(\Omega)} d\tau = 3AA_2$$

and the estimates (8), (19) and (21), we obtain estimates (27)–(29) for strong generalized solutions of the problem (1). And note that (29) is slightly better than estimates (19).

Since the boundary of the domain $\partial\Omega \in C^2$ is found to be an estimate (30), using the inequalities from [1; 26], which is valid for any functions $U(x) \in W^2_2(\Omega) \cap W^2_{2,0}(\Omega)$: The theorem 3 is proved.

Remark. As a result, we were convinced that the properties (7) together with the estimate (8) allows one to find answers to many problematic questions connected with the solvability of the problem (1). In addition, (8) confirms the validity of the maximum principle for (1a) shown in [3–6] and on the basis of which the obtained results from the same papers.

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Навье-Стокс теңдеулерінің табиғи жағдайда шешілетіндігі

Үшөлшемді Навье-Стокс теңдеулерінің (НСТ) деректері сыптығыр болғанымен, ұзақ уақыт бойы сыптығыр шешімдерінің табылатындығы дәлелденбегені және жалпылама шешімдер класында жалқылық теоремасының орындалмайтыны туралы мәліметтер белгілі. Үшөлшемді НСТ-не максимум қағидасын негіздеуге мақала авторының біраз жұмыстарында зерттеу ізденістерінің нәтижелері келтірілген. Бұл зерттеулер жылдар бойы жетілдіре дамытылып, нәтижесінде НСТ-ға максимум қағидасының өте жеңіл түрі орындалатындығы көрсетілген. Ізденісті жалғастыру барысында НСТ шешімдерінің қасиеттерінен қысым мен жылдамдық векторы модулі квадратының арақатынас байланысы табылған. Бұл нәтиже негізінде НСТ-ның шешілетіндігі жөніндегі көптеген өзекті мәселелерге жауап алынды. Зерттеушінің таңдаған кеңістігінде үшөлшемді НСТ-ға қойылған есептің әлсіз шешімінің жалқылығы мен әлді шешімінің ұзақ бойы табылатындығы дәлелденген.

Кілт сөздер: Навье-Стокс теңдеулері, Навье-Стокс теңдеулеріндегі қысым, әлсіз жалпылама шешімнің жалқылығы, әлді шешімнің табылатындығы.

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Естественная разрешимость уравнений Навье-Стокса

Известно, что для трехмерных уравнений Навье-Стокса (УНС) не доказаны существование в целом по времени гладких решений при наличии гладких данных, а в классе обобщенных решений о нарушении теорема единственности. Автором статьи ранее приведены результаты поисковых исследований по обоснованию принципа максимума для трехмерных УНС. Со временем результаты этих исследований улучшались, и впоследствии была доказана справедливость простейшего принципа максимума для трехмерных УНС. Дальнейшее исследование позволило установить из свойств решений УНС соотношение между давлением и квадратом модуля вектора скорости, на основе чего найдены ответы на многие проблемные вопросы, связанные с разрешимостью УНС. В частности, в выбранных пространствах доказаны единственность слабых и существование сильных решений задачи для трехмерных уравнений Навье-Стокса в целом по времени.

Ключевые слова: уравнения Навье-Стокса, давление в уравнениях Навье-Стокса, единственность слабых обобщенных решений, существование сильных решений.