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# On mathematical and analytical methods for solving problems on vibrations of membranes and plates

The problems about determination of the frequencies and forms of natural vibrations of plates and shells lead to the necessity of partial differential equations integration. The well-researched cases are those where it is possible to separate the variables. In particular, these include the vibrations of a rectangular plate hinged on opposite sides, umbrella and fan vibration of circular axisymmetric plates and vibrations of cylindrical shells, closed or hinged along generating curves. In this work, the vibration of a flat homogeneous membrane is investigated for the general case of boundary conditions.

*Keywords:* boundary value problem, membrane, plate, vibrations, spectrum problem, orthonormal system of functions, deflection function.

Film and membrane structures are highlighted among the thin-walled structures that combine lightness with high strength. Such thin-walled structures (films, membranes, coatings, etc.) find application in all branches of production and daily life [1].

To create new films, membranes and coatings with the specified performance and durability, one needs to investigate how the time and temperature, mechanical (including vibrational), chemical and other exposures may cause destructive processes in the material structure. Therefore, the necessary quality of films, membranes and coatings are usually provided by calculating the impact of these effects on the strength and the characteristics necessary for exploitation of the material [2].

Consider a flat homogeneous rectangular membrane fixed at the edges, with sides $b$ and $c$ in the plane $OXY$,$0 \leqslant x \leqslant b, 0 \leqslant y \leqslant c$. We denote the deflection function of the membrane, that is, its deviation from the equilibrium position at the point $(x, y)$ at the time $t$, by $u(x, y, t)$. Let us consider the process when the vibrations of the membrane are caused by a given initial deviation and a given initial velocity [3].

To find the function $u(x, y, t)$ we have the following boundary value problem: to find the solution of the partial differential equation describing the process of the membrane vibrations,

$$u_{tt} = a^2(u_{xx} + u_{yy}), \tag{1}$$

where $a^2 = \frac{T}{\rho}$, $T$ is the membrane tension; $\rho$ is the density of a membrane, in the region $0 < x < b$, $0 < y < c$, $t > 0$,

under the initial conditions

$$u(x, y, 0) = \varphi(x, y); \tag{2}$$

$$u_t(x, y, 0) = \psi(x, y). \tag{3}$$

and boundary conditions

$$\alpha_1 u(0, y, t) + \beta_1 u_x(0, y, t) = 0, \;\; \alpha_2 u(b, y, t) + \beta_2 u_x(b, y, t) = 0; \tag{4}$$

$$\gamma_1 u(x, 0, t) + \theta_1 u_y(x, 0, t) = 0, \;\; \gamma_2 u(x, c, t) + \theta_2 u_y(x, c, t) = 0, \tag{5}$$

where $\varphi$ and $\psi$ are the given functions; $\alpha_i$, $\beta_i$, $\gamma_i$, $\theta_i$ are the given numbers, and $\alpha_i^2 + \beta_i^2 \neq 0$, $\gamma_i^2 + \theta_i^2 \neq 0$; $i = 1, 2$.

We seek the solution of problem (1)–(5) by the method of variables separation as a function in a form [4]

$$u(x, y, t) = \nu(x, y) \cdot T(t). \tag{6}$$

This function (6) is not identically equal to zero. Dividing the variables, we obtain an equation for the function $T(t)$

$$T'' + a^2\sigma T = 0, \tag{7}$$

and for the function $\nu(x,y)$ we get the following boundary value problem

$$\nu_{xx} + \nu_{yy} + \sigma\nu = 0, \tag{8}$$

$$\alpha_1\nu(0,y) + \beta_1\nu_x(0,y) = 0, \;\; \alpha_2\nu(b,y) + \beta_2\nu_x(b,y) = 0, \tag{9}$$

$$\gamma_1\nu(x,0) + \theta_1\nu_y(x,0) = 0, \;\; \gamma_2\nu(x,c) + \theta_2\nu_y(x,c) = 0, \tag{10}$$

where $\sigma$ is a constant of variables separation. For the ease of calculations we take $\sigma$ with a minus sign, without assuming anything about its sign. The boundary conditions (9), (10) are obtained by the direct substitution of (6) in (4), (5).

To solve problem (8) - (10) we again apply the method of variables separation. We seek a solution of this problem in the form of a function $\nu(x,y) = X(x)\cdot Y(y)$ , which is not identically equal to zero. To define functions $X(x)$ and $Y(y)$ from (8) - (10) we obtain one-dimensional spectral problems

$$\begin{cases} X'' + \eta\cdot X = 0, \\ \alpha_1 X(0) + \beta_1 X'(0) = 0, \\ \alpha_2 X(b) + \beta_2 X'(b) = 0, \end{cases} \quad \begin{cases} Y'' + \tau\cdot X = 0, \\ \gamma_1 Y(0) + \theta_1 Y'(0) = 0, \\ \gamma_2 Y(c) + \theta_2 Y'(c) = 0, \end{cases} \tag{11}$$

where $\eta$ is constant variables separation, and $\tau = \sigma - \eta$ [5].

*Remark 1.* By direct calculation, we determine that the spectral problem for an equation with a parameter $\nu$

$$\begin{cases} Z'' + \nu\cdot Z = 0; \\ h_1 Z(0) + g_1 Z'(0) = 0; \\ h_2 Z(l) + g_2 Z'(l) = 0, \end{cases} \tag{12}$$

where $Z = Z(z)$; $0 < z < l$; $h_i$, $g_i$ $(i = 1,2)$ are the given numbers, and $h_i^2 + g_i^2 \neq 0$, $i = 1,2$, has non-trivial solutions in the following cases:

1) $\nu = 0$ when the condition holds

$$g_1 h_2 - h_1(h_2 l + g_2) = 0; \tag{13}$$

2) $\nu > 0$.

By remark 1, the spectral problems (11) have eigenvalues and eigenfunctions if $\eta = \tau = 0$ when the condition (13) holds for the corresponding parameters, and if $\eta > 0$, $\tau > 0$.

We introduce the notations $\eta = \lambda^2$, $\tau = \mu^2$ in (11). Solving spectral problems (11), we receive that the eigenvalues $\lambda_1,\ldots,\lambda_n,\ldots$ and $\mu_1$, $\ldots$, $\mu_m$, $\ldots$ of these problems are the roots of the following equations, respectively

$$\operatorname{tg}\lambda b = \frac{(\alpha_2\beta_1 - \alpha_1\beta_2)\lambda}{\alpha_1\alpha_2 + \beta_1\beta_2\lambda^2}, \;\; \operatorname{tg}\mu c = \frac{(\gamma_2\theta_1 - \gamma_1\theta_2)\mu}{\gamma_1\gamma_2 + \theta_1\theta_2\mu^2},$$

and the eigenfunctions are functions in the form

$$X_n(x) = A_n(\beta_1\lambda_n\cos\lambda_n x - \alpha_1\sin\lambda_n x), \;\; Y_m(y) = B_m(\theta_1\mu_m\cos\mu_m y - \gamma_1\sin\mu_m y),$$

where $A_n$, $B_m$ are constants.

Since $\sigma = \tau + \eta = \lambda^2 + \mu^2$, we obtain that the eigenvalues $\sigma_{n,m} = \lambda_n^2 + \mu_m^2$ correspond to eigenfunctions

$$\nu_{nm}(x,y) = A_{nm}(\beta_1\lambda_n\cos\lambda_n x - \alpha_1\sin\lambda_n x)(\theta_1\mu_m\cos\mu_m y - \gamma_1\sin\mu_m y), \tag{14}$$

where $A_{nm} = A_n\cdot B_m$ is a constant. We choose $A_{nm} = A_n\cdot B_m$ so that the norm of the function $\nu_{nm}$ with a weight of unit was equal to one, that is, we orthonormalize the functions $\nu_{nm}$

$$\int_0^b\int_0^c \nu_{nm}^2\,dxdy = A_{nm}^2\int_0^b(\beta_1\lambda_n\cos\lambda_n x - \alpha_1\sin\lambda_n x)^2\,dx\cdot\int_0^c(\theta_1\mu_m\cos\mu_m y - \gamma_1\sin\mu_m y)\,dy = 1,$$

$$A_{nm} = \frac{1}{\sqrt{\int_0^b(\beta_1\lambda_n\cos\lambda_n x - \alpha_1\sin\lambda_n x)^2\,dx\cdot\int_0^c(\theta_1\mu_m\cos\mu_m y - \gamma_1\sin\mu_m y)\,dy}}. \tag{15}$$

Calculation of the coefficients $A_{nm}$ in the general case, by the formula (15), is laborious and inexpedient. It is much more convenient and more rational to calculate the coefficients $A_{nm}$ in each case of the boundary conditions of the spectral problem, than to use the cumbersome and hard-to-remember formula obtained in calculations of the integrals in (15).

*Remark 2.* We investigate the spectral problem (12) by introducing the notation $\nu = p^2$. Under different boundary conditions, from (12) one can obtain nine spectral problems. We carry out the calculations for the case of boundary conditions of the third kind. Obviously, the remaining particular cases of problem (12) are investigated in a similar way with more elementary calculations. The general solution of the equation in (12) is $Z(z) = A\cos pz + +B\sin pz$. Its substitution into boundary conditions of the third kind of the problem (12) gives us a system of equations

$$\begin{cases} B = \dfrac{h}{p}A, \\ A(-p\sin pl + h\cos pl + \dfrac{h^2}{p}\sin pl) = 0. \end{cases}$$

To obtain non-trivial solutions, from condition $A \neq 0$ we receive that $p_k$ are the roots of the equation

$$\operatorname{ctg} pl = \frac{1}{2}\left(\frac{p}{h} - \frac{h}{p}\right), \tag{16}$$

and the eigenfunctions of the problem are the following functions

$$Z_k(z) = \widetilde{A_k}\Big(\cos p_k z + \frac{h}{p_k}\sin p_k z\Big) = A_k(p_k\cos p_k z + h\sin p_k z). \tag{17}$$

We normalize the functions (17), taking into account the relation (16) in the calculations,

$$\int_0^l Z_k^2(z)\,dz = \frac{A_k^2}{2}\int_0^l [p_k^2(1+\cos 2p_k z) + 2p_k h\sin 2p_k z + h^2(1-\cos 2p_k z)]\,dz =$$

$$= \frac{A_k^2}{2}\Big[p_k^2\Big(l + \frac{1}{2p_k}\sin 2p_k l\Big) + h(1-\cos 2p_k l) + h^2\Big(l - \frac{1}{2p_k}\sin 2p_k l\Big)\Big] =$$

$$= \frac{A_k^2}{2}\Big[p_k^2\Big(l + \frac{1}{2p_k}\cdot\frac{4p_k h(p_k^2-h^2)}{(p_k^2+h^2)^2}\Big) + h\Big(1 - \frac{(p_k^2-h^2)^2 - 4p_k^2h^2}{(p_k^2+h^2)^2}\Big) +$$

$$+h^2\Big(l - \frac{1}{2p_k}\cdot\frac{4p_k h(p_k^2-h^2)}{(p_k^2-h^2)^2}\Big)\Big] =$$

$$= \frac{A_k^2}{2(p_k^2+h^2)^2}\Big[p_k^2 l(p_k^2+h^2)^2 + 2p_k^2 h(p_k^2-h^2) + h(p_k^2+h^2)^2 - h(p_k^2-h^2)^2 +$$

$$+4p_k^2h^3 + h^2 l(p_k^2+h^2)^2 - 2h^3(p_k^2-h^2)\Big] =$$

$$= \frac{A_k^2}{2(p_k^2+h^2)^2}\Big[(p_k^2+h^2)^2(l\cdot(p_k^2+h^2)+h) + h(2p_k^4 - 2p_k^2h^2 - p_k^4 + 2p_k^2h^2 -$$

$$-h^4 + 4p_k^2h^2 - 2h^2p_k^2 + 2h^4)\Big] =$$

$$= \frac{A_k^2}{2(p_k^2+h^2)^2}\Big[(p_k^2+h^2)^2\Big(\Big[l\cdot(p_k^2+h^2)+h\Big] + h(p_k^4 + 2p_k^2h^2 + h^4)\Big)\Big] =$$

$$= \frac{A_k^2}{2(p_k^2+h^2)^2}(p_k^2+h^2)^2\Big[l\cdot((p_k^2+h^2)+2h)\Big] = A_k^2\cdot\frac{l(p_k^2+h^2)+2h}{2} = 1.$$

As a result, we obtain

$$A_k = \sqrt{\frac{2}{(p_k^2+h^2)+2h}},\quad Z_k(z) = A_k(p_k\cos p_k z + h\sin p_k z),\quad k = 1,2,...$$

We return to the original problem (1) - (5). We have from (14)

$$\nu_{nm}(x,y) = A_{nm}(\beta_1\lambda_n \cos\lambda_n x - \alpha_1 \sin\lambda_n x)(\theta_1\mu_m \cos\mu_m y - \gamma_1 \sin\mu_m y),$$

where the coefficients $A_{nm}$ are calculated as in remark 2 in each particular case of boundary conditions.

Let us find the general solution of equation (7) for $\sigma_{mn} = \lambda_n^2 + \mu_m^2$

$$T_{nm}(t) = C_{nm}\cos a\sqrt{\sigma_{nm}}t + D_{nm}\sin a\sqrt{\sigma_{nm}}t,$$

where $C_{nm}$, $D_{nm}$ are arbitrary constants. Returning to the original problem (1) - (5), we obtain that the particular solutions according to (6) will have the form

$$u_{nm}(x,y,t) = \nu_{nm}(x,y)\cdot T_{nm}(t) = \nu_{nm}(x,y)(C_{nm}\cos a\sqrt{\sigma_{nm}}t + D_{nm}\sin a\sqrt{\sigma_{nm}}t).$$

By the principle of superposition, the general solution of equation (1) with the boundary conditions (4), (5) has the form

$$u(x,y) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(C_{nm}\cos a\sqrt{\sigma_{nm}}t + D_{nm}\sin a\sqrt{\sigma_{nm}}t)\cdot\nu_{nm}(x,y). \tag{18}$$

Using the initial conditions (2), (3), relation (18) and the property of orthonormality of functions $\nu_{nm}$, we find the values of the constants $C_{nm}$ and $D_{nm}$

$$u(x,y,0) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}C_{nm}\nu_{nm}(x,y) = \varphi(x,y),\ \ C_{nm} = \int_0^b\int_0^c \varphi(x,y)\nu_{nm}(x,y)\,dxdy.$$

$$u_t(x,y,0) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}D_{nm}a\sqrt{\sigma_{nm}}\nu_{nm}(x,y) = \psi(x,y),\ \ D_{nm} = \frac{1}{a\sqrt{\sigma_{nm}}}\int_0^b\int_0^c \psi(x,y)\nu_{nm}(x,y)\,dxdy.$$

Hence, we obtain the solution of problem (1) - (5) in the analytical form

$$u(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(C_{nm}\cos a\sqrt{\sigma_{nm}}t + D_{nm}\sin a\sqrt{\sigma_{nm}}t)\cdot\nu_{nm}(x,y),$$

where

$$\nu_{nm}(x,y) = A_{nm}(\beta_1\lambda_n \cos\lambda_n x - \alpha_1 \sin\lambda_n x)(\theta_1\mu_m \cos\mu_m y - \gamma_1 \sin\mu_m y),$$

$\sigma_{mn} = \lambda_n^2 + \mu_m^2$; $\lambda_1, ..., \lambda_n, ...$ are the roots of the equation $\mathrm{tg}\,\lambda b = \frac{(\alpha_2\beta_1-\alpha_1\beta_2)\lambda}{\alpha_1\alpha_2+\beta_1\beta_2\lambda^2}$; $\mu_1, ..., \mu_m, ...$ are the roots of the equation $\mathrm{tg}\,\mu c = \frac{(\gamma_2\theta_1-\gamma_1\theta_2)\mu}{\gamma_1\gamma_2+\theta_1\theta_2\mu^2}$,

$$A_{nm} = \frac{1}{\sqrt{\int_0^b(\beta_1\lambda_n\cos\lambda_n x - \alpha_1\sin\lambda_n x)^2\,dx\cdot\int_0^c(\theta_1\mu_m\cos\mu_m y - \gamma_1\sin\mu_m y)\,dy}},$$

$$C_{nm} = \int_0^b\int_0^c \varphi(x,y)\nu_{nm}(x,y)\,dxdy,\ \ D_{nm} = \frac{1}{a\sqrt{\sigma_{nm}}}\int_0^b\int_0^c \psi(x,y)\nu_{nm}(x,y)\,dxdy.$$

Thus, function $u(x,y,t)$ is found in the general case of boundary conditions. This functiondescribes the deviation of the membrane from the equilibrium position.

*Statement and analytical methods of solving the problem on plate vibrations*

The Germain-Lagrange equation describing small transverse vibrations of an elastic isotropic plate $|x| < a$, $|y| < b$ of constant thickness $h$, has the form

$$D\triangle\triangle w + \rho\frac{\partial^2\omega}{\partial t^2} = 0, \quad \triangle = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

Here $w(x,y,t)$ is the transverse bending of the middle plane of the plate; $\triangle$ is the two-dimensional Laplace operator; $D = Eh^3/(12(1-\nu^2))$ is bending stiffness of the plate; $\nu$ is Poisson's coefficient; $E$ is Young's modulus; $\rho$ is the specific density per unit area ofthe plate; $t$ is time.

The problem of determining the natural frequencies and vibrations types of a plate with free edges is reduced to determining the deflection $W(x,y)$ (here and below the harmonic factor $e^{-i\omega t}$ is omitted) and the frequencies $k^4 = \rho h\omega^2/D$ from the homogeneous boundary value problem

$$\triangle\triangle W - k^4 W = 0 \tag{19}$$

with boundary conditions for $x = \pm a$:

$$\frac{\partial^2 W}{\partial x^2} + \nu\frac{\partial^2 W}{\partial y^2} = 0,$$

$$\frac{\partial^3 W}{\partial x^3} + (2-\nu)\frac{\partial^3 W}{\partial y^2 \partial x} = 0, \tag{20}$$

and for $y = \pm b$ :

$$\frac{\partial^2 W}{\partial y^2} + \nu\frac{\partial^2 W}{\partial x^2} = 0,$$

$$\frac{\partial^3 W}{\partial y^3} + (2-\nu)\frac{\partial^3 W}{\partial y^2 \partial x} = 0. \tag{21}$$

At present, two analytical approaches to the solution of the boundary value problem (19)–(21) have become most widely used. These approaches are the Ritz method and the superposition method. In his classic paper, Ritz pointed out that this boundary value problem is equivalent to finding the minimum value of the integral

$$J = \int_{-a}^{a}\int_{-b}^{b}\Big[\Big(\frac{\partial^2 W}{\partial x^2}\Big)^2 + \Big(\frac{\partial^2 W}{\partial y^2}\Big)^2 + 2\nu\frac{\partial^2 W}{\partial x^2}\frac{\partial^2 W}{\partial y^2} + 2(1-\nu)\Big(\frac{\partial^2 W}{\partial x\partial y}\Big)^2\Big]\,dxdy \tag{22}$$

provided that

$$\int_{-a}^{a}\int_{-b}^{b} W^2\,dxdy = A = const. \tag{23}$$

For the particular case of a square plate $a = 1$, $b = 1$, Ritz chose the representation

$$W_S = \sum_{m=0}^{S}\sum_{n=0}^{S} A_{nm} u_m(x)\nu_n(y), \tag{24}$$

in which $u_m(x)$ and $\nu_n(y)$ are eigenfunctions of bending vibrations of an elastic rod $\xi < 1$ with both free edges. In other words, these functions are solutions of the homogeneous equation

$$\frac{d^4 u}{d\xi^4} = k^4 u \tag{25}$$

with homogeneous boundary conditions

$$\frac{d^2 u}{d\xi^2} = 0, \qquad \frac{d^3 u}{d\xi^3} = 0, \qquad \xi = \pm 1. \tag{26}$$

The normalization is chosen so that for the solution number $m$ the following ratio holds

$$\int_{-1}^{1} u_m^2\,d\xi = 1.$$

Rayleigh studied the solution of the homogeneous boundary value problem (25), (26) in detail and showed that the required functions should be chosen as follows

• for even $m$

$$u_m(\xi) = \frac{chk_m \cos k_m\xi + \cos k_m chk_m\xi}{\sqrt{ch^2 k_m + \cos^2 k_m}},$$

where $k_m$ is a root of the equation $\operatorname{tg} k_m + thk_m = 0$,

- for odd $m$

$$u_m(\xi) = \frac{shk_m \sin k_m\xi + \sin k_m shk_m\xi}{\sqrt{sh^2 k_m - \sin^2 k_m}},$$

where $k_m$ is a root of the equation $tgk_m - thk_m = 0$.

It was assumed that

$$u_0(\xi) = \frac{1}{\sqrt{2}}, \qquad k_0 = 0,$$

$$u_1(\xi) = \sqrt{\frac{3}{2}}\xi, \qquad k_1 = 0.$$

When an indefinite Lagrange multiplier $\lambda = k^4$ is used, the integral $J$ is minimized and the condition (23) holds, the substitution of the expansion (24) into an quadratic with respect $A_{mn}$ expression (22) leads to a homogeneous system of linear algebraic equations with respect to $A_{mn}$. Hence $\lambda$ is defined in the standard way as a value that reduces to zero the determinant of this linear system. Moreover, all the boundary conditions (20) and (21) are satisfied identically.

For the Poisson's coefficient $\nu = 0.225$ (glass, as in the experiments of Khladny) and for the case of antisymmetric vibrations with respect to the diagonals of the square $y = \pm x$ (in this case $A_{mn} = A_{mn}$) Ritz took $s = 5$ in the representation (24), manually calculated all the necessary integrals and obtained a homogeneous system of linear algebraic equations of the sixth order. He also managed to find the first two roots of the determinant of this system.Further, a bold assumption was made that the vibrationstypes of a plate are determined only by the main summand $u_m(x)\nu_n(y) \pm u_n(x)\nu_m(y)$. In articles Hladni the table is given for calculating the first 35 eigenfrequencies and their comparison with the experimental data of Khladni. Ritz also cited figures of nodal lines for vibrations types at natural frequencies corresponding to all four types of symmetry with respect to the $x$ and y axes. He took $s = 5$ everywhere, represented the complete expression (24) with numerical coefficients, and emphasized the defining contribution of the principal terms. It was truly a titanic work, given the lack of a computer. A different approach, which is traditionally called the superposition method,represents a general solution of the differential equation (19) as the sum of solutions for bands $|x| \leqslant a$, $|y| \leqslant b$ in the form of trigonometric series. The solution is chosen in such a way as to satisfy the second boundary conditions in (20), (21) identically and to have enough arbitrariness to meet the remaining two conditions.

There are four types of symmetry of the plate deflection: the function $W_S(x, y)$ is even relative to $x$ and $y$; the function $W_{SA}(x, y)$ is even relative to $x$ and odd relative to $y$; the function $W_{AS}(x, y)$ is odd relative to $x$ and even relative to $y$; the function $W_A(x, y)$ is odd relative to $x$ and $y$. Using the standard method of variables separation, the solutions of equation (19) can be written in the form

$$W_S = \frac{bx_0^S}{k}\Big(\frac{\cos ky}{\sin kb} - \frac{chky}{shkb}\Big) + \frac{ay_0^S}{k}\Big(\frac{\cos kx}{\sin ka} - \frac{chkx}{shka}\Big) + b\sum_{n=1}^{\infty}(-1)^{n+1}x_n^S A(y, b, \alpha_n)\cos\alpha_n x+$$

$$+\alpha\sum_{n=1}^{\infty}(-1)^{n+1}y_n^S A(x, b, \beta_n)\cos\beta_n y, \tag{27}$$

$$W_{SA} = \frac{bx_0^{Sa}}{k^2}\Big(\frac{\sin ky}{\cos kb} + \frac{shky}{chkb}\Big) + b\sum_{n=1}^{\infty}(-1)^n x_n^{Sa} B(y, b, \alpha_n)\cos\alpha_n x - \alpha\sum_{n=1}^{\infty}(-1)^n y_n^{Sa} A(x, a, \delta_n)\sin\delta_n y; \tag{28}$$

$$W_A = b\sum_{n=1}^{\infty}(-1)^{n+1}x_n^a B(y, b, \gamma_n)\sin\gamma_n x + \sum_{n=1}^{\infty}(-1)^{n+1}y_n^a B(x, a, \delta_n)\sin\delta_n y, \tag{29}$$

where designations are thus introduced

$$\alpha_n = \frac{\pi n}{a}; \qquad \beta_n = \frac{\pi n}{b}; \qquad \gamma_n = \frac{\pi(2n-1)}{2a}; \qquad \delta_n = \frac{\pi(2n-1)}{2b};$$

$$A(z, h, \xi) = \frac{\xi^2 + k^2 - (2-\nu)\xi^2}{\sqrt{\xi^2 - k^2}}\frac{ch\sqrt{\xi^2 - k^2}z}{sh\sqrt{\xi^2 - k^2}h} - \frac{\xi^2 - k^2 - (2-\nu)\xi^2}{\sqrt{\xi^2 + k^2}}\frac{ch\sqrt{\xi^2 + k^2}z}{sh\sqrt{\xi^2 + k^2}h};$$

$$B(z, h, \xi) = \frac{\xi^2 + k^2 - (2-\nu)\xi^2}{\sqrt{\xi^2 - k^2}}\frac{sh\sqrt{\xi^2 - k^2}z}{ch\sqrt{\xi^2 - k^2}h} - \frac{\xi^2 - k^2 - (2-\nu)\xi^2}{\sqrt{\xi^2 + k^2}}\frac{sh\sqrt{\xi^2 + k^2}z}{ch\sqrt{\xi^2 + k^2}h}.$$

The expression for $W_{AS}$ is not written out because, due to the symmetry of the problem all the corresponding eigenvalues and forms are constructed by the solution of $SA$ with the substitution $x \leftrightarrow y$ and $a \leftrightarrow b$:

$$k_{SA}(a/b) = k_{AS}(b/a),$$
$$W_{SA}(x, y, a, b) = W_{AS}(y, x, b, a).$$

Note that for this reason, in the case of a square, the eigenvalues for the cases $AS$ and $SA$ coincide.

Substitution of solutions (27) - (29) into the first of the boundary conditions (20), (21) with the subsequent expansion of incoming functions into trigonometric series on the basis of formulas

$$\frac{chpz}{shph} = \frac{1}{ph} + \frac{2p}{h}\sum_{m=1}^{\infty}\frac{(-1)^m \cos\xi_m z}{\xi_m^2 + p^2}, \qquad \xi_m = \frac{m\pi}{h},$$

$$\frac{shpz}{chph} = \frac{2p}{h}\sum_{m=1}^{\infty}\frac{(-1)^{m+1} \sin\eta_m z}{\eta_m^2 + p^2}, \qquad \eta_m = \frac{(2m-1)\pi}{2h}$$

allows us to obtain homogeneous infinite systems of linear algebraic equations with respect to unknown coefficients from the equality under basic functions [6].

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## Мембраналар мен пластиналардың тербелісі бойынша мәселелерді шешудің математикалық және аналитикалық әдістері

Пластиналар мен қабықшалардың табиғи тербелістерінің жиіліктері мен формулаларын анықтау мәселелері жартылай дифференциалдық теңдеулерді интегралдаудың қажеттілігіне әкелді. Ең жақсы зерттелген жағдай — бұл айнымалыларды бөліп алуға болатын жағдай. Олардың ішінде, атап айтқанда, қарсы жағында бекітілген тік бұрышты пластинаның тербелісі, дөңгелек осьсимметрлік плиталардағы қолшатыр және желдеткіш тербелістер, цилиндр қабықшаларының тербелісі, жабық немесе генераторларға бекітілген. Мақалада шекаралық жағдайлардың жалпы жағдайында жазық біркелкі мембрананың тербелісі зерттелді.

*Кілт сөздер:* шекаралық есептер, мембрана, пластина, тербеліс, спектрлік есеп, ортонормалды функциялар жүйесі, иілу функциясы.

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# О математических и аналитических методах решения задач о колебании мембран и пластин

Задачи об определении частот и форм собственных колебаний пластин и оболочек приводят к необходимости интегрирования дифференциальных уравнений в частных производных. Наиболее хорошо изучены те случаи, когда оказывается возможным разделение переменных. К ним относятся, в частности, колебания прямоугольной пластины, шарнирно-опертой по противолежащим сторонам, зонтичные и веерные колебания круглых осесимметричных пластин, колебания цилиндрических оболочек, замкнутых или шарнирно-закрепленных вдоль образующих. В статье проведено исследование колебания плоской однородной мембраны для общего случая граничных условий.

*Ключевые слова:* краевая задача, мембрана, пластина, колебания, спектральная задача, ортонормированная система функций, функция прогиба.

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