UDC 517.95

M.T. Jenaliyev[1], M.M. Amangaliyeva[1], K.B. Imanberdiyev[2], M.I. Ramazanov[3,4]

[1]*Institute of Mathematics and Mathematical Modeling CS MES RK, Almaty, Kazakhstan;*
[2]*Al-Farabi Kazakh National University, Almaty, Kazakhstan;*
[3]*Ye.A. Buketov Karaganda State University, Kazakhstan;*
[4]*Institute of Applied Mathematics, Karaganda, Kazakhstan*
(E-mail: muvasharkhan@gmail.com)

On a stability of a solution of the loaded heat equation

Steadily growing interest in study of loaded differential equations is explained by the range of their applications and a circumstance that loaded equations make a special class of functional-differential equations with specific problems. These equations have applications in study of inverse problems of differential equations with important applied interests. In this paper solvability questions of stabilization problems with a boundary for the loaded heat equation are studied in the given bounded domain $\Omega \equiv (-\pi/2, \pi/2)$. The task is to choose boundary conditions (controls), that the solution of the obtained mixed boundary value problem tends to a given stationary solution with the prescribed speed $\exp(-\sigma_0 t)$ as $t \to \infty$. At this the control is required to be a feedback control, i.e. that it reacted to the unintended fluctuations of the system, suppressing the results of their impact on the stabilized solution. Stabilization problems have a direct connection with controllability problems. The paper proposes a mathematical formalization of the concept of feedback, and with its help it solves the problem of stabilizability of a loaded heat equation by dint of feedback control given on the part of the boundary is solved.

Keywords: stability, feedback control, loaded heat equation, boundary value problem, inverse problem, Green function, eigenvalue, eigenfunction.

Introduction. In recent years, an increasing interest in studying loaded differential equations is manifested. In this both the steadily extending field of their applications and the fact, that the loaded equations are a special class of equations with specific problems, played a role.

In this paper, the statement of the inverse problem on the stabilization of solutions for the loaded heat conduction equation using the boundary conditions is given. The theorem on the solvability of the inverse problem is proved and an algorithm for approximate constructing boundary controls in the form of synthesis is developed. The numerical calculations have been carried out, that show the effectiveness of the proposed algorithm (Fig. 1-3).

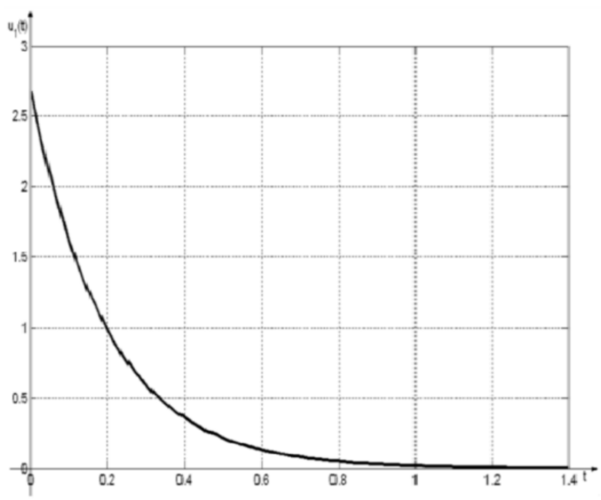

Figure 1. Graphic of $u_1(t)$

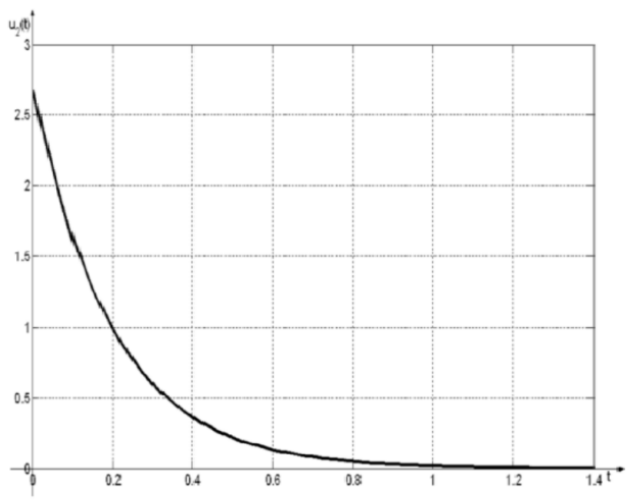

Figure 2. Graphic of $u_2(t)$

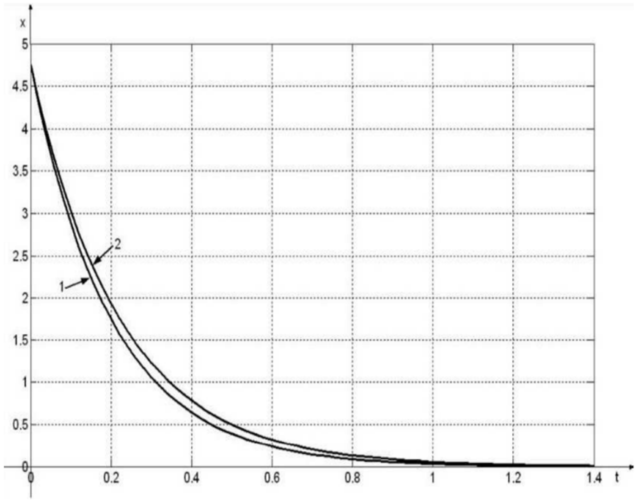

Figure 3. Graphics of $1 - \|y(x,t)\|_{L_2(-\pi/2,\pi/2)}$;
$2 - C_0 \cdot \exp\{-\sigma_0 t\}$, where $C_0 \approx 2,6808 \cdot \sqrt{\pi}$; $\alpha = 5,\ \sigma_0 = 4,5$

Statement of the boundary value problem. Search for such boundary controls $u_1(t), u_2(t) \in L_2(0,\infty)$, that the solution $y(x,t)$ of the boundary value problem

$$y_t(x,t) - y_{xx}(x,t) + \alpha \cdot y(0,t) = 0,\ \ \{x,t\} \in Q; \tag{1}$$

$$y(-\pi/2,t) = u_1(t),\ y(\pi/2,t) = u_2(t),\ y(x,0) = y_0(x), \tag{2}$$

as $t \to \infty$ approach zero as follows:

$$\|y(x,t)\|_{L_2(-\pi/2,\pi/2)} \le C_0 e^{-\sigma_0 t}, \tag{3}$$

where $Q = \{x,t|\ \frac{-\pi}{2} < x < \frac{\pi}{2},\ t > 0\}$, $\alpha \in \mathbf{C}$, σ_0 is the given positive number, $y_0(x) \in L_2(\frac{-\pi}{2}, \frac{\pi}{2})$ is the given function.

Equation (1) is called the loaded equation [1–3]. We note that the vast literature is devoted to the inverse problems of the differential equations. Among them, we want to acknowledge the recently published textbook for university students [4], which is apparently the first textbook dedicated to the inverse and ill-posed problems, and in which there is fairly detailed overview of statements current problems and unsolved problems.

On the solvability of the boundary value problem (1)–(2). We write the problem (1)–(2) in the operator form:

$$Ly = \{y_0, u_1, u_2\},$$

where

$$L : L_2(Q) \to E \equiv L_2(-\pi/2,\pi/2) \times L_2(0,\infty) \times L_2(0,\infty),$$

and we give the definition of a strong solution.

Definition 1. The function $y(x,t) \in L_2(Q)$ is called a strong solution of the boundary value problem (1)–(2), if there exists a sequence

$$\{y_s(x,t)\}_{s=1}^{\infty} \subset C^{2,1}_{x,t}(Q) \cap C(\overline{Q}),$$

such that

$$y_s(x,t) \to y(x,t) \text{ in } L_2(Q),\ \ Ly_s \to \{y_0,u_1,u_2\} \text{ in } E \text{ at } s \to \infty.$$

The following theorem holds

Theorem. For any given controls $u_1(t), u_2(t) \in L_2(0,\infty)$ and any initial function $y_0(x) \in L_2(\frac{-\pi}{2}, \frac{\pi}{2})$ of boundary value problem (1)–(2) has the unique strong solution $y(x,t) \in L_2(Q)$, and $y(x,t) \in W(0,\infty)$, where

$$W(0,\infty) = \{v|\, v \in L_2(0,\infty; W_2^1(-\pi/2,\pi/2)), v_t \in L_2(0,\infty; W_2^{-1}(-\pi/2,\pi/2))\}.$$

Proof. We transform boundary value problem (1)–(2) to the following loaded integral equation

$$y(x,t) = \int_{-\pi/2}^{\pi/2} y_0(\xi)G(x,\xi,t)d\xi - \alpha \int_0^t y(0,\tau) \int_{-\pi/2}^{\pi/2} G(x,\xi,t-\tau)d\xi\, d\tau +$$

$$+ \int_0^t u_1(\tau)H_1(x,t-\tau)d\tau - \int_0^t u_2(\tau)H_2(x,t-\tau)d\tau, \tag{4}$$

where the Green function G has the form

$$G(x,\xi,t) = \frac{2}{\pi}\sum_{n=1}^{\infty} \sin n(x+\pi/2)\sin n(\xi+\pi/2)\exp\{-n^2t\},$$

and the functions H_1 and H_2 are expressed in terms the Green function by the formulas:

$$H_1(x,t) = \frac{\partial}{\partial\xi}G(x,\xi,t)_{|\xi=-\pi/2}, \quad H_2(x,t) = \frac{\partial}{\partial\xi}G(x,\xi,t)_{|\xi=\pi/2}.$$

In turn, from (4) for the unknown function $\mu(t) = y(0,t)$ we obtain the following integral equation

$$\mu(t) + \alpha\int_0^t K(t-\tau)\mu(\tau)d\tau = F(t),\ t > 0, \tag{5}$$

where the kernel of the integral operator has the form

$$K(t) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n-1}\exp\{-(2n-1)^2t\}, \tag{6}$$

the right-hand side of the equation represents the sum $F(t) = F_0(t) + F_1(t) + F_2(t)$,
where

$$F_0(t) = \frac{2}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}\exp\{-(2n-1)^2t\}\int_{-\pi/2}^{\pi/2} y_0(\xi)\sin(2n-1)(\xi+\pi/2)d\xi, \tag{7}$$

$$F_j(t) = \int_0^t A(t-\tau)u_j(\tau)d\tau,\ j = 1,2, \tag{8}$$

the kernel $A(t)$ is determined by the formula:

$$A(t) = \frac{2}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}(2n-1)\exp\{-(2n-1)^2t\}. \tag{9}$$

We note that expressions (6) and (9) are called the Dirichlet series with real exponents [5; 111].

We show, that the function $K(t) \in L_1(0,\infty)$, and the functions $F_j(t)$, $j = 0,1,2$, belong to the space $L_2(0,\infty)$. Indeed, we have

$$\int_0^{\infty}|K(t)|dt = \frac{4}{\pi}\sum_{n=1}^{\infty}\int_0^{\infty}\left[\frac{\exp\{-(4n-3)^2t\}}{(4n-3)} - \frac{\exp\{-(4n-1)^2t\}}{(4n-1)}\right]dt < +\infty$$

according to the formula 0.234.4 from [6; 9]: $\sum\limits_{n=1}^{\infty}(-1)^{n-1}/(2n-1)^3 = \pi^3/32$.

We use the Cauchy inequality [7; 28]:

$$\left(\sum_{n=1}^{\infty} a_n b_n\right)^2 \leq \sum_{n=1}^{\infty} a_n^2 \sum_{n=1}^{\infty} b_n^2,$$

where

$$a_n = \frac{2}{\pi}\int\limits_{-\pi/2}^{\pi/2} y_0(\xi)\sin(2n-1)(\xi+\pi/2)\,d\xi, \quad b_n = \exp\{-(2n-1)^2 t\},$$

and also the equality 0.234.2 from [6; 9]: $\sum\limits_{n=1}^{\infty} 1/(2n-1)^2 = \pi^2/8$,

$$\int\limits_0^{\infty} |F_0(t)|^2 dt \leq C_0^2 \cdot \frac{4}{\pi^2}\sum_{n=1}^{\infty}\int\limits_0^{\infty} \exp\{-2(2n-1)^2 t\} dt \leq \frac{\|y_0\|_0^2}{4} < +\infty,$$

where

$$C_0^2 = \sum_{n=1}^{\infty}\left[\int\limits_{-\pi/2}^{\pi/2} y_0(\xi)\sin(2n-1)(\xi+\pi/2)\,d\xi\right]^2 \leq \|y_0\|_0^2.$$

Further, the functions $F_1,\ F_2$ are square-summable on the positive semiaxis, if the absolute value of functional series (9) is integrable. To prove the latter we rewrite series (9) as the sum of the differences:

$$A(t) = \frac{2}{\pi}\sum_{n=1}^{\infty}\left[(4n-3)\exp\{-(4n-3)^2 t\} - (4n-1)\exp\{-(4n-1)^2 t\}\right].$$

We note that each of these differences

$$(4n-3)\exp\{-(4n-3)^2 t\} - (4n-1)\exp\{-(4n-1)^2 t\} \tag{10}$$

represents a alternating function of the variable t, which changes sign once from negative to positive at the point $t_n = [8(2n-1)]^{-1}\ln(4n-1)/(4n-3)$, and it is evident that $t_1 > t_2 > ... > t_n > ...,\ t_n \to 0+$ at $n \to \infty$.

So, the integral of the absolute value of each difference (10) on the semiaxis is equal to:

$$I_n = \int\limits_0^{\infty}\left|(4n-3)\exp\{-(4n-3)^2 t\} - (4n-1)\exp\{-(4n-1)^2 t\}\right| dt =$$

$$= 2\left[\frac{1}{4n-3}\exp\{-(4n-3)^2 t_n\} - \frac{1}{4n-1}\exp\{-(4n-1)^2 t_n\}\right] - \left[\frac{1}{4n-3} - \frac{1}{4n-1}\right]. \tag{11}$$

Note that, by simple analysis to the maximum, we have that the following equality holds:

$$\frac{1}{4n-3}\exp\{-(4n-3)^2 t_n\} - \frac{1}{4n-1}\exp\{-(4n-1)^2 t_n\} =$$

$$= \max_{0\leq t\leq\infty}\left[\frac{1}{4n-3}\exp\{-(4n-3)^2 t\} - \frac{1}{4n-1}\exp\{-(4n-1)^2 t\right].$$

We take the sum of the right hand side of (11) from 1 to ∞ and multiply the result by $2/\pi$ (see formula (9). As a result, taking into account the well-known equality

$$\frac{\pi}{4} = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{2n-1},$$

we have

$$\int_0^{\infty} |A(t)|\,dt \leq \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n-1}\exp\{-(2n-1)^2\bar{t}_n\} - \frac{1}{2},$$

where for all odd $n : \bar{t}_n = \bar{t}_{n+1} = t_n$. The series on the right side of the last inequality converges by Leibnitz Theorem for alternating series [8; 302]. A positivity of the right side follows from relation (7).

Now it remains to use the convolution property (8) to obtain the desired properties of functions $F_j(t) \in L_2(0,\infty)$ [9; 9].

The existence and uniqueness of strong solution. Assume that functions $y_0(x)$, $u_j(t)$, $j = 1,2$, satisfy the conditions of Theorem. And assume that problem (1)–(2) has two distinct solutions $y_1(x,t)$ and $y_2(x,t)$. Then the difference $\tilde{y}(x,t) = y_1(x,t) - y_2(x,t)$ is the solution of the following homogeneous boundary value problem:

$$\begin{cases} \tilde{y}_t(x,t) - \tilde{y}_{xx}(x,t) + \alpha\tilde{y}(0,t) = 0, \;\; x,t \in Q; \\ \tilde{y}(x,0) = 0, \;\; \tilde{y}(-\pi/2,t) = \tilde{y}(\pi/2,t) = 0. \end{cases} \tag{12}$$

By taking a inner product of (12) with $\tilde{y}(x,t)$ in $L_2(-\pi/2,\pi/2)$, we have

$$\frac{1}{2}\frac{d}{dt}\|\tilde{y}(x,t)\|_0^2 + \|\tilde{y}_x(x,t)\|_0^2 \leq |\alpha|\sqrt{\pi}|\tilde{y}(0,t)| \cdot \|\tilde{y}(x,t)\|_0. \tag{13}$$

Here and further we denote by $\|\cdot\|_0$ and $(\cdot,\cdot)_0$ the norm and the inner product in $L_2(-\pi/2,\pi/2)$, respectively. Then, using the Friedrichs', Hölder's, Cauchy inequalities to the right side of (13), we have

$$\frac{d}{dt}\|\tilde{y}(x,t)\|_0^2 + \|\tilde{y}_x(x,t)\|_0^2 \leq |\alpha|^2\pi^2\|\tilde{y}(x,t)\|_0^2.$$

Hence, by Gronwall inequality [10], we have

$$\tilde{y}(x,t) \equiv 0 \in L_\infty((0,\infty); L_2(-\pi/2,\pi/2)) \cap L_2((0,\infty); \overset{\circ}{W}{}_2^1(-\pi/2,\pi/2)),$$

i.e., the boundary value problem (1)–(2) has no more than one solution.

Hence it follows that the integral equation (5) has no more than one solution. Otherwise, if the integral equation (5) has more than one solution, the boundary value problem (1)–(2) according to relation (4) would also have more than one solution, which is impossible, as we have just proved. This means that integral equation (5) in the class $L_2(0,\infty)$ can have only one solution. The uniqueness is proved.

The foregoing proof of the uniqueness without changes holds for the homogeneous boundary value problem adjoint to (12):

$$\begin{cases} -\tilde{p}_t(x,t) - \tilde{p}_{xx}(x,t) + \overline{\alpha}\cdot\delta(x) \otimes \int\limits_{-\pi/2}^{\pi/2} \tilde{p}(\xi,t)d\xi = 0, \;\; \{x,t\} \in Q; \\ \tilde{p}(x,\infty) = 0, \;\; \tilde{p}(-\pi/2,t) = \tilde{p}(\pi/2,t) = 0. \end{cases} \tag{14}$$

We transform the boundary value problem (14) to the following loaded integral equation

$$\tilde{p}(x,t) = -\overline{\alpha}\int_t^{\infty} G(0,x,\tau-t)\int_{-\pi/2}^{\pi/2}\tilde{p}(\xi,t)d\xi\,d\tau, \tag{15}$$

where the Green function G has the form:

$$G(\xi,x,t) = \frac{2}{\pi}\sum_{n=1}^{\infty}\sin n(\xi+\pi/2)\sin n(x+\pi/2)\exp\{-n^2t\}.$$

Implies that (15) integral equation the corresponding to the boundary-value problem (14) adjoint to the equation (5)

$$\nu(t) + \overline{\alpha}\int_t^{\infty} K(\tau-t)\nu(\tau)d\tau = 0, \;\; t > 0, \;\; \text{where} \;\; \nu(t) = \int_{-\pi/2}^{\pi/2}\tilde{p}(\xi,t)d\xi \tag{16}$$

and the kernel of the integral operator (in accordance with formula (6) has the form

$$K(t) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n-1}\exp\{-(2n-1)^2 t\}.$$

Since the uniqueness holds for the adjoint boundary value problem (14), then integral equation (16) in $L_2(0,\infty)$ can have only the trivial solution.

So, according to the theory of integral equations, in $L_2(0,\infty)$ integral equation (5) has a unique solution for all $F(t) \in L_2(0,\infty)$. Consequently, it follows the existence of a unique strong solution of boundary value problem (1)–(2). It remains to show that under the conditions of Theorem 1 the solution of problem (1)–(2), represented by formula (4) belongs to the class $L_2(Q)$.

We write a detailed expression for solution (4):

$$y(x,t) = \frac{2}{\pi}\sum_{n=1}^{\infty}\int\limits_{-\pi/2}^{\pi/2} y_0(\xi)\sin n(\xi+\pi/2)d\xi\cdot\sin n(x+\pi/2)\exp\{-n^2t\}-$$

$$-\frac{4\alpha}{\pi}\sum_{n=1}^{\infty}\frac{\sin(2n-1)(x+\pi/2)}{2n-1}\int\limits_0^t \exp\{-(2n-1)^2(t-\tau)\}y(0,\tau)\,d\tau+$$

$$+\frac{2}{\pi}\sum_{n=1}^{\infty} n\cdot\sin n(x+\pi/2)\int\limits_0^t \exp\{-n^2(t-\tau)\}u_1(\tau)\,d\tau+$$

$$+\frac{2}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}n\cdot\sin n(x+\pi/2)\int\limits_0^t \exp\{-n^2(t-\tau)\}u_2(\tau)\,d\tau \equiv \sum_{j=1}^{4} y_j(x,t). \tag{17}$$

Hence the required property of the solution follows. Indeed, the first summand is estimated as follows (using Cauchy inequality):

$$|y_1(x,t)| = \frac{2}{\pi}\left|\sum_{n=1}^{\infty}a_nb_n\right| \le \frac{2}{\pi}\left(\sum_{n=1}^{\infty}a_n^2\right)^{1/2}\left(\sum_{n=1}^{\infty}b_n^2\right)^{1/2},$$

where

$$a_n = \int\limits_{-\pi/2}^{\pi/2} y_0(\xi)\cdot\sin n(\xi+\pi/2)\,d\xi,\quad \sum_{n=1}^{\infty}|a_n|^2 \le \|y_0(x)\|_0^2 < \infty,$$

$$b_n(x,t) = \sin n(x+\pi/2)\exp\{-n^2t\},\quad \sum_{n=1}^{\infty}\|b_n(x,t)\|^2_{L_2(Q)} = \frac{\pi}{4}\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^3}{24}.$$

Thus, we obtain:

$$\|y_1(x,t)\|_{L_2(Q)} < \infty, \text{i.e.} y_1(x,t) \in L_2(Q).$$

For the second summand $y_2(x,t)$ we take:

$$a_n = \frac{\sin(2n-1)(x+\pi/2)}{2n-1},\quad b_n = \int\limits_0^t \exp\{-(2n-1)^2(t-\tau)\}y(0,\tau)\,d\tau.$$

Taking into account the recent notations and applying the Cauchy inequality as in the case of the first summand, as a result of simple calculations, we obtain:

$$\|y_2(x,t)\|_{L_2(Q)} < \infty,\ \text{ i.e. }\ y_2(x,t) \in L_2(Q).$$

We estimate the third summand. For this we rewrite it in the form:

$$y_3(x,t) = \frac{2}{\pi}\sum_{n=1}^{\infty} \sin n(x+\pi/2) \int_0^t n \exp\{-n^2(t-\tau)\} u_1(\tau)\, d\tau =$$

$$= \frac{2}{\pi}\sum_{m=1}^{\infty} \sin 2mx \cdot S_{1m}(t) + \frac{2}{\pi}\sum_{m=1}^{\infty} \cos{(2m-1)x} \cdot S_{2m}(t), \tag{18}$$

where

$$S_{1m}(t) = \int_0^t (-1)^m 2m \exp\{-4m^2(t-\tau)\} u_1(\tau)\, d\tau,$$

$$S_{2m}(t) = \int_0^t (-1)^{m-1}(2m-1) \exp\{-(2m-1)^2(t-\tau)\} u_1(\tau)\, d\tau.$$

Further, for the first summand of the right part of (18) we have:

$$\sum_{m=1}^{\infty} S_{1m}(t) = \sum_{n=1}^{\infty} \int_0^t \big[-2(2n-1)\exp\{-4(2n-1)^2(t-\tau)\} +$$

$$+ 4n \exp\{-16n^2(t-\tau)\}\big]\, u_1(\tau)\, d\tau = \sum_{n=1}^{\infty} S_{1n}^0(t). \tag{19}$$

We note that in the last representation each summand in the form of the integrand function enclosed in square brackets changes sign from positive to negative only once, and the point of changing the sign is determined by the formula:

$$t_n = \frac{1}{4(4n-1)} \ln \frac{2n}{2n-1}, \;\; t_1 > t_2 > \ldots > t_n \to 0 \;\text{ at }\; n \to \infty.$$

We estimate the norm of the first sum in (18) taking into account (19):

$$\frac{2}{\pi}\left\| \sum_{m=1}^{\infty} \sin 2mx \cdot S_{1m}(t) \right\|_{L_2(Q)} \leq \sqrt{\frac{2}{\pi}} \sum_{n=1}^{\infty} \|S_{1n}^0(t)\|_{L_2(0,\infty)}. \tag{20}$$

We now estimate each summand represented as a convolution $S_{1n}^0(t)$:

$$\|S_{1n}^0(t)\|^2_{L_2(0,\infty)} \leq \int_0^{\infty} \bigg| \int_0^t \big[-2(2n-1)\exp\{-4(2n-1)^2(t-\tau)\} \; +$$

$$+ \; 4n \exp\{-16n^2(t-\tau)\}\big]\, u_1(\tau)\, d\tau \bigg|^2 \, dt \leq$$

$$\leq \|u_1(t)\|^2_{L_2(0,\infty)} \cdot \left| \int_0^{\infty} \big|-2(2n-1)\exp\{-4(2n-1)^2 t\} \;+\; 4n\exp\{-16n^2 t\}\big| \, dt \right|^2 .$$

Now we compute the second factor on the right hand side in the last inequality:

$$\int_0^{\infty} \big|-2(2n-1)\exp\{-4(2n-1)^2(t-\tau)\} + 4n\exp\{-16n^2(t-\tau)\}\big| \, dt =$$

$$= \int\limits_0^{t_n} \left[-2(2n-1)\exp\{-4(2n-1)^2(t-\tau)\} + 4n\exp\{-16n^2(t-\tau)\}\right] dt +$$

$$+ \int\limits_{t_n}^{\infty} \left[2(2n-1)\exp\{-4(2n-1)^2(t-\tau)\} - 4n\exp\{-16n^2(t-\tau)\}\right] dt =$$

$$= \left[\frac{\exp\{-4(2n-1)^2 t_n\}}{2n-1} - \frac{\exp\{-16n^2 t_n\}}{2n}\right] - \left[\frac{1}{2(2n-1)} - \frac{1}{4n}\right] > 0,$$

since a simple research to maximum of the function shows

$$\varphi_n(t) = \frac{\exp\{-4(2n-1)^2 t\}}{2(2n-1)} - \frac{\exp\{-16n^2 t\}}{2\cdot 2n}, \ \varphi_n(t_n) = \max_{0\le t\le\infty}\{\varphi_n(t)\}.$$

Taking into account the calculations for the right hand side of (20), we obtain:

$$\sqrt{2/\pi}\sum_{n=1}^{\infty}\|S_{1n}^0(t)\|_{L_2(0,\infty)} \le \sqrt{2/\pi}\|u_1(t)\|_{L_2(0,\infty)}\times$$

$$\times\left\{\sum_{n=1}^{\infty}\left[\frac{\exp\{-4(2n-1)^2 t_n\}}{2n-1} - \frac{\exp\{-16n^2 t_n\}}{2n}\right]\right. < \infty,$$

since the series on the right hand side converge (by the Leibnitz theorem as alternating series).

Analogous calculations (carried out for the first summand of the right part of (18) are valid for the second summand of the right part of (18). Thus, we obtain the desired estimate for the third summand in (17) $\|y_3(t)\|_{L_2(Q)} < \infty$.

Now we show, that $y_4(x,t)$ in (17) belongs to $L_2(Q)$. We have:

$$y_4(x,t) = \frac{2}{\pi}\sum_{n=1}^{\infty}\sin n(x+\pi/2)\cdot S_n(t),$$

where

$$S_n(t) = \int\limits_0^t (-1)^{n-1} n\exp\{-n^2(t-\tau)\}u_2(\tau)\,d\tau.$$

Hence we obtain the following estimate:

$$\|y_4(x,t)\|_{L_2(Q)} \le \sqrt{2/\pi}\sum_{n=1}^{\infty}\|S_n(t)\|_{L_2(0,\infty)}.$$

To obtain an estimate for the convolution $S_n(t)$ we rewrite it in the form:

$$\sum_{n=1}^{\infty} S_n(t) = \sum_{n=1}^{\infty} S_n^0(t),$$

where

$$S_n^0(t) = \int\limits_0^t \left[(2n-1)\exp\{-(2n-1)^2(t-\tau)\} - 2n\exp\{-4n^2(t-\tau)\}\right] u_2(\tau)\,d\tau.$$

By arguing as in the estimate of $y_3(x,t)$, we obtain the required estimate $\|y_4(t)\|_{L_2(Q)} < \infty$.

Thus, the first assertion of Theorem is completely proved.

The next section is devoted to the proof of the second assertion of Theorem, that is to the establishment of additional differential properties of the solution of problem (1)–(2).

On additional smoothness of the solution. We note that according to the theorem on traces [11; 32–33, 265–269] for the given functions $u_j(t) \in L_2(0,\infty),\ j=1,2$, there exists a function $w(x,t) \in W(0,\infty)$, where

$$W(0,\infty) = \{v|\, v \in L_2(0,\infty; W_2^1(-\pi/2,\pi/2)), v_t \in L_2(0,\infty; W_2^{-1}(-\pi/2,\pi/2))\},$$

such that

$$w(-\pi/2,t) = u_1(t),\ w(\pi/2,t) = u_2(t).$$

The boundary value problem (1)–(2) takes the form:

$$\begin{cases} (y-w)_t(x,t) - (y-w)_{xx}(x,t) + \alpha(y-w)(0,t) = f_1(x,t),\ \{x,t\} \in Q; \\ (y-w)(x,0) = y_1(x),\ (y-w)(-\pi/2,t) = (y-w)(\pi/2,t) = 0, \end{cases} \tag{21}$$

where

$$\begin{cases} f_1 = -w_t(x,t) + w_{xx}(x,t) - \alpha w(0,t) \in L_2(0,\infty; W_2^{-1}(-\pi/2,\pi/2)); \\ y_1 = y_0(x) - w(x,0) \in L_2(-\pi/2,\pi/2). \end{cases} \tag{22}$$

Earlier, in [12] on the basis of a priori estimates established there and the application of Galerkin method it was proven that boundary value problem (21) for any given functions $f_1(x,t)$ and $y_1(x)$, satisfying the conditions (22), has a solution $(y-w)(x,t) \in W(0,\infty)$, namely corresponding to (21) boundary value problem (1)–(2) has a solution $y(x,t) \in W(0,\infty)$.

Hence the second assertion of Theorem follows. Thus, the proof of Theorem is completed.

However, relation (3) requires the choice of boundary controls that would provide the decrease of L_2-average values of the solution not slower than some exponent by time. Fourier method provides this requirement by choice of those exponents $\{\exp\{-\lambda_k t\},\ k \in \mathbf{Z}\}$ in the representation of solution through a series, where numbers λ_k, are defined by positive eigenvalues of the corresponding spectral problem, and which are not less than the exponent of decrease in the exponent of condition (3).

Thus inverse problem (1)–(2) will be solved, if we find a way of constructing the controls $u_j(t),\ j=1,2$, that provides the existence only the exponents of the form $\{\exp\{-\lambda_k t\},\ k \in \mathbf{Z}\}$ (where $\lambda_k \geq \sigma_0$ in (3)), in the presentation for the solution in the form of a series.

The following section of work is devoted to constructing and justifying the algorithm of choice of the desired boundary control functions $u_j(t),\ j=1,2$, in the problem (1)–(2) and its numerical realization.

Solving the problem of stabilization by extension of domain for independent variables. We consider in the domain $Q_1 = \{x,t| -\pi < x < \pi,\ t > 0\}$ the additional problem

$$z_t(x,t) - z_{xx}(x,t) + \alpha \cdot z(0,t) = 0,\ \ \{x,t\} \in Q_1; \tag{23}$$

$$z(-\pi,t) = z(\pi,t),\ z_x(-\pi,t) = z_x(\pi,t),\ z(x,t)|_{t=0} = z_0(x), \tag{24}$$

where $z_0(x)$ is a function that must be defined.

We will seek a solution of problem (23)–(24) in the form

$$z(x,t) = \sum_{k\in\mathbf{Z}} Z_k(t)\varphi_k(x). \tag{25}$$

where $\{\varphi_k(x),\ k \in \mathbf{Z}\}$ is the basis of the space $L_2(-\pi,\pi)$ and $\mathbf{Z} = \{0,\pm1,\pm2,...\}$.

For this, we consider the spectral problem corresponding to problem (23)–(24):

$$-\varphi''(x) + \alpha \cdot \varphi(0) = \lambda\varphi(x); \tag{26}$$

$$\varphi(-\pi) = \varphi(\pi),\ \varphi'(-\pi) = \varphi'(\pi). \tag{27}$$

We introduce the following notation $\mathbf{Z}' = \mathbf{Z}\backslash\{0\}$. For problem (26)–(27) we consider the following two cases.

1^0. *The case when there is no such $k \in \mathbf{Z}$, that $\alpha = k^2$.* The general solution of spectral problem (26)–(27) has the form:

$$\varphi_k(x) = A_k e^{i\sqrt{\lambda_k}x} + D_k \tag{28}$$

and substituting (28) into (26) we find $D_k = \frac{\alpha A_k}{\lambda_k - \alpha}$, here we take $A_k = 1$. Then it is necessary to take $\lambda_k = k^2$, to satisfy conditions (27). Hence we can write the final form of the solution of equation (26)

$$\varphi_k(x) = e^{ikx} + \frac{\alpha}{\lambda_k - \alpha}, \quad k \in \mathbf{Z}'.$$

For $k = 0:\ \varphi_0(x) = \text{const}$, $\lambda_0 = \alpha$; that is for the eigenvalue $\lambda_0 = \alpha$ it is possible to take the eigenfunction $\varphi_0(x) = 1$.

Thus, we have the following system of eigenfunctions and eigenvalues

$$\{\varphi_k(x),\ \lambda_k;\ k \in \mathbf{Z}\} = \left\{1,\ \lambda_0 = \alpha;\ e^{ikx} + \frac{\alpha}{k^2 - \alpha},\ \lambda_k = k^2,\ k \in \mathbf{Z}'\right\}. \tag{29}$$

We note that the obtained system of eigenfunctions (29) is complete in the space $L_2(-\pi, \pi)$, constitutes a basis, but it is not orthogonal. Completeness of the system of eigenfunctions (29) follows from the known theorem of Paley-Wiener [7; 224–227]. For (29) we will find a biorthogonal sequence in the following form

$$\{\psi_k(x),\ k \in \mathbf{Z}\} = \left\{f_0(x),\ e^{ikx},\ k \in \mathbf{Z}'\right\},$$

where $f_0(x)$ is unknown function.

Using basis (29) we will seek the unknown function $f_0(x)$ in the form:

$$f_0(x) = C_0 + \sum_{n \in \mathbf{Z}'} C_n \left(e^{inx} + \frac{\alpha}{n^2 - \alpha}\right),$$

from orthogonality conditions:

$$(1,\ f_0(x)) = 1; \quad \left(e^{ikx} + \frac{\alpha}{k^2 - \alpha},\ f_0(x)\right) = 0,\ \ k \in \mathbf{Z}'.$$

From these conditions, we have:

$$(1,\ f_0(x)) = \int_{-\pi}^{\pi} \left[C_0 + \sum_{n \in \mathbf{Z}'} C_n \left(e^{inx} + \frac{\alpha}{n^2 - \alpha}\right)\right] dx = 2\pi \cdot \left[C_0 + \sum_{n \in \mathbf{Z}'} C_n \cdot \frac{\alpha}{n^2 - \alpha}\right] = 1.$$

Hence it follows $C_0 = \frac{1}{2\pi} - \sum_{n \in \mathbf{Z}'} C_n \cdot \frac{\alpha}{n^2 - \alpha}$. Further

$$\left(e^{ikx} + \frac{\alpha}{k^2 - \alpha},\ C_0 + \sum_{n \in \mathbf{Z}'} C_n \left(e^{inx} + \frac{\alpha}{n^2 - \alpha}\right)\right) = 0,\ \ k \in \mathbf{Z}';$$

$$C_0 \cdot \frac{\alpha}{k^2 - \alpha} + C_k + \frac{\alpha}{k^2 - \alpha} \cdot \left(\frac{1}{2\pi} - C_0\right) = 0,\ \ k \in \mathbf{Z}'.$$

Here we find $C_k:\ C_k = -\frac{1}{2\pi} \cdot \frac{\alpha}{k^2 - \alpha},\ k \in \mathbf{Z}'$. Using the values C_k we rewrite C_0 :

$$C_0 = \frac{1}{2\pi} \cdot \left[1 + \sum_{n \in \mathbf{Z}'} \left(\frac{\alpha}{n^2 - \alpha}\right)^2\right].$$

Further, using the value C_0 we write the desired function f_0:

$$f_0(x) = \frac{1}{2\pi} \cdot \left[1 - \sum_{n \in \mathbf{Z}'} \frac{\alpha}{n^2 - \alpha} \cdot e^{inx}\right] = -\frac{1}{2\pi} \cdot \sum_{n \in \mathbf{Z}} \frac{\alpha}{n^2 - \alpha} \cdot e^{inx}.$$

Therefore, for basis (29) a biorthogonal sequence is the following sequence:

$$\{\psi_k(x),\ k \in \mathbf{Z}\} = \left\{-\frac{1}{2\pi} \sum_{n \in \mathbf{Z}} \frac{\alpha}{n^2 - \alpha} \cdot e^{inx},\ e^{ikx},\ k \in \mathbf{Z}'\right\}, \tag{30}$$

which defines in the space $L_2(-\pi, \pi)$ a biorthogonal basis.

2^0. *The case when there exists such number $k \in \mathbf{Z}$, that $\alpha = k^2$.* Let us k_0 be such number, namely $\alpha = k_0^2$. The general solution of spectral problem (26)–(27) has the form:

$$\varphi_k(x) = A_k e^{i\sqrt{\lambda_k}x} + D_k, \tag{31}$$

and substituting (31) into (26) we find $D_k = \frac{\alpha A_k}{\lambda_k - \alpha}$, here we take $A_k = 1$. Then it is necessary to take $\lambda_k = k^2$, to satisfy conditions (27). Hence we write the final form of the solution of equation (26)

$$\varphi_k(x) = e^{ikx} + \frac{\alpha}{\lambda_k - \alpha}; \quad k \in \mathbf{Z}'\backslash\{\pm k_0\}.$$

For $k_0 : \varphi_{k_0}(x) = \text{const}$, $\lambda_{k_0} = \alpha = k_0^2$; that is for the eigenvalue $\lambda_{k_0} = \alpha = k_0^2$ it is possible to take the eigenfunction $\varphi_{k_0}(x) = 1$. Further, the system of eigenfunctions and eigenvalues are complete, if we find the associated functions that satisfy the following conditions

$$-\overline{\varphi}''_{k_0}(x) + \alpha \cdot \overline{\varphi}_{k_0}(0) - k_0^2\overline{\varphi}_{k_0}(x) = k_0^2; \tag{32}$$

$$\overline{\varphi}_{k_0}(-\pi) = \overline{\varphi}_{k_0}(\pi),\ \overline{\varphi}'_{k_0}(-\pi) = \overline{\varphi}'_{k_0}(\pi). \tag{33}$$

The general solution of spectral problem (32)–(33) has the form

$$\overline{\varphi}_{k_0}(x) = \mathrm{C} + A_1 e^{ik_0x} + A_2 e^{-ik_0x}. \tag{34}$$

Substituting the general solution (34) into (32) we find $\alpha(A_1 + A_2) = k_0^2$, here we take $A_1 + A_2 = 1$. The associated functions are $\{e^{\pm ik_0x}\}$.

Thus, we have the eigenvalues and the corresponding eigenfunctions

$$\{\varphi_k(x),\ \lambda_k;\ k \in \mathbf{Z}\backslash\{\pm k_0\}\} =$$

$$= \left\{1,\ \lambda_0 = k_0^2;\ e^{ikx} + \frac{\alpha}{k^2 - \alpha},\ \lambda_k = k^2,\ k \in \mathbf{Z}'\backslash\{\pm k_0\}\right\} \tag{35}$$

and the associated functions

$$\{\varphi_{\pm k_0}(x),\ \lambda_0\} = \left\{e^{\pm ik_0x},\ \lambda_0 = k_0^2 = \alpha\right\}. \tag{36}$$

Here, the constant is an eigenfunction corresponding to the eigenvalue $\lambda_0 = k_0^2 = \alpha$. Furthermore, we note that zero is not an eigenvalue. In this case, the system of eigenfunctions is not complete and not orthogonal in the space $L_2(-\pi, \pi)$.

Combining (35) and (36), we obtain the complete system [7; 224–227]:

$$\{\varphi_k(x),\ \lambda_k;\ k \in \mathbf{Z}\} = \Big\{1,\ \lambda_0 = k_0^2;\ e^{\pm ik_0x},\ \lambda_0 = k_0^2 = \alpha;$$

$$e^{ikx} + \frac{\alpha}{k^2 - \alpha},\ \lambda_k = k^2,\ k \in \mathbf{Z}'\backslash\{\pm k_0\}\Big\}. \tag{37}$$

For (37) the biorthogonal sequence is

$$\{\psi_k(x);\ k \in \mathbf{Z}\} = \left\{f_0(x),\ e^{ikx},\ k \in \mathbf{Z}'\right\},$$

where it is necessary to find unknown function $f_0(x)$ by the following way:

$$f_0(x) = C_0 + \sum_{n \in \mathbf{Z}'\backslash\{\pm k_0\}} C_n \left(e^{inx} + \frac{\alpha}{n^2 - \alpha}\right) + C_{k_0}e^{ik_0x} + C_{-k_0}e^{-ik_0x},$$

from orthogonality conditions

$$(1, f_0(x)) = 1;\ \left(e^{ikx} + \frac{\alpha}{k^2 - \alpha}, f_0(x)\right) = 0,\ \ k \in \mathbf{Z}'\backslash\{\pm k_0\};$$

$$(e^{\pm ik_0x}, f_0(x)) = 0.$$

From these conditions we find:

$$(1, f_0(x)) = 2\pi \cdot \left[C_0 + \sum_{n\in\mathbf{Z}'\backslash\{\pm k_0\}} C_n \cdot \frac{\alpha}{n^2-\alpha} \right] = 1,$$

$$C_0 = \frac{1}{2\pi} - \sum_{n\in\mathbf{Z}'\backslash\{\pm k_0\}} C_n \cdot \frac{\alpha}{n^2-\alpha}.$$

Further

$$\left(e^{ikx} + \frac{\alpha}{k^2-\alpha}, f_0(x)\right) = \left(e^{ikx} + \frac{\alpha}{k^2-\alpha},\ C_0 + \sum_{n\in\mathbf{Z}'\backslash\{\pm k_0\}} C_n \left(e^{inx} + \frac{\alpha}{n^2-\alpha}\right)\right) = 0;$$

$$k \in \mathbf{Z}'\backslash\{\pm k_0\}.$$

Hence it follows

$$C_0 \cdot \frac{\alpha}{k^2-\alpha} + C_k + \frac{\alpha}{k^2-\alpha} \cdot \left(\frac{1}{2\pi} - C_0\right) = 0,\ \ k \in \mathbf{Z}'\backslash\{\pm k_0\}.$$

Here we find C_k:

$$C_k = -\frac{1}{2\pi} \cdot \frac{\alpha}{k^2-\alpha},\ \ k \in \mathbf{Z}'\backslash\{\pm k_0\}.$$

Using the values C_k we rewrite C_0:

$$C_0 = \frac{1}{2\pi} \cdot \left[1 + \sum_{n\in\mathbf{Z}'\backslash\{\pm k_0\}} \left(\frac{\alpha}{n^2-\alpha}\right)^2\right].$$

Next, using the values C_0 we write the desired function f_0:

$$f_0(x) = -\frac{1}{2\pi} \sum_{n\in\mathbf{Z}} \frac{\alpha}{n^2-\alpha} \cdot e^{inx}.$$

So for (37) the biorthogonal system is

$$\{\psi_k(x),\ k \in \mathbf{Z}\} = \left\{-\frac{1}{2\pi} \sum_{n\in\mathbf{Z}} \frac{\alpha}{n^2-\alpha} \cdot e^{inx},\ \ e^{ikx},\ \ k \in \mathbf{Z}'\right\}. \tag{38}$$

This system also defines a biorthogonal basis in the space $L_2(-\pi, \pi)$.

To determine the Fourier coefficients of expansion (25) we have Cauchy problem:

$$Z_k'(t) + \lambda_k Z_k(t) = 0,\ \ Z_k(0) = z_{0k},\ \ k \in \mathbf{Z}, \tag{39}$$

where z_{0k} are the expansion coefficients of the function $z_0(x)$ on system $\{\varphi_k(x)\}$.

The solution of Cauchy problem (39) has the form: $Z_k(t) = z_{0k}e^{-\lambda_k t},\ k \in \mathbf{Z}$.

We will further assume that in the space $L_2(-\pi, \pi)$ we have:

– basis $\{\varphi_k(x),\ k \in \mathbf{Z}\}$, composed of the system of eigenfunctions (29) or of the system of eigenfunctions and associated functions (37);

– and the corresponding biorthogonal basis $\{\psi_k(x),\ k \in \mathbf{Z}\}$, (30) or (38).

Then the solution of original initial-boundary value problem (23)–(24) can be written in form (25):

$$z(x,t) = z_{00}e^{-\alpha t}\varphi_0(x) + \sum_{k\in\mathbf{Z}'} z_{0k}e^{-k^2 t}\varphi_k(x), \tag{40}$$

where

$$z_{0k} = \int_{-\pi}^{\pi} \overline{\psi_k(x)} z_0(x)dx,\ \ k \in \mathbf{Z},$$

are Fourier coefficients $z_0(x)$, where $\psi_k(x),\ k\in\mathbf{Z}$, are defined by the formulas (30) and (38). From (39) and (40) it follows directly that if

$$z_{0k}=0 \ \text{ at } \ k^2<\sigma_0 \tag{41}$$

and

$$z_{00}\neq 0 \ \text{ at } \ \operatorname{Re}\alpha\geq\sigma_0;\ \ z_{00}=0 \ \text{ at } \ \operatorname{Re}\alpha<\sigma_0, \tag{42}$$

then solution (40) of problem (23)–(24) will satisfy the inequality

$$\|z(x,t)\|_{L_2(-\pi,\pi)}\leq Ce^{-\sigma_0 t}.$$

We denote by $\mathbf{Z}_0$ ($\mathbf{Z}_0\subset\mathbf{Z}$) the set of indices k that satisfy conditions (41) and (42).

Now, with the restriction operator $\zeta_{-\pi/2}$ and $\zeta_{\pi/2}$ we find the desired controls

$$u_1(t)=\zeta_{-\pi/2}\{z(x,t)\},\ \ u_2(t)=\zeta_{\pi/2}\{z(x,t)\}.$$

It remains to construct an extension operator of the function $y_0(x)$ up to the function $z_0(x)$, defined on the interval $(-\pi,\pi)$,

$$E:\ L_2(-\pi/2,\pi/2)\to L_2(-\pi,\pi),\ \text{i.e.}\ (\zeta_{(-\pi/2,\pi/2)}Ey_0)(x)\equiv y_0(x), \tag{43}$$

so that the Fourier coefficients z_{0k} of function $z_0=Ey_0$ (43) would satisfy conditions (41) and (42). Here we use the notation $\zeta_{(-\pi/2,\pi/2)}$ for the restriction operator $\zeta_{(-\pi/2,\pi/2)}:\ L_2(-\pi,\pi)\to L_2(-\pi/2,\pi/2)$.

By arguing as in the lemma of [13] we obtain the following lemma.

Lemma. For each $\sigma_0>0$ there exists a continuous extension operator E in (43), that for all $y_0(x)\in$ $\in L_2(-\pi/2,\pi/2)$ equality holds

$$\int\limits_{-\pi}^{\pi}\overline{\psi_k(x)}(Ey_0)(x)dx=0,\ \forall\, k\in\mathbf{Z}_0:\ \ |k|<\sqrt{\sigma_0}. \tag{44}$$

Proof. We define the operator E (43) by the formula

$$Ey_0(x)=\begin{cases} y_0(x), & x\in(-\pi/2,\pi/2);\\ z_1(x), & x\in(-\pi,-\pi/2)\cup(\pi/2,\pi),\end{cases}$$

where the function $z_1(x)$ to be determined. By virtue (44) $z_1(x)$ must satisfy the system of equations:

$$\int\limits_{(-\pi,-\pi/2)\cup(\pi/2,\pi)}\overline{\psi_k(x)}\cdot z_1(x)dx=-\int\limits_{(-\pi/2,\pi/2)}\overline{\psi_k(x)}\cdot y_0(x)dx\equiv-\widehat{y}_0(k),\ \ k\in\mathbf{Z}_0. \tag{45}$$

We seek the function $z_1(x)$ in the form:

$$z_1(x)=\sum_{j\in\mathbf{Z}_0}\widehat{z}_1(j)\psi_j(x). \tag{46}$$

Substituting (46) into (45), we obtain a system of equations to determine $\widehat{z}_1(j)$:

$$\sum_{j\in\mathbf{Z}_0}a_{kj}\widehat{z}_1(j)=-\widehat{y}_0(k),\ \ k\in\mathbf{Z}_0, \tag{47}$$

where $\widehat{y}_0(k)$ is defined in (45), and the coefficients a_{kj} are determined by relations:

$$a_{kj}=\int\limits_{(-\pi,-\pi/2)\cup(\pi/2,\pi)}\overline{\psi_k(x)}\cdot\psi_j(x)dx,\ \ k,\,j\in\mathbf{Z}_0. \tag{48}$$

The matrix $A=\|a_{kj}\|$ is positive. Indeed, if

$$\Psi=\{\widehat{\psi}_k,\ k\in\mathbf{Z}_0\}\ \text{and}\ \psi=\sum_{k\in\mathbf{Z}_0}\widehat{\psi}_k\cdot\psi_k(x),$$

then by virtue of (48)

$$(A\Psi,\Psi)=\sum_{k,j\in\mathbf{Z}_0}a_{kj}\cdot\widehat{\psi}_j\cdot\overline{\widehat{\psi}}_k=\sum_{k,j\in\mathbf{Z}_0}\int\limits_{(-\pi,-\pi/2)\cup(\pi/2,\pi)}\psi_j(x)\cdot\overline{\psi_k(x)}dx\cdot\widehat{\psi}_j\cdot\overline{\widehat{\psi}}_k=$$

$$=\int\limits_{(-\pi,-\pi/2)\cup(\pi/2,\pi)}\sum_{j\in\mathbf{Z}_0}\psi_j(x)\cdot\widehat{\psi}_j\cdot\sum_{k\in\mathbf{Z}_0}\overline{\widehat{\psi}}_k\cdot\overline{\psi_k(x)}dx=$$

$$=\int\limits_{(-\pi,-\pi/2)\cup(\pi/2,\pi)}\overline{\psi(x)}\cdot\psi(x)dx=\int\limits_{(-\pi,-\pi/2)\cup(\pi/2,\pi)}|\psi(x)|^2dx\geq 0. \tag{49}$$

If for some Ψ in (49) the equality holds, then

$$\psi(x)=\sum_{k\in\mathbf{Z}_0}\widehat{\psi}_k\psi_k(x)\equiv 0 \text{ and hence } \widehat{\psi}_k=0,\ \forall\, k\in\mathbf{Z}_0.$$

Hence $\det\|a_{kj}\|\neq 0$ and therefore system (47) and formula (46) uniquely determine operator (43), satisfying all the conditions of Lemma.

An algorithm for solving the inverse problem. The results of the preceding sections allow us to implement the following algorithms of approximate constructing the boundary control functions (and even in the form of synthesis, processing their random perturbations), providing monotonic decrease in time, not slower than the given exponent according to formula (4) in $L_2(-\pi/2,\pi/2)$–norm of the solution. The latter is achieved by fulfillment of requirements (41) and (42).

Step 1. According to original boundary value problem (1)–(2) at half-strip of the width π with non-homogeneous Dirichlet boundary conditions and initial condition on the interval $(-\pi/2,\pi/2)$, given by the function $y_0(x)$, auxiliary boundary value problem (23)–(24) is posed on the extended half-strip of the width which is equal to 2π, with periodicity conditions (instead of the Dirichlet conditions) and the initial function $z_0(x)$ on the interval $(-\pi,\pi)$. The function $z_0(x)$ will be defined as the continuation of the given function $y_0(x)$.

Thus, in auxiliary boundary problem (23)–(24) it is necessary to complete the definition of function $z_0(x)$ on the interval $(-\pi,\pi)$, so that for the solutions $z(x,t)$ of problem (23)–(24) requirement (4) would be fulfilled. In this case, condition (4) holds for its restriction $y(x,t)$ and the required boundary controls $u_1(t)$ and $u_2(t)$ will be determined as traces of the function $z(x,t)$ when $x=\pm\pi/2$.

Step 2. Constructing the complete biorthogonal systems of functions on the interval $(-\pi,\pi)$ by solving the corresponding spectral problems.

Step 3. We find the coefficients of the expansion of the required function $z_0(x)$ on the interval $(-\pi,\pi)$ by complete biorthogonal system that constructed in the preceding step, so that conditions (41) and (42) were hold. We note that conditions (41) and (42) provide the fulfillment of requirement (4) to solve auxiliary boundary value problem (23)–(24).

Step 4. According to solution $z(x,t)$ that is obtined of auxiliary boundary value problem (23)–(24) we find the solution $y(x,t)$ of original boundary value problem (1)–(2), satisfying required condition (4). We find the boundary controls $u_1(t)$ and $u_2(t)$ as traces of the solution $z(x,t)$, that is

$$u_1(t)=z(x,t)_{|x=-\pi/2},\quad u_2(t)=z(x,t)_{|x=\pi/2}.$$

The main step of the algorithm is the third. The constructive realizability of step 3 is mathematically justified by Lemma.

Conclusion. In this paper the statement of the inverse problem to stabilize the solution of the loaded heat conduction equation using boundary conditions is given. Theorem on solvability of the stated inverse problem is proved. An algorithm of approximate construction of boundary controls in the form of synthesis is developed. Numerical calculations were carried out, that showed the effectiveness of the proposed algorithm. We note that within this work the load is determined at the point $x=0$. This unessential condition, the results can be easily extended to the case of an arbitrary point in the interval $(-\pi/2,\pi/2)$.

References

1 Нахушев А.М. Уравнения математической биологии / А.М. Нахушев. — М.: Высш. шк., 1995. — 205 с.

2 Нахушев А.М. Задачи со смещением для уравнений в частных производных / А.М. Нахушев. — М.: Наука, 2006. — 288 с.

3 Дженалиев М.Т. Нагруженные уравнения как возмущения дифференциальных уравнений / М.Т. Дженалиев, М.И. Рамазанов. — Алматы: Ғылым, 2010. — 334 с.

4 Кабанихин С.И. Обратные и некорректные задачи / С.И. Кабанихин. — Новосибирск: Сибирское науч. изд-во, 2009. — 457 с.

5 Леонтьев А.Ф. Ряды экспонент / А.Ф. Леонтьев. — М.: Наука, 1976. — 536 с.

6 Hardy G.H. Inequalities / G.H. Hardy, J.E. Littlwood, G.Polia. — London: Cambridge University Press, 1934.

7 Riesz F. Lecons d'Analyse Fonctionnelle / F.Riesz, B.Sz.-Nagy. — Budapest: Akademiai Kiado, 2007

8 Фихтенгольц Г.М. Курс дифференциального и интегрального исчисления — Т. 2. / Г.М. Фихтенгольц. — М.: Физматлит, 2001. — 810 с.

9 Stein E.M. Introduction to Fourier Analysis on Euclidean Spaces / E.M.Stein, G.Weiss. — Princeton: Princeton University Press, 1971.

10 Gradshtein I.S. Table of Integrals, Series, and Products / I.S. Gradshtein, I.M. Ryzhik. — California: Elsevier, 2007. — 7th edn.

11 Lions J-L. Problemes aux Limites Nonhomogenes et Applications / J-L.Lions, E.Majenes. — Paris: Dunod, 1968. — Vol. 1.

12 Дженалиев М.Т. Оптимальное управление линейными нагруженными параболическими уравнениями / М.Т. Дженалиев // Дифференциальные уравнения. — 1989. — Т. 25. — № 4. — С. 641–651.

13 Фурсиков А.В. Стабилизируемость квазилинейного параболического уравнения с помощью граничного управления с обратной связью / А.В. Фурсиков // Математический сборник. — 2001. — Т. 192. — № 4. — С. 115–160.

М.Т. Жиенәлиев, М.М. Аманғалиева, Қ.Б. Иманбердиев, М.И. Рамазанов

Жүктелген жылуөткізгіштік теңдеуі шешімінің стабилизациясы туралы

Жүктелген дифференциалдық теңдеулерді зерттеуге үнемі артып келе жатқан қызығушылық жүктелген теңдеулер нақты есептерге қатысты функционалды-дифференциалдық теңдеулердің арнайы класын қалыптастыруы сынды қосымшалары мен жағдаяттарына байланысты түсіндіріледі. Бұл теңдеулер маңызды қолданбалы тартымдылығы бар дифференциалдық теңдеулердің кері есептерін зерттеуге арналған қосымшаларға ие. Мақалада $\Omega \equiv (-\pi/2, \pi/2)$ шектелген облысында жүктелген жылуөткізгіштік теңдеуі үшін шекара арқылы стабилизациялау есептерінің шешілетіндігі мәселелері зерттелген. Мәселе шекаралық шарттарды (басқаруды) таңдау кезінде алынған аралас шеттік есептің шешімі $t \to \infty$ болғанда берілген $\exp(-\sigma_0 t)$ жылдамдықпен белгілі стационар шешімге ұмтылуында. Сонымен қатар басқару кері байланысты болуы талап етіледі, яғни ол жүйенің күтілмеген флуктуацияларына жауап бере отырып, олардың шешімнің стабилизациясына әсер етуі нәтижелерін басуы керек. Стабилизация есептері басқарымдылық мәселелерімен тікелей байланысты. Авторлар кері байланыс ұғымын математикалық формализациялауды ұсынады және де оның көмегімен жүктелген жылуөткізгіштік теңдеуі шекара аймағында берілген кері байланысты басқару арқылы шешіледі.

Кілт сөздер: стабилизация, жүктелген жылуөткізгіштік теңдеуі, меншікті мән, меншікті функция.

М.Т. Дженалиев, М.М. Амангалиева, К.Б. Иманбердиев, М.И. Рамазанов

О стабилизации решения нагруженного уравнения теплопроводности

Постоянно растущий интерес к изучению нагруженных дифференциальных уравнений объясняется их приложением и тем обстоятельством, что нагруженные уравнения образуют особый класс функционально-дифференциальных уравнений с конкретными задачами. Эти уравнения имеют приложения для изучения обратных задач дифференциальных уравнений с важными прикладными интересами. В статье исследованы вопросы разрешимости задач стабилизации с границей для нагруженного уравнения теплопроводности в заданной ограниченной области $\Omega \equiv (-\pi/2, \pi/2)$. Задача заключается в выборе граничных условий (управлений); решение полученной смешанной краевой задачи стремится к заданному стационарному решению с заданной скоростью $\exp(-\sigma_0 t)$ при $t \to \infty$. При этом требуется, чтобы управление было с обратной связью, т.е. чтобы оно реагировало на непредусмотренные флуктуации системы, подавляя результаты их воздействия на стабилизируемое решение. Задачи стабилизации имеют непосредственную связь с проблемами управляемости. В работе предложена математическая формализация понятия обратной связи, и с его помощью решается задача о стабилизируемости нагруженного уравнения теплопроводности посредством управления с обратной связью, заданного на части границы.

Ключевые слова: стабилизация, управление с обратной связью, нагруженное уравнение теплопроводности, краевая задача, обратная задача, функция Грина, собственное значение, собственная функция.

References

1 Nakhushev, A.M. (1995). *Uravneniia matematicheskoi biolohii [Equations of Mathematical Biology].* Moscow: Vyschaia shkola [in Russian].

2 Nakhushev, A.M. (2006). *Zadachi so smeshcheniem dlia uravnenii v chastnykh proizvodnykh [Problems with Shift for the Partial Differential Equations].* Moscow: Nauka [in Russian].

3 Jenaliyev, (Dzhenaliev) M.T., & Ramazanov, M.I. (2010). Nahruzhennye uravneniia kak vozmushcheniia differentsialnykh uravnenii [Loaded Equations as Perturbations of Differential Equations]. Almaty: Hylym [in Russian].

4 Kabanikhin, S.I. (2009). *Obratnye i nekorektnye zadachi [Inverse and Ill-posed Problems].* Novosibirsk: Sibirskoe nauchnoe izdatelstvo [in Russian].

5 Leont'ev, A.F. (1976). *Riady eksponent [Series of Exponents].* Moscow: Nauka [in Russian].

6 Hardy, G.H., Littlwood, J.E., & Polia, G. (1934). *Inequalities.* London: Cambridge University Press.

7 Riesz, F., & Sz.-Nagy, B. (2007). *Lecons d'Analyse Fonctionnelle.* Budapest: Akademiai Kiado.

8 Fichtengol'z, G.M. (2001). Kurs differentsialnoho i intehralnoho ischisleniia [Differential and Integral Calculus]. (Vol. 2). Moscow: Fizmatlit [in Russian].

9 Stein, E.M., & Weiss G. (1971). *Introduction to Fourier Analysis on Euclidean Spaces.* Princeton: Princeton University Press.

10 Gradshtein, I.S., & Ryzhik, I.M. (2007). *Table of Integrals, Series, and Products.* (7th edn.). California: Elsevier.

11 Lions, J-L., & Majenes, E. (1968). *Problemes aux Limites Nonhomogenes et Applications, Vol. 1.* Paris: Dunod.

12 Jenaliyev (Dzhenaliev), M.T. (1989). Optimalnoe upravlenie lineinymi nahruzhennymi parabolicheskimi uravneniiami [Optimal control of the linear loaded parabolic equations]. *Differentsialnye uravneniia – Differential equations, Vol. 25,* 641–651 [in Russian].

13 Fursikov, A.V. (2001). Stabiliziruemost kvazilineinoho parabolicheskoho uravneniia s pomoshchiu hranichnoho upravleniia s obratnoi sviaziu [Stabiliting of quasilinear parabolic equation by boundary controls with feedback]. *Matematicheskii sbornik – Mathematical collection, Vol. 192,* 4, 115–160 [in Russian].