https://doi.org/10.31489/2025M3/22-33 *Research article*

On the Convergence of the Approximate Solution to the Optimization Problem for Oscillatory Processes

E.F. Abdyldaeva[1,*], A. Kerimbekov[2], T.K. Yuldashev[3], M. Kamali[4]

[1] *Kyrgyz-Turkish Manas University, Bishkek, Kyrgyzstan;*
[2] *Kyrgyz Russian Slavic University, Bishkek, Kyrgyzstan;*
[3] *Tashkent State Transport University, Tashkent, Uzbekistan;*
[4] *Kyrgyz-Turkish Manas University, Bishkek, Kyrgyzstan*
(E-mail: elmira.abdyldaeva@manas.edu.kg, akl7@rambler.ru, tursun.k.yuldashev@gmail.com, muhammet.kamali@manas.edu.kg)

This article addresses the non-linear optimization problem of oscillatory processes governed by partial integro-differential equations involving a Fredholm integral operator. A distinctive feature of the problem is that both the objective functional and the functions describing external and boundary influences are non-linear with respect to the vector controls. The integro-differential equation describing the state of the oscillatory process includes Fredholm integral operator, which has a significant impact on the structure and properties of the solutions. The algorithm for constructing the complete solution to this problem, as well as the effect of the Fredholm integral operator on the solution of the corresponding boundary value problem, has been published in previous studies. This article is dedicated to the investigation of the convergence of approximate solutions to the exact solution of the considered non-linear optimization problem. The influence of the Fredholm integral operator on the convergence behavior of the approximations is examined. It is demonstrated that the presence of the integral operator necessitates the construction of three distinct types of approximations of the optimal process: "Resolvent" approximations, based on the resolvent of the kernel of the integral operator; Approximations by optimal controls, constructed through the approximation of control functions; Finite-dimensional approximations.

Keywords: optimal control, optimal process, minimal value of functional, non-linear optimization problem, approximations of complete solution, resolvent approximation, finite-dimensional approximation, convergence.

2020 Mathematics Subject Classification: 49K20.

Introduction

Optimal control of systems with distributed parameters is one of the intensively developing scientific directions of Optimal control theory. Dynamics of systems with distributed parameters is described by partial differential equations, integral, integro-differential and more complex functional equations. Methods for solving linear optimization problems in programming control of systems with distributed parameters are based on the methods of classical variational calculus, the maximum principle, and they have been developed in studies [1–3]. The mathematical model [4, 5] of many applied problems need to solve non-linear optimization problems, for which methods for solving them are not sufficiently developed [6, 7]. A research group of Kyrgyz mathematicians, led by Professor A. Kerimbekov, is actively investigating the solvability of non-linear optimization problems [8–10] and the convergence of their approximate solutions [11, 12]. The results of the authors' research on solutions to non-linear optimization problems are presented in works [13, 14].

*Corresponding author. *E-mail: elmira.abdyldaeva@manas.edu.kg*
This research was funded by Kyrgyz-Turkish Manas University, project number KTMU-BAP-2025.FB.14.
Received: 28 January 2025; *Accepted:* 4 June 2025.

In the paper [14], we have considered the non-linear optimization problem of oscillation processes described by integro-differential equations in partial derivatives with the integral Fredholm operator and an algorithm was developed for constructing a complete solution to this problem. It is established that the presence of the integral operator significantly affects the solvability of the non-linear optimization problem, in particular, when constructing a generalized solution to the boundary value problem of the controlled process and when proving the existence and uniqueness of a solution to system of non-linear integral equations.

In [14], the problem of non-linear optimization for oscillatory processes described by integro-differential equations with the participation of the Fredholm integral operator was investigated. An algorithm for constructing a complete solution to this problem was developed. It was established that the presence of the Fredholm integral operator has a significant effect on the solvability of the non-linear optimization problem, in particular, on the construction of a generalized solution to the boundary value problem of the controlled process and on the proof of the existence and uniqueness of a solution to a system of non-linear integral equations with respect to optimal controls.

This paper continues the study of the complete solution of the non-linear optimization problem developed in [14], in particular, with the aim of studying the convergence of its approximations. It is shown that the presence of the Fredholm integral operator necessitates constructing three types of approximations of the optimal process: approximation through the resolvent of the kernel of the integral operator, approximation by optimal controls, and finite-dimensional approximation. Accordingly, three types of approximations of the minimum value of the objective functional are also considered. Sufficient conditions are established for the convergence of approximations of both distributed and boundary vector optimal controls, three types of approximations of the optimal process, and approximations of the minimum value of the functional.

1 Formulation of the Non-linear Optimization Problem and Its Complete Solution

Consider the following non-linear optimization problem, where it is required to minimize the quadratic integral functional [14].

$$J[\bar{u}(t,x),\bar{\vartheta}(t,x)] = \int_Q [V(T,x)-\xi_1(x)]^2\,dx + \int_Q [V_t(T,x)-\xi_2(x)]^2\,dx + \\ + \int_0^T \left[\alpha\int_Q h^2[t,x,\bar{u}(t,x)]dx + \beta\int_\gamma b^2[t,x,\bar{\vartheta}(t,x)]dx\right] dt, \quad \alpha,\beta>0, \tag{1}$$

on the set of solutions to the boundary value problem

$$V_{tt}(t,x) - AV(t,x) = \lambda\int_0^T K(t,\tau)V(\tau,x)d\tau + f[t,x,\bar{u}(t,x)], \quad x\in Q\subset\mathbb{R}^n, \quad 0<t<T, \tag{2}$$

$$V(0,x)=\psi_1(x), \quad V_t(0,x)=\psi_2(x), \quad x=(x_1,x_2,\dots,x_n)\in Q, \tag{3}$$

$$\Gamma V(t,x) \equiv \sum_{i,k=1}^n a_{ik}(x)V_{x_k}(t,x)\cos(\delta,x_i) + a(x)V(t,x) = p[t,x,\bar{\vartheta}(t,x)], \quad x\in\gamma, \quad 0<t<T. \tag{4}$$

It should be noted that the characteristics of the data in problem (1)–(4) are preserved as presented in [14]. It is assumed that the functions describing external and boundary influences satisfy the following monotonicity conditions with respect to the functional variables:

$$f_{u_i}[t,x,\bar{u}(t,x)] \neq 0, \quad i=1,2,\dots,m, \quad \forall (t,x)\in H(Q_T), \tag{5}$$

$$p_{\vartheta_i}[t,x,\bar{\vartheta}(t,x)] \neq 0, \quad i=1,2,\ldots,r, \quad \forall (t,x) \in H(\gamma_T).$$

The conditions stated in (5) guarantee a one-to-one correspondence between the elements of the space of controls $(\bar{u}^0(t,x), \bar{\vartheta}^0(t,x))$ and the space of states $V(t,x)$ the controlled process.

The complete solution of nonlinear optimization problem (1)–(4) is defined in the form of a triple $\big((\bar{u}^0(t,x), \bar{\vartheta}^0(t,x)), V^0(t,x), J[\bar{u}^0(t,x), \bar{\vartheta}^0(t,x)]\big)$ [14], where:

1) the distributed vector optimal control $\bar{u}^0(t,x)$ and the boundary vector optimal control $\bar{\vartheta}^0(t,x)$ are determined by the formulas

$$\bar{u}^0(t,x) = \bar{\varphi}[t,x,\theta_1^0(t,x),\alpha], \quad \theta_1^0(t,x) = \lim_{n\to\infty} \theta_1^{(n)}(t,x), \quad x \in Q, \tag{6}$$

$$\bar{\vartheta}^0(t,x) = \bar{v}[t,x,\theta_2^0(t,x),\beta], \quad \theta_2^0(t,x) = \lim_{n\to\infty} \theta_2^{(n)}(t,x), \quad x \in \gamma, \tag{7}$$

where functions $\theta_1^{(n)}(t,x)$ and $\theta_2^{(n)}(t,x)$ are defined as solutions of the operator equation

$$\theta^n(t,x) = F\left[\theta^{n-1}(t,x)\right], \quad n=1,2,3,\ldots,$$

with

$$\theta^{(n)}(t,x) = \begin{cases} \theta_1^{(n)}(t,x), & x \in Q, \\ \theta_2^{(n)}(t,x), & x \in \gamma, \end{cases}$$

and satisfy the estimate

$$\|\theta^{(0)}(t,x) - \theta^{(n)}(t,x)\|_{H(\bar{Q}_T)} \leq \frac{C^n(\alpha,\beta)}{1-C(\alpha,\beta)} \|F\left(\theta_0(t,x)\right) - \theta_0(t,x)\|_{H(\bar{Q}_T)}, \tag{8}$$

where

$$\theta_0(t,x) = \begin{cases} \theta_{10}(t,x), & x \in Q, \\ \theta_{20}(t,x), & x \in \gamma, \end{cases}$$

is an arbitrary vector function in the space $H(\bar{Q}_T)$, and

$$C(\alpha,\beta) = \sqrt{f_0^2 m \varphi_0^2(\alpha) + p_0^2 r v_0^2(\beta)}\sqrt{2} E_0 G_0 T < 1, \tag{9}$$

with constants f_0, p_0, $\varphi_0(\alpha)$, $v_0(\beta)$, E_0, and G_0 defined appropriately.

2) $V^0(t,x)$ is an optimal process, determined by the following formula

$$V^0(t,x) = \sum_{n=1}^{\infty} \Big(\psi_n(t,\lambda) + \frac{1}{\lambda_n}\int_0^T E_n(t,\eta,\lambda)\Big(\int_Q f[\eta,\xi,\bar{u}^0(\eta,\xi)] z_n(\xi) d\xi + \\ + \int_\gamma p[\eta,\xi,\bar{\vartheta}^0(\eta,\xi)] z_n(\xi) d\xi\Big) d\eta\Big) z_n(x), \tag{10}$$

where

$$\psi_n(t,\lambda) = \psi_{1n}\left[\cos\lambda_n t + \lambda\int_0^T R_n(t,s,\lambda)\cos s ds\right] + \frac{\psi_{2n}}{\lambda_n}\left[\sin\lambda_n t + \lambda\int_0^T R_n(t,s,\lambda)\sin\lambda_n s ds\right],$$

$$E_n(t,\eta,\lambda) = \begin{cases} \sin\lambda_n(t-\eta) + \lambda\int_\eta^T R_n(t,s,\lambda)\sin\lambda_n(s-\eta)ds, & 0 \leq \eta \leq t, \\ \lambda\int_\eta^T R_n(t,s,\lambda)\sin\lambda_n(s-\eta)ds, & t \leq \eta \leq T. \end{cases}$$

3) $J[\bar{u}^0(t,x), \bar{\vartheta}^0(t,x)]$ is a minimum value of the functional determined by the following formula

$$J[\bar{u}^0(t,x),\bar{\vartheta}^0(t,x)] = \int_Q \Big(\big[V^0(T,x)-\xi_1(x)\big]^2 + \big[V_t^0(T,x)-\xi_2(x)\big]^2 \Big)\, dx +$$
$$+ \Big(\alpha \int_Q h^2[t,x,\bar{u}^0(t,x)]dx + \beta \int_\gamma b^2[t,x,\bar{\vartheta}^0(t,x)]dx \Big)\, dt, \quad \alpha>0, \quad \beta>0. \tag{11}$$

2 Approximations of the Complete Solution to a Non-linear Optimization Problem

The main objective of this work is to investigate the construction of approximate solutions to non-linear optimization problem (1)–(4) and to analyze their convergence. Since the complete solution to the problem is represented as a triple $\big((\bar{u}^0(t,x),\bar{\vartheta}^0(t,x)),V^0(t,x),J[\bar{u}^0(t,x),\bar{\vartheta}^0(t,x)]\big)$ consisting of the optimal control, the optimal process, and the minimum value of the functional, we consider approximations of each of these components separately.

2.1 Convergence of Approximations of Vector Optimal Controls

In formulas (6) and (7), replacing functions $\theta_1^0(t,x)$ and $\theta_2^0(t,x)$ with functions $\theta_1^{(k)}(t,x)$ and $\theta_2^{(k)}(t,x)$, we find the k-th approximation of the vector distributed control by the formula

$$\bar{u}^{(k)}(t,x) = \bar{\varphi}[t,x,\theta_1^{((k))}(t,x),\alpha], \quad x\in Q, \quad k=1,2,3,\ldots,$$

and similarly, we find the k-th approximation of the boundary vector control by the formula

$$\bar{\vartheta}^{(k)}(t,x) = \bar{\upsilon}[t,x,\theta_2^{(k)}(t,x),\beta], \quad x\in\gamma, \quad k=1,2,3,\ldots,$$

where $\bar{\varphi}[t,x,\theta_1^{(k)}(t,x),\alpha]$ and $\bar{\upsilon}[t,x,\theta_2^{(k)}(t,x),\beta]$ are known vector functions.

Lemma 1. The k-th approximations of the distributed and boundary vector controls for non-linear optimization problem (1)–(4) converge to the optimal distributed and boundary vector controls, respectively, in the norms of the Hilbert spaces $H^m(Q_T)$ and $H^r(\gamma_T)$.

Proof. Let us introduce the notation

$$\bar{U}(t,x) = \begin{cases} \bar{u}(t,x), & x\in Q, \\ \bar{\vartheta}(t,x), & x\in\gamma. \end{cases}$$

Using inequalities (8) and (9), we calculate the following norm:

$$\|\bar{U}(t,x)-\bar{U}^n(t,x)\|^2_{H(\bar{Q}_T)} = \|\bar{u}^0(t,x)-\bar{u}^n(t,x)\|^2_{H^m(\bar{Q}_T)} + \|\bar{\vartheta}^0(t,x)-\bar{\vartheta}^n(t,x)\|^2_{H^k(\bar{\gamma}_T)} \le$$
$$\le \varphi_0^2(\alpha)\|\theta_1^0(t,x)-\theta_1^n(t,x)\|^2_{H(Q_T)} + \upsilon_0^2(\beta)\|\theta_2^0(t,x)-\theta_2^n(t,x)\|^2_{H(\gamma_T)} \le \Psi^2(\alpha,\beta)\|\theta^0(t,x)-\theta^n(t,x)\|^2_{H(\bar{Q}_T)},$$

$\Psi^2(\alpha,\beta)=\max\{\varphi_0^2(\alpha),\upsilon_0^2(\beta)\}$, from which the assertion of the lemma follows.

2.2 Approximations of the Optimal Process and Their Convergence

The presence of the Fredholm integral operator in boundary value problem (2)–(4), according to formula (10), leads to the construction of the following three types of approximations of the optimal process: approximations based on the resolvent of the kernel of the integral operator; approximations induced by the approximations of the optimal controls; finite-dimensional approximations. Each of these approximation types will be considered separately below.

2.2.1 "Resolvent" Approximations of the Optimal Process and Their Convergence

Functions defined by the formulas

$$V^{(m)}(t,x)=\sum_{n=1}^{\infty}\Big(\psi_n^{(m)}(t,\lambda)+\frac{1}{\lambda_n}\int_0^T E_n^{(m)}(t,\eta,\lambda)\times$$
$$\times\Big(\int_Q f[\eta,\xi,\bar{u}^0(\eta,\xi)]z_n(\xi)d\xi+\int_\gamma p[\eta,\xi,\bar{\vartheta}^0(\eta,\xi)]z_n(\xi)d\xi\Big)d\eta\Big)z_n(x),\quad m=1,2,3,...,$$

are called m-th approximations of the optimal process with respect to the resolvent or "resolvent" approximations of the optimal Process. Here,

$$\psi_n^{(m)}(t,\lambda)=\psi_{1n}\left[\cos\lambda_n t+\lambda\int_0^T R_n^{(m)}(t,s,\lambda)\cos\lambda_n s ds\right]+\frac{\psi_{2n}}{\lambda_n}\left[\sin\lambda_n t+\lambda\int_0^T R_n^{(m)}(t,s,\lambda)\sin\lambda_n s ds\right],$$

$$E_n^{(m)}(t,\eta,\lambda)=\begin{cases}\sin\lambda_n(t-\eta)+\lambda\int_\eta^T R_n^{(m)}(t,s,\lambda)\sin\lambda_n(s-\eta)ds, & 0\le\eta\le t,\\ \lambda\int_\eta^T R_n^{(m)}(t,s,\lambda)\sin\lambda_n(s-\eta)ds, & t\le\eta\le T,\end{cases}$$

$$R_n^{(m)}(t,s,\lambda)=\sum_{n=0}^{m}\lambda^{i-1}K_{n,i}(t,s),\quad n=1,2,3,\ldots$$

Lemma 2. "Resolvent" approximations $V^{(m)}(t,x)$ of the optimal process under the conditions of non-linear optimization problem (1)–(4) converge to the optimal process $V^0(t,x)$ in the norm of the Hilbert space $H(Q_T)$.

Proof. We evaluate the following norm

$$\|V^0(t,x)-V^{(m)}(t,x)\|^2_{H(Q_T)}\le 2T\frac{2\lambda^2T^2K_0}{\lambda_1^2}\left(|\lambda|\,\frac{T\sqrt{K_0}}{\lambda_1}\right)^{2m}\left(1-\frac{1}{\ln|\lambda|\,\frac{T\sqrt{K_0}}{\lambda_1}}\right)^2\times$$

$$\times\Big(\|\psi_1(x)\|^2_{H(Q)}+\frac{1}{\lambda_1^2}\|\psi_2(x)\|^2_{H(Q)}+\|f[\eta,\xi,\bar{u}^0(\eta,\xi)]\|^2_{H(Q_T)}+\|p[\eta,\xi,\bar{\vartheta}^0(\eta,\xi)]\|^2_{H(\gamma_T)}\Big)\sum_{n=1}^{\infty}\frac{1}{\lambda_n^2}\to 0,\ m\to\infty,$$

from which, by virtue of the condition $|\lambda|\,\frac{T\sqrt{K_0}}{\lambda_1}<1$, the assertion of the lemma follows.

2.2.2 m,k-th Approximations of the Optimal Process and Their Convergence

Functions defined by following the formula

$$V_k^{(m)}(t,x)=\sum_{n=1}^{\infty}\Big(\psi_n^{(m)}(t,\lambda)+\frac{1}{\lambda_n}\int_0^T E_n^{(m)}(t,\eta,\lambda)\times$$
$$\times\left(\int_Q f[\eta,\xi,\bar{u}^{(k)}(\eta,\xi)]z_n(\xi)d\xi+\int_\gamma p[\eta,\xi,\bar{\vartheta}^{(k)}(\eta,\xi)]z_n(\xi)d\xi\right)d\eta\Big)z_n(x),$$

are called m,k-th approximations of the optimal process with respect to controls, where $\bar{u}^{(k)}(t,x)$ are k-th approximations of the distributed vector control, and $\bar{\vartheta}^{(k)}(t,x)$ are k-th approximations of the boundary vector control.

Lemma 3. m,k-th approximations $V_k^{(m)}(t,x)$ of the optimal process under the conditions of non-linear optimization problem (1)–(4) converge to the "resolvent" approximations $V^{(m)}(t,x)$ when $k\to\infty$ for any $m=1,2,3,\ldots$ in the norm of the space $H(Q_T)$.

Proof. The evaluation of the following norm leads directly to the conclusion of Lemma 3.

$$\|V^{(m)}(t,x)-V_k^{(m)}(t,x)\|^2_{H(Q_T)} \le 4T\left(1+\frac{\lambda^2T^2K_0}{\left(\lambda_1-|\lambda|\,T\sqrt{K_0}\right)^2}\right)\times$$

$$\times\sum_{n=1}^{\infty}\frac{1}{\lambda_n^2}\left(f_0^2\|\bar u_0(\eta,\xi)-\bar u_{(m)}(\eta,\xi)\|^2_{H(Q_T)}+p_0^2\|\bar\vartheta_0(\eta,\xi)-\bar\vartheta_{(m)}(\eta,\xi)\|^2_{H(\gamma_T)}\right)\to 0,\quad k\to\infty,$$

which is obtained taking into account the estimate

$$\int_0^T\left(E_n^{(m)}(t,\eta,\lambda)\right)^2 d\eta \le 2T\left(1+\lambda^2\frac{T^2K_0}{\left(\lambda_n-\lambda T\sqrt{K_0}\right)^2}\right).$$

2.2.3 *Finite-Dimensional Approximations of the Optimal Process and Their Convergence*

Functions defined by the following formula

$$\begin{aligned} V_{k,l}^{(m)}(t,x)=&\sum_{n=1}^{l}\Big(\psi_n^{(m)}(t,\lambda)+\frac{1}{\lambda_n}\int_0^T E_n^{(m)}(t,\eta,\lambda)\times\\ &\times\Big(\int_Q f[\eta,\xi,\bar u^{(k)}(\eta,\xi)]z_n(\xi)d\xi+\int_\gamma p[\eta,\xi,\bar\vartheta^{(k)}(\eta,\xi)]z_n(\xi)d\xi\Big)d\eta\Big)z_n(x),\\ &m=1,2,3,\dots,\ \mu_m<\infty,\quad k=1,2,3,\dots,\ \mu_k<\infty,\quad l=1,2,3,\dots,\ \mu_l<\infty,\end{aligned}\tag{12}$$

are called m,k,l-th approximations or finite-dimensional approximations of the optimal process.

Lemma 4. m,k,l-th approximations $V_{k,l}^{(m)}(t,x)$ or finite-dimensional approximations of the optimal process under the conditions of non-linear optimization problem (1)–(4) converge to m,k-th approximations $V_k^{(m)}(t,x)$ when $l\to\infty$ for any m, k in the norm of the space $H(Q_T)$.

Proof. The assertion of the lemma follows from the following relation:

$$\|V_k^{(m)}(t,x)-V_{k,l}^{(m)}(t,x)\|^2_{H(Q_T)} \le \sum_{n=l+1}^{\infty}\int_0^T\int_Q\left(V^{(m)}(t,x)-V_k^{(m)}(t,x)\right)^2dxdt\le$$

$$\le 4T\left(1+\frac{\lambda^2T^2K_0}{\left(\lambda_1-|\lambda|\,T\sqrt{K_0}\right)^2}\right)\times$$

$$\times\sum_{n=l+1}^{\infty}\frac{1}{\lambda_n^2}\left(f_0^2\|\bar u_0(\eta,\xi)-\bar u_{(m)}(\eta,\xi)\|^2_{H(Q_T)}+p_0^2\|\bar\vartheta_0(\eta,\xi)-\bar\vartheta_{(m)}(\eta,\xi)\|^2_{H(\gamma_T)}\right)\to 0,\quad l\to\infty,$$

which holds due to the convergence of the remainder terms of the convergent series for each fixed m,k.

2.3 *Approximations of the Generalized Derivative of the Optimal Process and Their Convergence*

Similarly, the convergence of approximations was investigated for the generalized derivative of the optimal process determined by the following formula

$$\begin{aligned}V_t^0(t,x)=&\sum_{n=1}^{\infty}\Big(\psi'_{nt}(t,\lambda)+\frac{1}{\lambda_n}\int_0^T E'_{nt}(t,\eta,\lambda)\Big(\int_Q f[\eta,\xi,\bar u^0(\eta,\xi)]z_n(\xi)d\xi+\\ &+\int_\gamma p[\eta,\xi,\bar\vartheta^0(\eta,\xi)]z_n(\xi)d\xi\Big)d\eta\Big)z_n(x),\end{aligned}$$

where

$$\psi'_{nt}(t,\lambda)=\psi_{1n}\left(-\lambda_n\sin\lambda_n t+\lambda\int_0^T R'_{nt}(t,s,\lambda)\cos\lambda_n s\,ds\right)+\frac{\psi_{2n}\lambda_n}{\left(\lambda_n\cos\lambda_n t+\lambda\int_0^T R'_{nt}(t,s,\lambda)\sin\lambda_n s\,ds\right)},$$

$$E'_{nt}(t,\eta,\lambda)=\begin{cases}\lambda_n\cos\lambda_n(t-\eta)+\lambda\int_\eta^T R'_{nt}(t,s,\lambda)\sin\lambda_n(s-\eta)ds, & 0\le\eta\le t,\\ \lambda\int_\eta^T R'_{nt}(t,s,\lambda)\sin\lambda_n(s-\eta)ds, & t\le\eta\le T,\end{cases}$$

and it is an element of the space $H(Q_T)$ [14].

2.3.1 "Resolvent" Approximations of the Generalized Derivative of the Optimal Process and Their Convergence

Functions defined by the following formula

$$V_t^m(t,x)=\sum_{n=1}^{\infty}\Big(\psi'^m_{nt}(t,\lambda)+\frac{1}{\lambda_n}\int_0^T E'^m_{nt}(t,\eta,\lambda)\times$$

$$\times\Big(\int_Q f[\eta,\xi,\bar u^0(\eta,\xi)]z_n(\xi)d\xi+\int_\gamma p[\eta,\xi,\bar\vartheta^0(\eta,\xi)]z_n(\xi)d\xi\Big)d\eta\Big)z_n(x),\qquad m=1,2,3,...,$$

are called m-th approximations or "resolvent" approximations of the generalized derivative of the optimal process.

Lemma 5. "Resolvent" approximations $V'^m_t(t,x)$ of the generalized derivative of the optimal process under the conditions of non-linear optimization problem (1)–(4), converge to the generalized derivative optimal process $V'^0_t(t,x)$ in the norm of the Hilbert space $H(Q_T)$.

Proof. The assertion of Lemma 5 follows from the following relation

$$\|V_t^0(t,x)-V_t^m(t,x)\|^2_{H(Q_T)}\le 4T\lambda^2T^2K_0\left(|\lambda|\sqrt{\frac{K_0T^2}{\lambda_1^2}}\right)^{2m}\left(1-\frac{1}{\ln\left(|\lambda|\,T\sqrt{K_0}\right)}\right)\times$$
$$\times\left(\|\psi_1(x)\|^2_{H(Q)}+\frac{1}{\lambda_1^2}\|\psi_2(x)\|^2_{H(Q)}+\right.\tag{13}$$
$$\left.+\left(\|f[\eta,\xi,\bar u^0(\eta,\xi)]\|^2_{H(Q_T)}+\|p[\eta,\xi,\bar\vartheta^0(\eta,\xi)]\|^2_{H(Q_T)}\right)\sum_{n=1}^{\infty}\frac{1}{\lambda_n^2}\right)\to 0,\quad m\to\infty$$

which holds due to the condition $\frac{|\lambda|\sqrt{K_0T^2}}{\lambda_1}<1$.

2.3.2 m,k-th Approximations of the Generalized Derivative of the Optimal Process and Their Convergence

Functions defined by the following formula

$$V'^m_{tk}(t,x)=\sum_{n=1}^{\infty}\Big(\psi'^m_{nt}(t,\lambda)+\frac{1}{\lambda_n}\int_0^T E'^m_{nt}(t,\eta,\lambda)\times$$
$$\times\Big(\int_Q f[\eta,\xi,\bar u^{(k)}(\eta,\xi)]z_n(\xi)d\xi+\int_\gamma p[\eta,\xi,\bar\vartheta^{(k)}(\eta,\xi)]z_n(\xi)d\xi\Big)d\eta\Big)z_n(x),\tag{14}$$

are called m,k-th approximations of the generalized derivative of the optimal process.

Lemma 6. m, k-th approximations $V_{tk}^{\prime m}(t,x)$ of the generalized derivative of the optimal process under the conditions of non-linear optimization problem (1)–(4) converge to the m-th approximations $V_t^{\prime(m)}(t,x)$ of the generalized derivative of the optimal process when $k \to \infty$ for any value of $m = 1, 2, 3, \dots$ in the norm of the space $H(Q_T)$.

Proof. Proof of the lemma follows from the following relation:

$$\|V_t^m(t,x) - V_{tk}^m(t,x)\|_{H(Q_T)}^2 \leq 4T^3 \left(1 + \frac{\lambda^2 K_0 T}{\lambda_1^2}\right) \cdot \Big(f_0^2 \left\|\bar{u}^0(t,x) - \bar{u}^{(k)}(t,x)\right\|_{H(Q_T)}^2 + $$
$$+ p_0^2 \left\|\bar{\vartheta}^0(t,x) - \bar{\vartheta}^{(k)}(t,x)\right\|_{H(\gamma_T)}^2\Big) \to 0, \quad k \to \infty.$$

2.3.3 Finite-Dimensional Approximations of the Generalized Derivative of the Optimal Process and Their Convergence

Functions defined by the following formula

$$V_{tk,l}^{\prime m}(t,x) = \sum_{n=1}^{l} \Big(\psi_{nt}^{\prime m}(t,\lambda) + \frac{1}{\lambda_n}\int_0^T E_{nt}^{\prime m}(t,\eta,\lambda) \times$$
$$\times \Big(\int_Q f[\eta,\xi,\bar{u}^{(l)}(\eta,\xi)] z_n(\xi) d\xi + \int_\gamma p[\eta,\xi,\bar{\vartheta}^{(l)}(\eta,\xi)] z_n(\xi) d\xi\Big) d\eta\Big) z_n(x),$$

are called m, k, l-th approximations or finite-dimensional approximations of the generalized derivative of the optimal process.

Lemma 7. Finite-dimensional approximations $V_{tk,l}^{\prime m}(t,x)$ of the generalized derivative of the optimal process under the conditions of non-linear optimization problem (1)–(4) converge to m, k-th approximations $V_{tk}^{\prime m}(t,x)$ of the generalized derivative of the optimal process when $l \to \infty$ for any value of m, k in the norm of space $H(Q_T)$.

Proof. Proof of the lemma follows from the following inequality

$$\left\|V_{tk}^m(t,x) - V_{tk,l}^m(t,x)\right\|_{H(Q_T)}^2 \leq 8T \left(1 + \frac{\lambda^2}{\lambda_n^2} \cdot \frac{K_0 T^2 \lambda_n^2}{\left(\lambda_n |\lambda| \sqrt{K_0 T^2}\right)^2}\right) \cdot \left(\sum_{n=i+1}^{\infty} \lambda_n^2 \psi_{1n}^2 + \sum_{n=i+1}^{\infty} \psi_{1n}^2 + \right.$$
$$\left. + \sum_{n=i+1}^{\infty} \int_0^T f_n^2[\eta, \bar{u}^k]\, d\eta + \sum_{n=i+1}^{\infty} \int_0^T p_n^2[\eta, \bar{\vartheta}^k]\, d\eta\right) \to 0, \quad l \to \infty,$$

which hold due to the convergence of the remainder terms of convergent series.

2.4 Approximations of the Minimum Value of the Functional and Their Convergence

The minimum value of functional (11), in accordance with the approximations of the optimal process, has three types of approximations.

Let us first derive the following formula that will be repeatedly used in proving the convergence of approximations of the minimum value of the functional:

$$\begin{aligned}|J[\hat{u},\hat{\vartheta}] - J[\tilde{u},\tilde{\vartheta}]| \leq{}& \|V(T,x) + W(T,x) - 2\xi_1(x)\|_{H(Q)} \cdot \|V(T,x) - W(T,x)\|_{H(Q)} \\ &+ \|V_t(T,x) + W_t(T,x) - 2\xi_2(x)\|_{H(Q)} \cdot \|V_t(T,x) - W_t(T,x)\|_{H(Q)} \\ &+ \alpha h_0 \cdot \|h[t,x,\hat{u}(t,x)] + h[t,x,\tilde{u}(t,x)]\|_{H(Q_T)} \cdot \|\hat{u}(t,x) - \tilde{u}(t,x)\|_{H(Q_T)} \\ &+ \beta b_0 \cdot \|b[t,x,\hat{\vartheta}(t,x)] + b[t,x,\tilde{\vartheta}(t,x)]\|_{H(\gamma_T)} \cdot \|\hat{\vartheta}(t,x) - \tilde{\vartheta}(t,x)\|_{H(\gamma_T)},\end{aligned}$$

2.4.1 Finite-dimensional approximations of the functional minimum value and their convergence

According to formulas (12) and (13), finite-dimensional approximations of the functional minimum value are calculated by the formula

$$J_m^{k,j}[\bar{u}^{(k)}(t,x),\bar{\vartheta}^{(k)}(t,x)]=\int_Q\left[\left(V_{k,j}^{(m)}(T,x)-\xi_1(x)\right)^2+\left(V_{t_{k,j}}(T,x)-\xi_2(x)\right)^2\right]dx+$$
$$+\int_0^T\left[\alpha\int_Q h^2(t,x,\bar{u}^{(k)}(t,x))\,dx+\beta\int_\gamma b^2(t,x,\bar{\vartheta}^{(k)}(t,x))\,dx\right]dt.$$

Lemma 8. Finite-dimensional approximations $J_m^k[\bar{u}^0(t,x),\bar{\vartheta}^0(t,x)]$ of the functional minimal value under the conditions of the non-linear optimization problem (1)–(4) converge to the m-th approximations of the functional minimal value when $k\to\infty$ for all fixed values of m,k in the norm of real numbers space R.

Proof. In formula (14), by replacing

$$V(t,x)\to V_{k,l}^{(m)}(t,x),\quad V_t(t,x)\to V_{tk}^{(m)}(t,x),\quad W(t,x)\to V_{k,j}^{(m)}(t,x),\quad W_t(t,x)\to V_{tk,j}^{(m)}(t,x),$$

we obtain the inequality

$$\left|J_m^{(k)}[\bar{u}^k(t,x),\bar{\vartheta}^{(k)}(t,x)]-J_m^{k,j}[\bar{u}^{(k)}(t,x),\bar{\vartheta}^{(k)}(t,x)]\right|\le C^{(2)}\left\|V_k^m(T,x)-V_{k,j}^m(T,x)\right\|_{H(Q)}+$$
$$+C^{(3)}\left\|V_{tk}^m(T,x)-V_{tk,j}^m(T,x)\right\|_{H(\gamma)}\to 0,\quad k\to\infty,$$

where $C^{(2)}$, $C^{(3)}$ are constants.

3 Main results

Theorem 1. (Convergence of Finite-Dimensional Approximations to the Optimal Process). Let the following conditions be satisfied:

1) Functions of external and boundary influences satisfy the Lipschitz condition for functional variables (for controls):

$$\|f[\eta,\xi,\hat{u}(\eta,\xi)]-f[\eta,\xi,\tilde{u}(\eta,\xi)]\|^2_{H(Q_T)}\le f_0^2\|\hat{u}(\eta,\xi)-\tilde{u}(\eta,\xi)\|^2_{H(Q_T)},\quad f_0^2=\text{const},$$

$$\|p[\eta,\xi,\hat{\vartheta}(\eta,\xi)]-p[\eta,\xi,\tilde{\vartheta}(\eta,\xi)]\|^2_{H(Q_T)}\le p_0^2\|\hat{\vartheta}(\eta,\xi)-\tilde{\vartheta}(\eta,\xi)\|^2_{H(Q_T)},\quad p_0^2=\text{const}.$$

2) The intermediate vectors $\bar{\varphi}[t,x,\theta_1(t,x),\alpha],\ x\in Q$, and $\bar{\upsilon}[t,x,\theta_2(t,x),\beta],\ x\in\gamma$, of the functions satisfy the Lipschitz condition with respect to functional variables:

$$\|\bar{\varphi}[t,x,\hat{\theta}_1(t,x),\alpha]-\bar{\varphi}[t,x,\tilde{\theta}_1(t,x),\alpha]\|_{H(Q_T)}\le\varphi_0(\alpha)\|\hat{\theta}_1(t,x)-\tilde{\theta}_1(t,x)\|_{H(Q_T)},\quad\varphi_0(\alpha)>0,$$

$$\|\bar{\upsilon}[t,x,\hat{\theta}_2(t,x),\beta]-\bar{\upsilon}[t,x,\tilde{\theta}_2(t,x),\beta]\|_{H(Q_T)}\le\upsilon_0(\beta)\|\hat{\theta}_2(t,x)-\tilde{\theta}_2(t,x)\|_{H(Q_T)},\quad\upsilon_0(\beta)>0.$$

3) With respect to the parameters of non-linear optimization problem (1)–(4), the following inequality holds:

$$C(\alpha,\beta)=\sqrt{f_0^2m\varphi_0^2(\alpha)+p_0^2r\upsilon_0^2(\beta)}\sqrt{2}E_0G_0T<1.$$

Then finite-dimensional approximations $V_{k,l}^{(m)}(t,x)$ of the optimal process $V^0(t,x)$ under the conditions of the non-linear optimization problem (1)–(4) converge to the optimal process when $m,k,l\to\infty$ in the norm of the space $H(Q_T)$.

Proof. Based on Lemmas 1–4, the assertion of the theorem follows from the inequality:

$$\|V^0(t,x) - V_{k,l}^{(m)}(t,x)\|_{H(Q_T)} \leq \|V^0(t,x) - V^{(m)}(t,x)\|_{H(Q_T)} + \|V^{(m)}(t,x) - V_k^{(m)}(t,x)\|_{H(Q_T)} + $$
$$+ \|V_k^{(m)}(t,x) - V_{k,l}^{(m)}(t,x)\|_{H(Q_T)} \to 0, \quad m,k,l \to \infty.$$

Theorem 2. (Convergence of finite-dimensional approximations of the generalized derivative to the generalized derivative of the optimal process). Let the conditions of Theorem 1 be satisfied. Then Finite-dimensional approximations $V_{tk,l}^{(m)}(t,x)$ of the generalized derivative of the optimal under the conditions of non- linear optimization problem (1)–(4)converge to generalized derivative $V_t^0(t,x)$ of the optimal process when $m,k,l \to \infty$ in the norm of the space $H(Q_T)$.

Proof. Proof of the lemma follows from following inequality

$$\|V_t^0(t,x) - V_{tk,l}^{(m)}(t,x)\|_{H(Q_T} = \|V_t^0(t,x) - V_t^{(m)}(t,x)\|_{H(Q_T)} + \|V_t^{(m)}(t,x) - V_{tk}^{(m)}(t,x)\|_{H(Q_T)} + $$
$$+ \|V_{tk}(t,x) - V_{tk,l}^{(m)}(t,x)\|_{H(Q_T)} \to 0, \quad l \to \infty.$$

Theorem 3. (Convergence of finite-dimensional approximations of the functional minimum value to the minimum value of the functional). Let the conditions of Theorem 1 be satisfied, then Finite-dimensional approximations $J_m^k[\bar{u}^0(t,x), \bar{\vartheta}^0(t,x)]$ of the functional minimal value under the conditions of non-linear optimization problem (1)–(4) converge to functional minimal value $J[\bar{u}^0(t,x), \bar{\vartheta}^0(t,x)]$ when $m,k,l \to \infty$ in the norm of real numbers space R.

Proof. Proof of Theorem 3 follows from the inequality

$$|J[\bar{u}^0(t,x), \bar{\vartheta}^0(t,x)] - J_m^{k,j}[\bar{u}^{(k)}(t,x), \bar{\vartheta}^{(k)}(t,x)]| \leq |J[\bar{u}^0(t,x), \bar{\vartheta}^0(t,x)] - J_m[\bar{u}^0(t,x), \bar{\vartheta}^0(t,x)]| +$$
$$+ |J_m[\bar{u}^0(t,x), \bar{\vartheta}^0(t,x)] - J_m^k[\bar{u}^{(k)}(t,x), \bar{\vartheta}^{(k)}(t,x)]| +$$
$$+ |J_m^k[\bar{u}^{(k)}(t,x), \bar{\vartheta}^{(k)}(t,x)] - J_m^{k,j}[\bar{u}^{(k)}(t,x), \bar{\vartheta}^{(k)}(t,x)]| \to 0, \quad m,k,l \to \infty.$$

Conclusion

In this paper, the influence of the Fredholm integral operator in the integro-differential equation on the convergence of approximate solutions to a nonlinear optimization problem is investigated. It is established that the presence of the Fredholm integral operator leads to the identification of three distinct types of approximations of the optimal process (“Resolvent” approximations, based on the resolvent of the kernel of the integral operator; Approximations by optimal controls, constructed through the approximation of control functions; Finite-dimensional approximations) and corresponding approximations of the minimum value of the functional.

Acknowledgments

The research was supported by the Commission for the Scientific Research Projects of Kyrgyz-Turkish Manas University, project number KTMU-BAP-2025.FB.14

Author Contributions

Conceptualization and Formal analysis: Elmira Faizuldaevna Abdyldaeva; Methodology: Akylbek Kerimbekov and Elmira Faizuldaevna Abdyldaeva; Preparation and editing of the manuscript, translation into English, and other technical works: Elmira Faizuldaevna Abdyldaeva, Tursun Kamaldinovich Yuldashev, Muhammet Kamali

All authors participated in the revision of the manuscript and approved the final submission.

Conflict of Interest

The authors declare no conflict of interest.

References

1 Butkovskii, A.G. (1965). *Teoriia optimalnogo upravleniia sistemami s raspredelennymi parametrami [Theory of optimal control of systems with distributed parameters*]. Moscow: Nauka [in Russian].

2 Egorov, A.I. (1978). *Optimalnoe upravlenie teplovymi i diffuzionnymi protsessami [Optimal control of thermal and diffusion processes].* Moscow: Nauka [in Russian].

3 Sirazetdinov, T.K. (1987). *Ustoichivost sistemy s raspredelennymi parametrami [Stability of systems with distributed parameters].* Novosibirsk: Nauka [in Russian].

4 Li, J., Xu, Z., Zhu, D., Dong, K., Yan, T., Zeng, Z., & Yang, S.X. (2021). Bio-inspired intelligence with applications to robotics: a survey. *Intelligence and Robotics, 1*(1), 58–83. https://doi.org/10.20517/ir.2021.08

5 Saveriano, M., Abu-Dakka, F.J., Kramberger, A., & Peternel, L. (2023). Dynamic movement primitives in robotics: A tutorial survey. *The International Journal of Robotics Research, 42*(13), 1133–1184. https://doi.org/10.1177/02783649231201196

6 Plotnikov, V.I. (1968). An energy inequality and the overdeterminacy property of a system of eigenfunctions. *Mathematics of the USSR-Izvestiya, 2*(4), 695–707. https://doi.org/10.1070/IM1968v002n04ABEH000656

7 Raskin, L., & Sira, O. (2019). Construction of the fractional-non-linear optimization method. *Eastern-European Journal of Enterprise Technologies, 4*(4 (100)), 37–43. https://doi.org/10.15587/1729-4061.2019.174079

8 Krasnichenko, L.S. (2012). Reshenie zadachi nelineinoi optimizatsii teplovykh protsessov pri granichnom upravlenii [Solution of the problem of nonlinear optimization of thermal processes under boundary control. *Candidate's thesis.* Bishkek [in Russian].

9 Asanova, Zh.K. (2012). Tochechnoe podvizhnoe nelineinoe optimalnoe upravlenie protsessom teploperedachi [Pointwise moving nonlinear optimal control of heat transfer process]. *Candidate's thesis.* Bishkek [in Russian].

10 Ermekbaeva, A. (2022). Optimalnoe tochnoe upravlenie teplovymi protsessami, opisyvaemymi fredgolmovo integro-differentsialnymi uravneniiami [Optimal exact control of thermal processes described by Fredholm integro-differential equations]. *Candidate's thesis.* Bishkek [in Russian].

11 Baetov, A.K. (2014). Priblizhennye resheniia zadach nelineinoi optimizatsii kolebatelnykh protsessov [Approximate solutions of problems of nonlinear optimization of oscillatory processes]. *Candidate's thesis.* Bishkek [in Russian].

12 Uryvskaya, T.Yu. (2010). Priblizhennoe reshenie zadachi nelineinoi optimizatsii teplovykh protsessov [Approximate solution of the problem of nonlinear optimization of thermal processes]. *Candidate's thesis.* Bishkek [in Russian].

13 Kerimbekov, A., & Abdyldaeva, E. (2016). On the Solvability of a Nonlinear Tracking Problem Under Boundary Control for the Elastic Oscillations Described by Fredholm Integro-Differential Equations. In: *Bociu, L., Desideri, JA., Habbal, A. (Eds.). System Modeling and Optimization. 27th IFIP TC 7 Conference. CSMO 2015,* 312–321. Springer. https://doi.org/10.1007/978-3-319-55795-3_29

14 Abdyldaeva, E.F., & Kerimbekov, A. (2023). On features of the nonlinear optimization problem of oscillation processes under distributed and boundary vector controls. *Journal of the Institute of Automation and Information Technologies of the National Academy of Sciences of the Kyrgyz Republic: Problems of Automation and Control*, 14–25.

*Author Information**

Elmira Faizuldaevna Abdyldaeva (*corresponding author*) — Candidate of Physical and Mathematical Sciences, Associate Professor, Department of Mathematics, Kyrgyz-Turkish Manas University, 56 Chyngyz Aitmatov avenue, Bishkek, Kyrgyz Republic; e-mail: *elmira.abdyldaeva@manas.edu.kg*; https://orcid.org/0000-0002-3874-9055

Akylbek Kerimbekov — Doctor of Physical and Mathematical Sciences, Professor, Head of the Department of Applied Mathematics and Computer Science, Kyrgyz Russian Slavic University, 44 Kyiv street, Bishkek, 720000, Kyrgyz Republic; e-mail: *akl7@rambler.ru*; https://orcid.org/0000-0002-7401-4312

Tursun Kamaldinovich Yuldashev — Doctor of Physical and Mathematical Sciences, Professor, Department of Higher Mathematics, Tashkent State Transport University, 1 Temiryolchilar street, Tashkent, 100167, Uzbekistan; e-mail: *tursun.k.yuldashev@gmail.com*; https://orcid.org/0000-0002-9346-5362

Muhammet Kamali — Doctor, Professor, Department of Mathematics, Kyrgyz-Turkish Manas University, 56 Chyngyz Aitmatov avenue, Bishkek, Kyrgyz Republic; e-mail: *muhammet.kamali@manas.edu.kg*; https://orcid.org/0000-0002-6230-6836

*Authors' names are presented in the order: first name, middle name, and last name.