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On the properties for families of function classes over harmonic intervals and their embedding relation with Besov spaces

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The article is dedicated to the issues of studying the approximation of functions by trigonometric polynomials with a spectrum from special sets. In this paper, these special sets are harmonic intervals. To study the approximation of functions over harmonic intervals, families of function classes have been created, designed as a subsidiary tool. These families of function classes are characterized through the best approximations of functions by trigonometric polynomials over such sets and are used in the research. For these families of function classes, their properties and the connection with classical Besov spaces are shown. The results of the study are presented in the form of theorems and lemmas. In carrying out the research presented in the article, the main apparatus for proving theorems are the fundamentals of approximation theory, the method of real interpolation of spaces, and the fundamentals of the theory of embedding classes of functions and functional spaces. The article is destined for mathematicians and can be used by researchers and specialists whose interests lie in the indicated areas of mathematics.

Keywords: harmonic interval, spectrum, best approximation of a function by trigonometric polynomials with a spectrum from harmonic intervals, family of classes of functions, Besov spaces, embedding theorems.

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Introduction

In approximation theory, the spectrum of approximating functions plays an important role. This spectrum can be selected from sets of different shapes, types, contours, structures, locations, etc. The spectrum can contain a wide variety of sets [1, 2].

A person's perception of information occurs through his sense organs, readings from various devices of measuring and observation, etc. And they all have a limited finite, not infinite, range. Based on this, when modeling diverse, multifaceted practical and applied problems [3–5], it is necessary to find a solution on some finite set that reflects this finite range. Special sets called harmonic intervals I_k^N [6, 7], where N defines exactly such a finite range of perception, to some extent help to conduct research and solve such problems.

The approximative properties of a function are usually characterized by the magnitude of the best approximation or the speed of approximation by a linear method. Often, during the course of research, there is a need to create auxiliary elements and tools. One of them is the families of classes of functions $\{B_{p,q,N}^N\}_N$, the definition of which is expressed in terms of the best approximations of functions by trigonometric polynomials with spectrum from sets I_k^N. These families of function classes are used, for example, in the theorems on the boundedness of the partial sum operator of the Fourier series, etc.

In carrying out the proofs of the statements described in the article, the fundamentals of approximation theory [8–10], the method of real interpolation of spaces [11–13] and the fundamentals of embedding theory [14] for classes of functions and function spaces are used.

In the article the properties of families of function classes $\left\{B_{p,q,N}^r\right\}_N$ and the relationship of these families of function classes to the classical Besov spaces [15, 16] are studied.

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1 Definitions and auxiliary results

We construct families of classes of functions related to the best approximations of functions by trigonometric polynomials with a spectrum from harmonic intervals.

Definition 1. [17] Let $1 \le p, q \le \infty$, $r > 0$, $N \in \mathrm{N}$, $f \in L_p[0; 2\pi)$. The family of function classes is denoted by $\left\{B^r_{p,q,N}\right\}_N$ and is defined in terms of the best approximations over harmonic intervals using the relation

$$B^r_{p,q,N} = \left\{f : \|f\|_{B^r_{p,q,N}} < \infty\right\},$$

where

$$\|f\|_{B^r_{p,q,N}} = \left(\sum_{k=1}^{N} k^{rq-1} \left(E^N_{k-1}(f)_p\right)^q\right)^{\frac{1}{q}}.$$

Definition 2. [17] Let A^N and B^N be two classes of functions depending on a parameter N. The class of functions A^N is embedded in the class of functions B^N and denote it by

$$A^N \hookrightarrow B^N$$

if the following conditions are met:

1) $A^N \subset B^N$;

2) there is a positive parameter C such that the inequality

$$\|f\|_{B^N} \le C\|f\|_{A^N}$$

is correct for any $f \in A^N$, and the parameter C does not depend on f and N.

Definition 3. [17] Let two families of function classes $\left\{A^N\right\}_N$ and $\left\{B^N\right\}_N$ be given, where $N \in \mathrm{N}$ and $\left\{A^N\right\}_N \bigcap \left\{B^N\right\}_N = \varnothing$. The classes of functions $\left\{A^N\right\}_N$ and $\left\{B^N\right\}_N$ is said to be equivalent

$$\|f\|_{A^N} \sim \|f\|_{B^N}$$

if there are positive parameters C_1, C_2 such that for any $f \in A^N$ the relation

$$C_1\|f\|_{B^N} \le \|f\|_{A^N} \le C_2\|f\|_{B^N}$$

is correct. Moreover, the parameters C_1, C_2 do not depend on f and N. In this case, it is assumed that the families of function classes $\left\{A^N\right\}_N$ and $\left\{B^N\right\}_N$ coincide

$$\left\{A^N\right\}_N = \left\{B^N\right\}_N.$$

Definition 4. [11] Let (A_0, A_1) be an interpolation pair. For any t such that $0 < t < \infty$, the functional is defined by the equality

$$K\left(t, a; \bar{A}\right) = \inf_{a=a_0+a_1}\left(\|a_0\|_{A_0} + t\|a_1\|_{A_1}\right),$$

where $a = A_0 + A_1$, and $\bar{A}$=(A_0, A_1) is a compatible pair of spaces. This functional is called Petre's K-functional or simply Petre's functional.

Definition 5. [11] Let (A_0, A_1) be an interpolation pair, $0 < \theta < 1$. For 1$\le q < \infty$, we have

$$\bar{A}_{\theta,q} = (A_0, A_1)_{\theta,q} =$$

$$= \left\{a : a \in A_0 + A_1, \quad \|a\|_{\bar{A}_{\theta,q}} = \left(\int_0^\infty \left[t^{-\theta}K\left(t, a; \bar{A}\right)\right]^q \frac{dt}{t}\right)^{\frac{1}{q}} < \infty\right\},$$

and for $q = \infty$ the equality

$$\bar{A}_{\theta,\infty} = (A_0, A_1)_{\theta,\infty} = \{a : a \in A_0 + A_1,\ \|a\|_{\bar{A}_{\theta,\infty}} = \sup_{0<t<\infty} t^{-\theta} K\left(t, a; \bar{A}\right) < \infty\}$$

holds. The space $\bar{A}_{\theta,q}$ is defined by the following relation

$$\bar{A}_{\theta,q} = \{a : \|a\|_{\bar{A}_{\theta,q}} < \infty\}.$$

Theorem 1. [7] Let $m \in \mathrm{N}$, $1 \le p, q, q_0, q_1 \le \infty$, $0 < \theta < 1$, $r > 0$, $r_0 > r_1 > 0$. Then there is the embedding of the form

$$B^r_{p,q,2^m} \hookrightarrow \left(B^{r_0}_{p,q_0,2^m}; B^{r_1}_{p,q_1,2^m}\right)_{\theta,q},$$

where $r = (1-\theta)\, r_0 + \theta r_1$.

Theorem 2. [7] Let $N \in \mathrm{N}$, $1 \le p, q, q_1 \le \infty$, $r > 0$, then the following embedding

$$B^r_{p,q,N} \hookrightarrow B^r_{p,q_1,N}$$

is performed for $q < q_1$.

Theorem 3. [7] Let $f \in B^r_{p,q,2^m}$, $m \in \mathrm{N}$, $\sum\limits_{v\in \mathrm{Z}} a_v e^{ivx}$ be the trigonometric Fourier series of the function f. Then for $1 \le p, q \le \infty$, $r > 0$ the following relation

$$\|f\|_{B^r_{p,q,2^m}} \sim \left(\sum_{k=1}^{m} 2^{rqk} \left(\delta_k (f)_p\right)^q\right)^{\frac{1}{q}}$$

is correct, where

$$\delta_k (f)_p = \left\|\sum_{s\in \mathrm{Z}} \sum_{\tau=2^{k-1}}^{2^k-1} a_{\tau+s\cdot 2^m} \cdot e^{i(\tau+s\cdot 2^m)x}\right\|_p.$$

Lemma 1. [6] Let $B = [-k, k]$ be the segment in Z; $k, d, h \in \mathrm{N}$, $k < h$, $\left\{I^{h,d}_B\right\}_{d=0}^{\infty}$ be the sequence of harmonic segments in Z, converging to the harmonic interval I^h_B, where $I^h_B = \bigcup\limits_{v=-\infty}^{\infty} [B + vh]$. If $f \in L_p\,[0;\ 2\pi)$, $1 \le p \le \infty$, $\sum\limits_{v\in \mathrm{Z}} a_v e^{ivx}$ is the Fourier series of the function f, then the sequence of partial sums of the Fourier series of the function f over harmonic segments

$$S^{h,d}_B (f) = \sum_{v\in I^{h,d}_B} a_v e^{ivx}$$

converges in the space $L_p\,[0;\ 2\pi)$ as $d \to \infty$ to the function

$$S^h_B (f) = \frac{1}{h} \sum_{r=0}^{h-1} f\left(x + \frac{2\pi r}{h}\right) D_B \left(\frac{2\pi r}{h}\right), \tag{1}$$

where $D_B (x) = \sum\limits_{m\in B} e^{imx}$ is the Dirichlet kernel corresponding to the segment B from Z, and its Fourier series is the function $\sum\limits_{v\in I^h_B} a_v e^{ivx}$.

Theorem 4. [7] Let $m, d, N \in \mathrm{N}$, $f \in L_p[0;2\pi)$, $1 < p < \infty$, $S_m^N(f)$ and $E_m^N(f)_p$ be the partial sum of the Fourier series and the best approximation of the function f over the harmonic interval I_m^N, respectively, then the following relation is correct

$$E_m^N(f)_p \sim \left\| f - S_m^N(f) \right\|_p .$$

Theorem 5. [7] Let $f \in B_{p,q,2^m}^r, m \in \mathrm{N}$, then for $1 \le p, q \le \infty$, $r > 0$, we have

$$\|f\|_{B_{p,q,2^m}^r} \sim \left(\sum_{k=1}^m 2^{rqk} \left(E_{2^k-1}^{2^m}(f)_p \right)^q \right)^{\frac{1}{q}} .$$

2 *Properties for families of function classes* $\left\{ B_{p,q,N}^r \right\}_N$

Embedding Theorem 6 is the inverse of Embedding Theorem 1 for the same parameter values.

Theorem 6. Let $m \in \mathrm{N}$, $1 \le p, q, q_0, q_1 \le \infty$, $0 < \theta < 1$, $r > 0$, $r_0 > r_1 > 0$, then the embedding of the following type

$$\left(B_{p,q_0,2^m}^{r_0}; B_{p,q_1,2^m}^{r_1} \right)_{\theta,q} \hookrightarrow B_{p,q,2^m}^r$$

is satisfied, where $r = (1-\theta)\, r_0 + \theta\, r_1$.

Proof. From Theorem 2 for families of function classes $\left\{ B_{p,q,N}^r \right\}_N$, it follows that

$$B_{p,q,N}^r \hookrightarrow B_{p,\infty,N}^r .$$

Then, to prove the statement of the theorem, it is sufficient to prove the following embedding

$$\left(B_{p,\infty,2^m}^{r_0}; B_{p,\infty,2^m}^{r_1} \right)_{\theta,q} \hookrightarrow B_{p,q,2^m}^r .$$

Let $f = f_0 + f_1$ be an arbitrary representation of a function f, where $f_0 \in B_{p,\infty,2^m}^{r_0}$, $f_1 \in B_{p,\infty,2^m}^{r_1}$. Using Theorem 3 and Definition 1, we estimate the following expression

$$2^{r_0 k} \delta_k(f)_p \le 2^{r_0 k} \delta_k(f_0)_p + 2^{(r_0 - r_1)k} 2^{r_1 k} \delta_k(f_1)_p \le$$

$$\le \|f_0\|_{B_{p,\infty,2^m}^{r_0}} + 2^{(r_0 - r_1)k} \|f_1\|_{B_{p,\infty,2^m}^{r_1}} .$$

Given the arbitrariness of the representation for the function f and using Definition 4, we receive the ratio

$$2^{r_0 k} \delta_k(f)_p \le \inf_{f = f_0 + f_1} \left(\|f_0\|_{B_{p,\infty,2^m}^{r_0}} + 2^{(r_0 - r_1)k} \|f_1\|_{B_{p,\infty,2^m}^{r_1}} \right) =$$

$$= K\left(2^{(r_0 - r_1)k}, f; B_{p,\infty,2^m}^{r_0}, B_{p,\infty,2^m}^{r_1} \right),$$

where K is Petre's functional.

Therefore, we obtain the following inequality

$$\|f\|_{B_{p,q,2^m}^r} \le \left(\sum_{k=1}^m 2^{rqk} \left(\delta_k(f)_p \right)^q \right)^{\frac{1}{q}} \le$$

$$\le \left(\sum_{k=1}^m 2^{(r - r_0)qk} \left\{ K\left(2^{(r_0 - r_1)k}, f; B_{p,\infty,2^m}^{r_0}, B_{p,\infty,2^m}^{r_1} \right) \right\}^q \right)^{\frac{1}{q}} . \tag{2}$$

Taking into account that $r - r_0 = -\theta\,(r_0 - r_1)$, considering the partition of the interval $(0;\infty)$ into half-intervals $\left[2^{k(r_0-r_1)}; 2^{(k+1)(r_0-r_1)}\right)$, $k \in \mathrm{Z}$, and using Definition 5, we transform inequality (2) in the following way

$$\|f\|_{B^r_{p,q,2^m}} \le \left(\sum_{k\in \mathrm{Z}} 2^{-\theta(r_0-r_1)qk}\left\{K\left(2^{(r_0-r_1)k},\ f;\ B^{r_0}_{p,\infty,2^m},\ B^{r_1}_{p,\infty,2^m}\right)\right\}^q\right)^{\frac{1}{q}} \le$$

$$\le \left(\sum_{k\in Z}\left\{2^{-\theta(r_0-r_1)k}K\left(2^{(r_0-r_1)k}, f;\ B^{r_0}_{p,\infty,2^m},\ B^{r_1}_{p,\infty,2^m}\right)\right\}^q \times\right.$$

$$\left.\times 2^{-(r_0-r_1)k}\left(2^{(r_0-r_1)(k+1)} - 2^{(r_0-r_1)k}\right)\right)^{\frac{1}{q}} \le$$

$$\le \left(\int_0^\infty \left\{t^{-\theta}K\left(t, f;\ B^{r_0}_{p,\infty,2^m},\ B^{r_1}_{p,\infty,2^m}\right)\right\}^q \frac{dt}{t}\right)^{\frac{1}{q}} = \|f\|_{\left(B^{r_0}_{p,\infty,2^m};B^{r_1}_{p,\infty,2^m}\right)_{\theta,q}}.$$

As a result, we have

$$\|f\|_{B^r_{p,\infty,2^m}} \le C\,\|f\|_{\left(B^{r_0}_{p,\infty,2^m};B^{r_1}_{p,\infty,2^m}\right)_{\theta,q}}.$$

This inequality determines the required statement of the theorem. The theorem is proved.

The following corollary follows directly from Theorems 1, 6 and Definitions 2, 3.

Corollary 1. Let $m \in \mathrm{N}$, $1 \le p, q, q_0, q_1 \le \infty$, $0 < \theta < 1$, $r > 0$, $r_0 > r_1 > 0$, then the equality

$$\left(B^{r_0}_{p,q_0,2^m}; B^{r_1}_{p,q_1,2^m}\right)_{\theta,q} = B^r_{p,q,2^m}$$

is correct, where $r = (1-\theta)\ r_0 + \theta\, r_1$.

3 Relationship for the families of function classes with Besov spaces

The following theorems show the connection between families of function classes $\left\{B^r_{p,q,N}\right\}_N$ and the classical Besov spaces.

Theorem 7. [17] If $N \in \mathrm{N}$, $1 \le p, q, \le \infty$, $r > 0$, then the following relation is fulfilled

$$\bigcap_{N=1}^{\infty} B^r_{p,q,N} = B^r_{p,q}.$$

Theorem 8. Let $1 \le p \le q \le \infty$, $1 \le r \le \infty$, $m \in \mathrm{N}$, $\alpha > 0$, $\beta > 0$, $\alpha - \beta = \frac{1}{p} - \frac{1}{q}$, then for any value of m the following embedding

$$B^\alpha_{p,r} \hookrightarrow B^\beta_{q,r,2^m}$$

holds, that is, the inequality

$$\|f\|_{B^\beta_{q,r,2^m}} \le C\,\|f\|_{B^\alpha_{p,r}}$$

is satisfied, where C is a constant that does not depend on f and m.

Proof. First, we show that the following inequality

$$\|f\|_{B^\beta_{q,\infty,2^m}} \le \|f\|_{B^\beta_{q,\infty}} \tag{3}$$

holds. Indeed, the above inequality is correct, since it follows from the relation

$$\|f\|_{B^{\beta}_{q,\infty,2^m}} = \max_{1\le k\le m} 2^{\beta k} E^{2^m}_{2^{k-1}}(f)_q \le \sup_{k\ge 1} 2^{\beta k} E_{2^k}(f)_q = \|f\|_{B^{\beta}_{q,\infty}}.$$

According to the Besov embedding theorem [16], for $1 \le p \le q \le \infty$, $\alpha - \beta = \frac{1}{p} - \frac{1}{q}$, we have such inequality

$$\|f\|_{B^{\beta}_{q,\infty}} \le C\,\|f\|_{B^{\alpha}_{p,\infty}}, \tag{4}$$

where C is a constant that does not depend on f and m. Based on (3) and (4), we get an inequality of the form

$$\|f\|_{B^{\beta_i}_{q,\infty,2^m}} \le C_i\,\|f\|_{B^{\alpha_i}_{p,\infty}}, \tag{5}$$

where C_i are constants that do not depend on f and m, $i = 0,\ 1$.

With the parameters α, β, p, q specified in the conditions of the theorem, we define the operator I as follows

$$If = f.$$

Let us take pairs (α_0, α_1) and (β_0, β_1), where the parameters α_0, β_0, α_1, β_1 satisfy the conditions

$$\alpha_0 < \alpha < \alpha_1,\ \beta_0 < \beta < \beta_1,\ \alpha_0 - \beta_0 = \alpha_1 - \beta_1 = \frac{1}{p} - \frac{1}{q}.$$

Taking into account inequality (5), we have

$$I : B^{\alpha_0}_{p,\infty} \to B^{\beta_0}_{q,\infty,2^m}$$

with the norm C_0, and

$$I : B^{\alpha_1}_{p,\infty} \to B^{\beta_1}_{q,\infty,2^m}$$

with the norm C_1. Then, by the interpolation theorem [11], we obtain

$$I : \left(B^{\alpha_0}_{p,\infty}; B^{\alpha_1}_{p,\infty}\right)_{\theta,r} \to \left(B^{\beta_0}_{q,\infty,2^m}; B^{\beta_1}_{q,\infty,2^m}\right)_{\theta,r}. \tag{6}$$

According to Petre's theorem [11], the equality

$$\left(B^{\alpha_0}_{p,\infty}; B^{\alpha_1}_{p,\infty}\right)_{\theta,r} = B^{\alpha_\theta}_{p,r} \tag{7}$$

is valid, where

$$\alpha_\theta = (1-\theta)\,\alpha_0 + \theta\alpha_1,$$

$0 < \theta < 1$. Using a similar equality for families of classes of functions $\left\{B^r_{p,q,N}\right\}_N$, we have the following relation

$$\left(B^{\beta_0}_{q,\infty,2^m}; B^{\beta_1}_{q,\infty,2^m}\right)_{\theta,r} = B^{\beta_\theta}_{q,r,2^m}, \tag{8}$$

where

$$\beta_\theta = (1-\theta)\,\beta_0 + \theta\beta_1,$$

$0 < \theta < 1$. Since

$$\alpha_\theta - \beta_\theta = (1-\theta)(\alpha_0 - \beta_0) + \theta(\alpha_1 - \beta_1) = \frac{1}{p} - \frac{1}{q},$$

then there exists $\theta \in (0;1)$ such that

$$\beta_\theta = \beta \ \Rightarrow\ \alpha_\theta = \alpha. \tag{9}$$

As a result, taking into account (7)–(9), relation (6) determines that

$$I : B^{\alpha}_{p,r} \to B^{\beta}_{q,r,2^m},$$

and besides

$$\|f\|_{B^{\beta}_{q,r,2^m}} \leq C \, \|f\|_{B^{\alpha}_{p,r}} .$$

This inequality proves the statement of the theorem. The theorem is proved.

In particular, the following lemma is correct for the conditions of Theorem 6.

Lemma 2. Let $1 \leq p \leq q \leq \infty$, $1 \leq r \leq \infty$; $m, n \in \mathrm{N}$, $\alpha > 0$, $\beta > 0$, $\alpha - \beta = \frac{1}{p} - \frac{1}{q}$. For functions f that are summable to the p-th power, the inequality

$$\left(\sum_{k=1}^{m} 2^{\beta\, rk} \left(\left\| f - \frac{1}{2^{m+1}} \sum_{n=0}^{2^{m+1}-1} f\left(x + \frac{\pi\, n}{2^m}\right) D_{2^k-1}\left(\frac{\pi\, n}{2^m}\right) \right\|_q \right)^r \right)^{\frac{1}{r}} \leq C \, \|f\|_{B^{\alpha}_{p,r}}$$

holds, where $D_{2^k-1}\left(\frac{\pi\, n}{2^m}\right)$ is the Dirichlet kernel.

Proof. By successively applying (1), Theorems 4 and 5, we obtain the relation

$$\left(\sum_{k=1}^{m} 2^{\beta rk} \left(\left\| f - \frac{1}{2^{m+1}} \sum_{n=0}^{2^{m+1}-1} f\left(x + \frac{\pi n}{2^m}\right) D_{2^k-1}\left(\frac{\pi n}{2^m}\right) \right\|_q \right)^r \right)^{\frac{1}{r}} =$$

$$= \left(\sum_{k=1}^{m} 2^{\beta\, rk} \left(\left\| f - S^{2^m}_{2^k-1}(f) \right\|_q \right)^r \right)^{\frac{1}{r}} \sim$$

$$\sim \left(\sum_{k=1}^{m} 2^{\beta\, rk} \left(E^{2^m}_{2^k-1}(f)_q \right)^r \right)^{\frac{1}{r}} \sim \|f\|_{B^{\beta}_{q,r,2^m}}.$$

From this relation, taking into account the ratio

$$\|f\|_{B^{\beta}_{q,r,2^m}} \leq C \, \|f\|_{B^{\alpha}_{p,r}} ,$$

the required inequality follows. The lemma is proved.

Conclusion

In the article the properties of families of function classes $\left\{ B^{r}_{p,q,N} \right\}_N$ are presented. Embedding theorems for the specified families of function classes and the lemma on estimating the norm of the Besov space are proved. The embedding theorem showing the relationship of these families of function classes to the classical Besov spaces is proved by the method of real interpolation. The results presented in the article will be used in future directions of research on the approximation of functions over harmonic intervals.

Author Contributions

L.A. Serikova conducted the research presented in the article under the scientific supervision of G.A. Yessenbayeva. All authors contributed equally to this work and participated in the revision of the manuscript and approved the final submission.

Conflict of Interest

The authors declare no conflict of interest.

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