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CONTENT

MATHEMATICS

Recent advances in analysis and applied mathematics and their applications
PREFACE

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Keywords: partial differential equations, integro-differential equations, boundary value problems, Dirichlet problem; well-posedness, regular solutions, identification problems, delay differential operators with involution, numerical methods and solutions, difference schemes, involution, stability.

2020 Mathematics Subject Classification: 34K28, 35A35, 35G15, 35J15, 35J25, 35K10, 35K60, 35L05, 35L10, 35L35, 35M10, 35M12, 35N25, 35R30, 35S15, 39A14, 42A10, 42B10, 45J05, 45J99, 47A62, 47B39, 54E15, 54D20, 58J05, 58J32, 58J99, 58JXX, 60H30, 65M06, 65H10, 65H30, 65J22, 65N06, 82D75

This issue is a collection of 14 selected papers of foreign and national scientists. All of these have been accepted after peer review and contain numerous new results in the fields of analysis and applied mathematics, including their applications to constructing and investigating solutions for well-posed and ill-posed boundary value problems for partial differential equations. The authors of the selected papers are from different countries: Turkey, Kazakhstan, Sweden, Russian Federation, Azerbaijan, Kirgizistan, Uzbekistan, and Turkmenistan. Especially, we are pleased with the fact that many articles are written by co-authors who work in different universities around the world.

Guest-Editors: *A. Ashyralyev, C. Ashyralyyev* and *M. Sadybekov*

June 12, 2024

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MATHEMATICS

https://doi.org/10.31489/2024M3/5-12 *Research article*

The boundary value problem for an ordinary linear half-order differential equation

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This study is devoted to the study of the solution of a boundary value problem for an ordinary linear differential equation of half order with constant coefficients. Using of the fundamental solution of the main part of the considered equation, we obtained the principal relations, from which we obtain the necessary conditions for the Fredholm property of the original problem. Further, using the Mittag-Leffler function, a general solution of the homogeneous equation is obtained. Finally, the problem under consideration is reduced to an integral Fredholm equation of the second kind with a non-singular kernel, i.e., the Fredholm property of the stated problem is proved.

Keywords: half-order equations, boundary value problem, fundamental solution, basic relation, integral equations, Fredholm property, Mittag-Leffler functions, general solution of a homogeneous half-order equation.

2020 Mathematics Subject Classification: 34A08.

Introduction

Most investigations in different fields of science and engineering are modeled with the help of differential equations (or systems of equations) with fractional derivatives. The concept of fractional calculus has gained considerable popularity and importance during the past half decades. The concept of the fractional calculus takes beginning from outstanding learned as Marquis de L'Hopital, G.W. Leibniz, Fourier, Laplace, Liouville, Riemann, Letnikov etc, as gained considerable popularity and importance during the past half decades, in [1–5].

The study of solving boundary value problems is closely related to the Green's function. The construction of the Green's function is not an easy task, since it is related to the considered equations and the boundary condition [6–8].

Problems of the Cauchy type for an ordinary linear differential equation of fractional order, in particular half-order, are studied in [1, 2, 4, 5], where these problems are reduced to Volterra integral equations of the second kind. Constructing of a fundamental solution is much easier than constructing

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the Green's function, since it is associated only with the equation under consideration. In [9] and [10] solutions of some classes of Cauchy problems containing fractional differential operators are established.

In [11–13] for some class of the fractional order equations of fundamental solutions constructed. This article is devoted to the study of a boundary value problem for an ordinary linear differential equation of half order. We used the fundamental solution proposed in [6], where a fundamental solution was constructed for a wide class of differential equations.

With the help of the fundamental solution of the main part of the considered equation, the main relations is obtained, from which the necessary conditions for Fredholm property are proved. Further, with the help of the Mittag-Leffler function the general solution of the homogeneous equation is obtained.

Let us consider the following problem

$$D^{\frac{1}{2}}_{x_1-}y(x) - ay(x) = f(x), \quad 0 < x_0 < x < x_1, \tag{1}$$

$$y(x_1) + \alpha y(x_0) = 0, \tag{2}$$

where a, x_0, x_1 and α are given constants, $f(x)$ is a known continuous function defined on $[x_0, x_1]$ and $y(x)$ is an unknown function that is required to define, and

$$D^{\frac{1}{2}}_{x_1-}y(x) = -\frac{d}{dx}\int_x^{x_1} \frac{(x-t)^{-\frac{1}{2}}}{(-\frac{1}{2})!}y(t)dt, \quad x < x_1$$

is left half order derivative of the function $y(x)$ [1],

$$\Big(-\frac{1}{2}\Big)! = \Gamma\Big(\frac{1}{2}\Big) = \int_0^\infty e^{-t}t^{-\frac{1}{2}}dt.$$

Here, Γ is Euler's gamma function.

In order to construct solution of the considered problem we use of the fundamental solution of the conjugate equation corresponding to Eq. (1)

$$D^{\frac{1}{2}}_{x_0}y(x) - ay(x) = f(x), \quad 0 < x_0 < x < x_1, \tag{3}$$

where

$$D^{\frac{1}{2}}_{x_0}y(x) = \frac{d}{dx}\int_{x_0}^{x} \frac{(x-t)^{-\frac{1}{2}}}{(-\frac{1}{2})!}y(t)dt, \quad x > x_0.$$

Let $f(x), g(x) \in C[x_0, x_1]$ and $D^{\frac{1}{2}}_{x_0}g(x)$, $D^{\frac{1}{2}}_{x_1-}f(x)$ exist on $[x_0, x_1]$, where $C[x_0, x_1]$ is a class of continuously functions in $[x_0, x_1]$. According to [13] the following equality holds

$$\int_{x_0}^{x_1}\Big(D^{\frac{1}{2}}_{x_0}f(x)\Big)g(x)dx = \int_{x_0}^{x_1} f(x)\Big(D^{\frac{1}{2}}_{x_1-}g(x)\Big)dx. \tag{4}$$

Easy to see that $\frac{x^{-\frac{1}{2}}}{(-\frac{1}{2})!}$ is a fundamental solution for the main part of Eq. (3). Indeed [1],

$$D^{\frac{1}{2}}_{x_0}\frac{x^{-\frac{1}{2}}}{(-\frac{1}{2})!} = \frac{x^{-1}}{(-1)!} = \delta(x).$$

Here $\delta(x)$ is Dirac's function.

Then multiplying Eq. (1) by

$$\frac{(t-\xi)^{-\frac{1}{2}}}{\left(-\frac{1}{2}\right)!}+Cy_h(t),$$

and integrating over x on the open interval (x_0, x_1), we have

$$\int_{x_0}^{x_1}\left(\frac{(x-\xi)^{-\frac{1}{2}}}{\left(-\frac{1}{2}\right)!}+Cy_h(x)\right)D_{x_1-}^{\frac{1}{2}}y(x)dx-a\int_{x_0}^{x_1}\left(\frac{(x-\xi)^{-\frac{1}{2}}}{\left(-\frac{1}{2}\right)!}+Cy_h(x)\right)y(x)dx=$$

$$=\int_{x_0}^{x_1}\left(\frac{(x-\xi)^{-\frac{1}{2}}}{\left(-\frac{1}{2}\right)!}+Cy_h(x)\right)f(x)dx. \tag{5}$$

Here,

$$y_h(x)=\sum_{k=0}^{\infty}a^k\frac{x^{\frac{k-1}{2}}}{\left(\frac{k-1}{2}\right)!} \tag{6}$$

is a partial solution and is the homogeneous solution corresponding to Eq. (3). Indeed,

$$D_{x_0}^{\frac{1}{2}}y(x)=\sum_{k=0}^{\infty}a^k\frac{x^{\frac{k-2}{2}}}{\left(\frac{k-2}{2}\right)!}=\frac{x^{-1}}{(-1)!}+a\frac{x^{-\frac{1}{2}}}{\left(-\frac{1}{2}\right)!}+a^2\frac{x^0}{(0)!}+a^3\frac{x^{\frac{1}{2}}}{\left(\frac{1}{2}\right)!}+\cdots=$$

$$=\delta(x)+a\left[\frac{x^{-\frac{1}{2}}}{\left(-\frac{1}{2}\right)!}+a\frac{x^0}{(0)!}+a^2\frac{x^{\frac{1}{2}}}{\left(\frac{1}{2}\right)!}+\cdots\right]=ay(x).$$

Then the general solution of Eq. (3) has the form

$$y_h(x)=C\sum_{k=0}^{\infty}a^k\frac{x^{\frac{k-1}{2}}}{\left(\frac{k-1}{2}\right)!},$$

where C is an arbitrary constant.

Taking into account Eq. (4), we get from Eq. (5)

$$\int_{x_0}^{x_1}y(x)dxD_{x_0}^{\frac{1}{2}}\left[\frac{(x-\xi)^{-\frac{1}{2}}}{\left(-\frac{1}{2}\right)!}+Cy_h(x)\right]-a\int_{x_0}^{x_1}\left[\frac{(x-\xi)^{-\frac{1}{2}}}{\left(-\frac{1}{2}\right)!}+Cy_h(x)\right]y(x)dx=$$

$$=\int_{x_0}^{x_1}\left[\frac{(x-\xi)^{-\frac{1}{2}}}{\left(-\frac{1}{2}\right)!}+Cy_h(x)\right]f(x)dx$$

or

$$\int_{x_0}^{x_1}D_{x_0}^{\frac{1}{2}}\left(\frac{(x-\xi)^{-\frac{1}{2}}}{\left(-\frac{1}{2}\right)!}\right)y(x)dx+\int_{x_0}^{x_1}D_{x_0}^{\frac{1}{2}}\left[Cy_h(x)\right]y(x)dx-$$

$$-a\int_{x_0}^{x_1}\left(\frac{(x-\xi)^{-\frac{1}{2}}}{\left(-\frac{1}{2}\right)!}\right)y(x)dx-a\int_{x_0}^{x_1}Cy_h(x)y(x)dx=\int_{x_0}^{x_1}\left[\frac{(x-\xi)^{-\frac{1}{2}}}{\left(-\frac{1}{2}\right)!}+Cy_h(x)\right]f(x)dx.$$

According to

$$D_{x_0}^{\frac{1}{2}}\left(\frac{(x-\xi)^{-\frac{1}{2}}}{\left(-\frac{1}{2}\right)!}\right)=\frac{(x-\xi)^{-1}}{(-1)!}=\delta(x-\xi)$$

from the last relation, we have

$$\int_{x_0}^{x_1} y(x)\delta(x-\xi)dx + \int_{x_0}^{x_1} C\left[D_{x_0}^{\frac{1}{2}}\big(y_h(x)\big) - ay_h(x)\right]y(x)dx -$$

$$-a\int_{x_0}^{x_1}\left(\frac{(x-\xi)^{-\frac{1}{2}}}{\left(-\frac{1}{2}\right)!}\right)y(x)dx = \int_{x_0}^{x_1}\left[\frac{(x-\xi)^{-\frac{1}{2}}}{\left(-\frac{1}{2}\right)!} + Cy_h(x)\right]f(x)dx.$$

From here we get the following main relation

$$a\int_{x_0}^{x_1}\frac{(x-\xi)^{-\frac{1}{2}}}{\left(-\frac{1}{2}\right)!}y(x)dx + \int_{x_0}^{x_1}\left[\frac{(x-\xi)^{-\frac{1}{2}}}{\left(-\frac{1}{2}\right)!} + Cy_h(x)\right]f(x)dx =$$

$$= \begin{cases} y(\xi), & \xi \in (x_0, x_1), \\ \frac{1}{2}y(x_0), & \xi = x_0, \\ \frac{1}{2}y(x_1), & \xi = x_1, \end{cases}$$

or

$$a\int_{x_0}^{x_1}\frac{(x-\xi)^{-\frac{1}{2}}}{\left(-\frac{1}{2}\right)!}y(x)dx + \int_{x_0}^{x_1}\left[\frac{(x-\xi)^{-\frac{1}{2}}}{\left(-\frac{1}{2}\right)!} + C\sum_{k=0}^{\infty}a^k\frac{x^{\frac{k-1}{2}}}{\left(\frac{k-1}{2}\right)!}\right]f(x)dx =$$

$$= \begin{cases} y(\xi), & \xi \in (x_0, x_1), \\ \frac{1}{2}y(x_0), & \xi = x_0, \\ \frac{1}{2}y(x_1), & \xi = x_1. \end{cases} \tag{7}$$

Thus, based on the fundamental solution for the main part of Eq. (3), for the fractional order equation, we obtained the main relation (7), which consists of two parts. The first part, where $\xi \in (x_0, x_1)$ gives any solution of Eq. (3), and the second part, where $\xi = x_0$, or $\xi = x_1$ gives us the necessary conditions. With this, for each solution of the inhomogeneous Eq. (1), the boundary values are obtained in the main relation (7).

Thus, for $x \in (x_0, x_1)$ for the general solution of Eq. (3) we have the following representation

$$y(\xi) = a\int_{x_0}^{x_1}\frac{(x-\xi)^{-\frac{1}{2}}}{\left(-\frac{1}{2}\right)!}y(x)dx + \int_{x_0}^{x_1}\left[\frac{(x-\xi)^{-\frac{1}{2}}}{\left(-\frac{1}{2}\right)!} + C\sum_{k=0}^{\infty}a^k\frac{x^{\frac{k-1}{2}}}{\left(\frac{k-1}{2}\right)!}\right]f(x)dx \tag{8}$$

and for boundary points $\xi = x_0$, and $\xi = x_1$ we get relations

$$\begin{cases} \frac{1}{2}y(x_0) = \int_{x_0}^{x_1}\frac{(x-x_0)^{-\frac{1}{2}}}{\left(-\frac{1}{2}\right)!}y(x)dx + \int_{x_0}^{x_1}\left[\frac{(x-x_0)^{-\frac{1}{2}}}{\left(-\frac{1}{2}\right)!} + C\sum_{k=0}^{\infty}a^k\frac{x^{\frac{k-1}{2}}}{\left(\frac{k-1}{2}\right)!}\right]f(x)dx, \\ \frac{1}{2}y(x_1) = \int_{x_0}^{x_1}\frac{(x-x_1)^{-\frac{1}{2}}}{\left(-\frac{1}{2}\right)!}y(x)dx + \int_{x_0}^{x_1}\left[\frac{(x-x_1)^{-\frac{1}{2}}}{\left(-\frac{1}{2}\right)!} + C\sum_{k=0}^{\infty}a^k\frac{x^{\frac{k-1}{2}}}{\left(\frac{k-1}{2}\right)!}\right]f(x)dx. \end{cases} \tag{9}$$

Putting (9) in boundary condition (2), we can define the arbitrary constant C

$$2a\int_{x_0}^{x_1}\frac{(x-x_1)^{-\frac{1}{2}}}{\left(-\frac{1}{2}\right)!}y(x)dx + 2\int_{x_0}^{x_1}\left[\frac{(x-x_1)^{-\frac{1}{2}}}{\left(-\frac{1}{2}\right)!} + C\sum_{k=0}^{\infty}a^k\frac{x^{\frac{k-1}{2}}}{\left(\frac{k-1}{2}\right)!}\right]f(x)dx +$$

$$+2\alpha\left[a\int_{x_0}^{x_1}\frac{(x-x_0)^{-\frac{1}{2}}}{(-\frac{1}{2})!}y(x)dx+\int_{x_0}^{x_1}\left[\frac{(x-x_0)^{-\frac{1}{2}}}{(-\frac{1}{2})!}+C\sum_{k=0}^{\infty}a^k\frac{x^{\frac{k-1}{2}}}{(\frac{k-1}{2})!}\right]f(x)dx\right]=0$$

or

$$2a\int_{x_0}^{x_1}\frac{(x-x_1)^{-\frac{1}{2}}}{(-\frac{1}{2})!}y(x)dx+2\alpha a\int_{x_0}^{x_1}\frac{(x-x_0)^{-\frac{1}{2}}}{(-\frac{1}{2})!}y(x)dx+$$

$$+2\int_{x_0}^{x_1}\left[\frac{(x-x_1)^{-\frac{1}{2}}}{(-\frac{1}{2})!}\right]f(x)dx+2\alpha\int_{x_0}^{x_1}\left[\frac{(x-x_0)^{-\frac{1}{2}}}{(-\frac{1}{2})!}\right]f(x)dx+$$

$$+2\int_{x_0}^{x_1}\left[C\sum_{k=0}^{\infty}a^k\frac{x^{\frac{k-1}{2}}}{(\frac{k-1}{2})!}\right]f(x)dx+2\alpha\int_{x_0}^{x_1}\left[C\sum_{k=0}^{\infty}a^k\frac{x^{\frac{k-1}{2}}}{(\frac{k-1}{2})!}\right]f(x)dx=0.$$

Grouping similar terms, we have

$$a\int_{x_0}^{x_1}\frac{(x-x_1)^{-\frac{1}{2}}+\alpha(x-x_0)^{-\frac{1}{2}}}{(-\frac{1}{2})!}y(x)dx+\int_{x_0}^{x_1}\frac{(x-x_1)^{-\frac{1}{2}}+\alpha(x-x_0)^{-\frac{1}{2}}}{(-\frac{1}{2})!}f(x)dx+$$

$$+C(1+\alpha)\int_{x_0}^{x_1}\sum_{k=0}^{\infty}a^k\frac{x^{\frac{k-1}{2}}}{(\frac{k-1}{2})!}f(x)dx=0,$$

or

$$C(1+\alpha)\int_{x_0}^{x_1}\sum_{k=0}^{\infty}a^k\frac{x^{\frac{k-1}{2}}}{(\frac{k-1}{2})!}f(x)dx=-a\int_{x_0}^{x_1}\frac{(x-x_1)^{-\frac{1}{2}}+\alpha(x-x_0)^{-\frac{1}{2}}}{(-\frac{1}{2})!}y(x)dx-$$

$$-\int_{x_0}^{x_1}\frac{(x-x_1)^{-\frac{1}{2}}+\alpha(x-x_0)^{-\frac{1}{2}}}{(-\frac{1}{2})!}f(x)dx. \tag{10}$$

If

$$\Delta=\int_{x_0}^{x_1}\sum_{k=0}^{\infty}a^k\frac{x^{\frac{k-1}{2}}}{(\frac{k-1}{2})!}f(x)dx\neq 0, \tag{11}$$

then from Eq. (10) we obtain

$$C=-\frac{1}{\Delta(1+\alpha)}\left[a\int_{x_0}^{x_1}\frac{(x-x_1)^{-\frac{1}{2}}+\alpha(x-x_0)^{-\frac{1}{2}}}{(-\frac{1}{2})!}y(x)dx+\right.$$

$$\left.+\int_{x_0}^{x_1}\frac{(x-x_1)^{-\frac{1}{2}}+\alpha(x-x_0)^{-\frac{1}{2}}}{(-\frac{1}{2})!}f(x)dx\right]. \tag{12}$$

Finally, substituting Eq. (12) in Eq. (8), we have

$$y(\xi)=a\int_{x_0}^{x_1}\frac{(x-\xi)^{-\frac{1}{2}}}{(-\frac{1}{2})!}y(x)dx+\int_{x_0}^{x_1}\left[\frac{(x-\xi)^{-\frac{1}{2}}}{(-\frac{1}{2})!}\right]f(x)dx-$$

$$-\int_{x_0}^{x_1}\sum_{k=0}^{\infty}a^k\frac{x^{\frac{k-1}{2}}}{(\frac{k-1}{2})!}f(x)dx\left\{\frac{1}{\Delta(1+\alpha)}\left[a\int_{x_0}^{x_1}\frac{(x-x_1)^{-\frac{1}{2}}+\alpha(x-x_0)^{-\frac{1}{2}}}{(-\frac{1}{2})!}y(x)dx+\right.\right.$$

$$\left.\left.+\int_{x_0}^{x_1}\frac{(x-x_1)^{-\frac{1}{2}}+\alpha(x-x_0)^{-\frac{1}{2}}}{(-\frac{1}{2})!}f(x)dx\right]\right\}$$

or

$$y(\xi) = a\int_{x_0}^{x_1} \frac{(x-\xi)^{-\frac{1}{2}}}{(-\frac{1}{2})!} y(x)dx + \int_{x_0}^{x_1} \left[\frac{(x-\xi)^{-\frac{1}{2}}}{(-\frac{1}{2})!}\right] f(x)dx -$$

$$-\frac{a}{1+\alpha}\int_{x_0}^{x_1} \frac{(x-x_1)^{-\frac{1}{2}} + \alpha(x-x_0)^{-\frac{1}{2}}}{(-\frac{1}{2})!} y(x)dx - \frac{a}{1+\alpha}\int_{x_0}^{x_1} \frac{(x-x_1)^{-\frac{1}{2}} + \alpha(x-x_0)^{-\frac{1}{2}}}{(-\frac{1}{2})!} f(x)dx.$$

Thus, the solution of problem (3), (2), we reduce to the following integral equation

$$y(\xi) = a\int_{x_0}^{x_1} \frac{(x-\xi)^{-\frac{1}{2}} - \frac{1}{1+\alpha}\left((x-x_1)^{-\frac{1}{2}} + \alpha(x-x_0)^{-\frac{1}{2}}\right)}{(-\frac{1}{2})!} y(x)dx +$$

$$+\int_{x_0}^{x_1} \frac{(x-\xi)^{-\frac{1}{2}} - \frac{1}{1+\alpha}\left((x-x_1)^{-\frac{1}{2}} + \alpha(x-x_0)^{-\frac{1}{2}}\right)}{(-\frac{1}{2})!} f(x)dx.$$

Let us denote by $K(x,\xi)$ kernel in the last integral

$$K(x,\xi) = \frac{(x-\xi)^{-\frac{1}{2}} - \frac{1}{1+\alpha}\left((x-x_1)^{-\frac{1}{2}} + \alpha(x-x_0)^{-\frac{1}{2}}\right)}{(-\frac{1}{2})!}$$

then the solution of problem (3), (2) is reduced to the second type integral equation of the Fredholm with regular kernel as

$$y(\xi) = a\int_{x_0}^{x_1} K(x,\xi)y(x)dx + \int_{x_0}^{x_1} K(x,\xi)f(x)dx, \tag{13}$$

and so the following theorem is true.

Theorem 1. Let a and α be given positive constants and $f(x)$ by $x \in (x_0, x_1)$ known a continuous function. If series (6) is convergent and take place (11), then the boundary value problem (3), (2) has the Fredholm property.

The actual solution of problem (1), (2) can be obtained from Eq. (13) either by the method of successive approximations [14], or by replacing the integral entering in (13) with any approximate integration formula, for example method of trapeze, Simpson, etc.

Conclusion

For the first time, using the fundamental solution of the main part of the conjugate corresponding to the main equation, we obtained the main relations from which the necessary conditions for the Fredholm property of the original problem are obtained.

With this, for each solution of the inhomogeneous Eq. (3) the boundary values are obtained mainly from relation (9).

Author Contributions

N. Aliyev drew attention to the issue. B. Sinsoysal researched relevant literature and assisted in the research process. M. Rasulov and B. Sinsoysal together did all the theoretical work and implemented the process of writing the article. All authors participated in the revision of the manuscript and approved the final submission. All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

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On stability of nonlinear difference equations and some of their applications

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The issues of stability in solving nonlinear difference equations were considered. Based on a generalized difference analog of the well-known Bihari lemma, stability conditions for a trivial solution based on initial data were obtained, and an a priori estimate of stability under permanent disturbances was determined. The results were used to study the stability of solving explicit and implicit difference schemes approximating nonlinear parabolic equations.

Keywords: nonlinear difference equations, difference schemes, a priori estimates, stability.

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Introduction

Many applied problems are reduced to the solution of nonlinear difference equations. One of the main issues in the theory of difference equations is the study of the stability of their solution. Therefore, it is of particular interest to study the stability of solutions of linear and nonlinear difference equations. The concept of stability of solutions to difference equations was first formulated by O. Perron [1] as an analog of the stability of differential equations. Then, numerous works appeared devoted to the study of the stability of difference equations. Currently, methods for studying the stability of linear difference equations with constant coefficients are quite well-developed (we do not consider equations with periodic coefficients). However, the study of the stability of linear difference equations with variable coefficients and nonlinear difference equations were not sufficient, since there were no effective criteria for the stability of their solutions. It should be noted that many problems are reduced to the solution of difference equations with variable coefficients and nonlinear difference equations. For example, such problems are posed when numerically solving differential equations using finite difference or finite element methods [2–6].

The stability of systems of linear difference equations with constant and variable coefficients was studied in [7, 8]. O. Perron [7] formulated the concept of stability of solutions of a system of difference equations with constant coefficients by analogy with this concept for differential equations. In [8] P.I. Koval studied the stability of linear difference equations with variable coefficients. He considered the difference equation in vector-matrix form:

$$y_{n+1} = Ay_n + b_n, \ n = 1, 2, ..., \tag{1}$$

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where $\{y_n\}$ is the sought-for sequence of vectors, $\{A_n\}$ is the given sequence of matrices, and $\{b_n\}$ is the given norm-bounded sequence of vectors. It was proven that the solution of system (1) is stable if matrix A_n of the corresponding homogeneous system

$$y_{n+1} = A_n y_n$$

satisfies condition $\|A_n\| \leq 1 + q_n$, where $\sum\limits_{n=n_0}^{\infty} q_n < \infty$, and asymptotically stable, if $\|A_n\| \leq a < 1$ $(n > n_0)$. Next, the so-called limiting matrix $A = \lim\limits_{n\to\infty} A_n$ is introduced and, on its basis, the stability and instability of difference equations of the form (1) are studied. In [8] P.I. Koval considered linear difference equations that could be reduced to almost triangular form using linear substitutions. The asymptotic behavior of linear difference equations with almost triangular matrices was also studied there.

M.A. Skalkina in [9] showed the connection between the stability of differential and difference equations. V.B. Demidovich in [10, 11] studied the stability of nonlinear difference equations based on the first Lyapunov method. At that point, the concept of characteristic numbers of a system of linear difference equations was introduced. The concepts of reducible and regular systems of linear difference equations were introduced. In particular, it was shown that every reducible system is regular. In addition, stability under the first approximation was studied. The main result of these studies is the theorem on the asymptotic stability of the system

$$y_{n+1} = S_n y_n + f_n(y_n),$$

where $f_n(y_n)$ is the nonlinear term, S_n is the transition operator.

Nonlinear difference equations, the right-hand sides of which are linear combinations of power functions of phase variables, were studied in [12]. In addition, similar studies for differential and difference equations were carried out in [13–15].

In this article, issues of stability of the solution of nonlinear difference equations are studied. Various stability criteria are obtained, based on which nonlinear two-layer difference schemes are studied. A theorem on the stability of a trivial solution with respect to initial data is proven. The difference analog of Behari's lemma is generalized and, on its basis, an a priori estimate of the stability under permanent disturbances of a nonlinear difference equation is obtained. Examples of application of the theorem to explicit and implicit difference schemes approximating nonlinear parabolic equations are considered. Examples are given that confirm the theoretical results obtained.

1 Statement of the problem

Let us consider the Cauchy problem

$$\dot{u}(t) + A(t)u(t) = f(t, u), \;\; u(0) = u_0, \;\; \dot{u} = dt/du, \tag{2}$$

where A is a slowly varying matrix.

Equation (1) is obtained by spatial discretization of a partial differential equation of parabolic type

$$\partial t/\partial u = Lu, \;\; u(0) = u_0, \tag{3}$$

where $Lu \in H$ is some general form of a nonlinear differential operator. Here, H is the Hilbert space with scalar product (u, ϑ) and norm $\|u\| = \sqrt{(u, u)}$. Such problems arise in the mathematical modeling of processes of chemical kinetics, combustion theory, biophysics, various kinds of biochemical reactions (reaction-diffusion), convection-diffusion, processes of population growth and migration, etc.

Any two-layer difference scheme [1] that approximates problem (2) or (3) can be written in the following form of the difference equation:

$$y_{n+1} = S_n y_n + \tau f_n(y_n), \;\; y(0) = y_0, \;\; n = 0, 1, ..., \tag{4}$$

where y is a grid function that approximates function u, $y_n = y(t_n)$, $t_n \in \overline{\omega}_\tau$, $\overline{\omega}_\tau = \{t_n = n\tau, n = 0, 1, ...\}$, $\tau > 0$ is a uniform grid in time $t \in [0, T]$, S_n is a certain operator (transition operator), $f_n(y_n)$ is a nonlinear term.

Let us study the stability of the trivial solution of equation (4).

Along with (4), we consider the following homogeneous equation:

$$y_{n+1} = S_n y_n, \;\; y(0) = y_0, \;\; n = 0, 1, ... \tag{5}$$

The stability of solutions of the nonlinear non-homogeneous equation (4) is completely determined by the stability of the trivial solution of its homogeneous equation (5) [10].

We consider the difference equation (4), where the nonlinear disturbance $f_n(y_n)$ satisfies the following conditions:

$$\|f_n(y_n)\| = K_n \|y_n\|^r, \;\; r > 1, \;\; f_n(0) = 0, \;\; \sum_{m=0}^{n-1} K_m \le M_1 < \infty, \tag{6}$$

where M_1 is some positive constant. In this case, the trivial sequence $y_n = 0$ is a solution to equation (4).

2 Stability theorems

Lemma 1. (The discrete analogue of Bihari's lemma) [10]. Let

$$0 \le y_0 \le c \;\; (c > 0) \tag{7}$$

and

$$y_n \le c + \sum_{v=0}^{n-1} a_v \varphi(y_v), \;\; n = 1, 2, ...,$$

where c is a positive constant, the sequence $y_i \ge 0$, $a_i \ge 0$, $i = 0, 1, ...$, $\varphi(y)$ is a continuous monotonically increasing positive function for $y > 0$, and $\varphi(0) \ge 0$, and let the inequality $\sum\limits_{v=0}^{n-1} a_v < \varphi(\infty)$ be satisfied, where $\varphi(z) = \int\limits_c^z \frac{dz_1}{\varphi(z_1)}$. Then the following estimate is valid:

$$y_n \le \varphi^{-1}\left(\sum_{v=0}^{n-1} a_v\right), \;\; n = 1, 2, ...$$

Corollary 1. Let $\varphi(y) = y^r$ $(r > 0)$, i.e. inequalities (7) be satisfied and

$$y_n \le c + \sum_{v=0}^{n-1} a_v y_v^r, \;\; n = 1, 2, ... \;,$$

where the sequence $y_i \ge 0$, $a_i \ge 0$, $i = 0, 1, ...$ Then, based on Lemma 1, we have:

$$y_n \le c \Big/ \left[1 - (r-1)c^{r-1} \sum_{v=0}^{n-1} a_v\right]^{1/(r-1)},$$

only if

$$\sum_{v=0}^{n-1} a_v < 1/\left[(r-1)c^{r-1}\right].$$

Let us generalize Lemma 1.

Lemma 2. Let the following inequality hold

$$\begin{gathered} 0 \le y_0 \le c_0 \ \ (c_0 > 0), \\ y_n \le c_n + \sum_{v=0}^{n-1} a_v \varphi(y_v), \ \ n = 1, 2, ..., \end{gathered} \tag{8}$$

where $y_i \ge 0$, $c_i > 0$, $a_i \ge 0$, $i = 0, 1, ...$, c_i is a non-decreasing sequence ($c_{i+1} \ge c_i$), $\varphi(y)$ is a homogeneous continuous monotonically increasing function ($\varphi(0) \ge 0$) of r-th order; and let the following inequality be satisfied:

$$c_n^{-1} \sum_{v=0}^{n-1} c_v^r a_v < \phi(\infty),$$

where

$$\phi(z) = \int_1^z \frac{dz_1}{\varphi(z_1)}.$$

Then the following estimate holds:

$$y_n \le c_n \phi^{-1}\left(c_n^{-1} \sum_{v=0}^{n-1} \widetilde{a}_v\right), \ \ n = 1, 2, ..., \tag{9}$$

where $\widetilde{a}_v = c_v^r a_v$.

Proof. We divide (8) by $c_n > 0$:

$$\frac{y_n}{c_n} \le 1 + \sum_{v=0}^{n-1} \frac{a_v}{c_n} \varphi(y_v), \quad \frac{y_0}{c_0} \le 1.$$

Since $c_n \ge c_v$, then from the last inequality considering homogeneity of $\varphi(y)$ it follows that

$$\frac{y_n}{c_n} \le 1 + \frac{1}{c_n} \sum_{v=0}^{n-1} c_v^r a_v \varphi\left(\frac{y_v}{c_v}\right), \quad \frac{y_0}{c_0} \le 1. \tag{10}$$

For inequality (10), we apply Lemma 1, which gives the following estimate:

$$\frac{y_n}{c_n} \le \phi^{-1}\left(\frac{1}{c_n} \sum_{v=0}^{n-1} c_v^r a_v\right),$$

where $\phi^{-1}(z)$ is the inverse function of $\phi(z)$. This gives us estimate (9).

Corollary 2. Let $\varphi(y) = y^r$ ($r > 1$) and the following inequalities be satisfied

$$0 < y_0 < c_0 \ \ (c_0 > 0),$$

$$y_n \le c_n + \sum_{v=0}^{n-1} a_v y_v^r, \ \ r > 1, \ \ n = 1, 2, ... \ ,$$

where $y_i \geq 0$, $c_i > 0$, $a_i \geq 0$, $i = 0, 1, ...$ Therefore, if

$$\sum_{v=0}^{n-1} c_v^{r-1} a_v < \frac{1}{r-1},$$

then based on Lemma 2, we have that

$$y_n \leq c_n / \left[1 - (r-1) c_n^{-1} \sum_{v=0}^{n-1} c_v^r a_v \right]^{1/(r-1)}.$$

Thus, the following theorem holds.

Theorem 1. Let the following conditions be satisfied:

a) the trivial solution of equation (5) is uniformly stable, i.e. $\forall j > 0,\ j \leq n$, estimate $\|y_n\| \leq M_2 \|y_j\|$ holds; M_2 is a positive constant;

b) the nonlinear right-hand side of equation (4) satisfies conditions (6);

c) the initial disturbance y_0 is small.

Then the trivial solution of equation (4) is stable, i.e. the following estimate holds:

$$\|y_n\| \leq \widetilde{M_2} \|y_0\|, \ \ \forall n = 0, 1, ..., \tag{11}$$

where $\widetilde{M_2}$ is a positive constant.

Proof. The solution of equation (4) satisfies the following relationship:

$$y_n = T_{n,0} y_0 + \sum_{m=0}^{n-1} T_{n,m} f_m(y_m),$$

where $T_{n,m} = S_{n-1} S_n \cdots S_m$ is the resolving operator of equation (5) from layer m to layer n. Due to assumptions a) and b) for the solution (4), we have the following estimates

$$\|T_{n,m}\| \leq M_2, \ \ \|y_n\| \leq M_2 \|y_0\| + \sum_{m=0}^{n-1} M_2 K_m \|y_m\|^r.$$

Applying the discrete analogue of Bihari's lemma (Lemma 1) to this inequality, we obtain

$$\|y_n\| \leq \frac{M_2 \|y_0\|}{\varphi(\|y_0\|)}, \tag{12}$$

where

$$\varphi(\|y_0\|) = \left[1 - (r-1)(M_2 \|y_0\|)^{r-1} M_2 \sum_{m=0}^{n-1} K_m \right]^{1/(r-1)}, \ \ \varphi(0) = 1.$$

Let us estimate the lower bound $\varphi(\|y_0\|)$. We assume that

$$(r-1) M_2^r \|y_0\|^{r-1} \sum_{m=0}^{n-1} K_m \leq \delta, \ \ 0 < \delta < 1, \tag{13}$$

i.e. y_0 is a small value. Then $\varphi(\|y_0\|) \geq (1-\delta)^{1/(r-1)}$. Inequality (13) is satisfied, for example, if $\sum\limits_{m=0}^{n-1} K_m \leq M_3,\ \forall n > 1$, and the initial data satisfies the following condition

$$\|y_0\| \leq (\delta / [(r-1) M_3 M_2^r])^{1/(r-1)}, \tag{14}$$

where M_3 is a positive constant. From (12) and (14) for the solution (4), we obtain estimate (11), which means stability based on the initial data of difference equation (4), where $\widetilde{M_2} = M_2 / \left[(1-\delta)^{1/(r-1)} \right]$.

3 Stability under permanent disturbances

Let us study the stability of the trivial solution of the difference equation (4) under permanent disturbances g_n, i.e. consider the following difference equation:

$$y_{n+1} = S_n y_n + f_n(y_n) + g_n, \;\; y(0) = y_0, \;\; g_n(0) \neq 0, \;\; n = 0, 1, ... \tag{15}$$

The nonlinear disturbance $f_n(y_n)$ satisfies condition (6) and the permanent disturbance g_n is such that

$$\sum_{m=0}^{n=1} \|g_m\| \leq \delta_0, \;\; \forall m, \;\; \delta_0 > 0, \tag{16}$$

where δ_0 is quite small.

The following theorem holds.

Theorem 2. Let the conditions of Theorem 1 be satisfied. In addition, a permanent disturbance satisfies condition (16). Then the trivial solution of equation (15) is stable under permanent disturbances and the following estimate is valid for its solution

$$\|y_n\| \leq \widetilde{M}_2 \left(\|y_0\| + \sum_{m=0}^{n=1} \|g_m\| \right). \tag{17}$$

Proof. The solution of equation (15) satisfies the following relationship

$$y_n = T_{n,0} y_0 + \sum_{m=0}^{n-1} T_{n,m} \left[f_m(y_m) + g_m \right].$$

Hence, considering conditions of the theorem, we have

$$\|y_n\| \leq M_2 \left(\|y_0\| + \sum_{m=0}^{n-1} \|g_m\| + \sum_{m=0}^{n-1} K_m \|y_m\|^r \right), \;\; r > 1. \tag{18}$$

Let $\|y_0\| + \sum\limits_{m=0}^{n=1} \|g_m\| = c_n$. Then, applying Lemma 2 to inequality (18), we obtain

$$\|y_n\| \leq M_2 \left(\|y_0\| + \sum_{m=0}^{n=1} \|g_m\| \right) / \left[1 - (r-1) \sum_{m=0}^{n=1} K_m M_2^r \left(\|y_0\| + \sum_{v=0}^{m=1} \|g_v\| \right)^{r-1} \right]^{1/(r-1)},$$

i.e.

$$\|y_n\| \leq \frac{M_2}{\widetilde{\varphi}(\|y_0\|)} \left(\|y_0\| + \sum_{m=0}^{n=1} \|g_m\| \right),$$

where $\widetilde{\varphi}(\|y_0\|) \geq (1-\delta)^{1/(r-1)}$, if

$$\delta = (r-1) \sum_{m=0}^{n=1} M_2^r K_m \left(\|y_0\| + \sum_{v=0}^{m=1} \|g_v\| \right)^{r-1} \leq \delta_1, \;\; 0 \leq \delta_1 < 1. \tag{19}$$

We rewrite inequality (19) in the following form

$$(r-1)M_2^r \sum_{m=0}^{n=1} K_m \left(\|y_0\| + \sum_{v=0}^{m=1} \|g_v\| \right)^{r-1} \leq \delta_1, \ \ 0 \leq \delta_1 < 1.$$

Then

$$\|y_0\| + \sum_{v=0}^{m=1} \|g_v\| \leq (\delta_1 / \left[(r-1) M_2^r M_3 \right])^{1/(r-1)},$$

where $\sum\limits_{v=0}^{m=1} K_m \leq M_3 < \infty$, for all m.

Consequently, estimate (17) of Theorem 2 holds.

4 Study of the stability of nonlinear difference schemes

Let us consider the Cauchy problem

$$\frac{\partial u}{\partial t} = f(u), \ \ u(0) = u_0, \tag{20}$$

where u is a certain variable describing the state of the system, $f(u)$ is a nonlinear operator (functional). Similar problems include equations of the following form:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + q(u),$$

with linear derivative terms, but containing a nonlinear in u term. For example, the following semilinear equations [16, 17]:

– Zeldovich's equation, found in combustion theory, for which

$$q(u) = ku^v(1-u), \ \ v > 1, \, q(u) > 0, \ \ 0 < u < 1,$$

$$q(0) = q(1) = 0, q'(0) = 0, \ \ q'(1) < 0;$$

– Semenov's equation describing autocatalytic chain reactions:

$$q(u) = u(u-\alpha)(1-u), \, 0 < u < 1, \ \ 0 < \alpha < 1,$$

$$q(0) = q(\alpha) = q(1) = 0, q'(0) < 0, \ \ q'(\alpha) > 0, \ \ q'(1) < 0;$$

– Fisher's equation (or Kolmogorov-Petrovsky-Piskunov's equation) found in problems of mathematical biology, for which

$$q(u) = ku(1-u), \ \ q(u) > 0, \ \ 0 < u < 1,$$

$$q(0) = q(1) = 0, \ \ q'(0) > 0, \ \ q'(1) < 0,$$

$k > 0$ is the constant.

1°. Let us approximate (20) with an explicit difference scheme of the following form (Eulerian scheme)

$$y_t = f(y_n), \ \ y(0) = y_0, \tag{21}$$

where $y_t = (y_{n+1} - y_n)/\tau$.

The error of scheme (21) $z = y - u$ $(y = z + u)$ satisfies the following equation:

$$z_t = f(y_n) - f(u_n) + g_n, \ \ z_0 = 0, \tag{22}$$

where $g_n = O(\tau)$ is the approximation error.

Using the Fréchet derivative for functional f, we obtain:

$$f(y) - f(u) = f(u + z) - f(u) = f'(u)z + O(z),$$

i.e. from (22) it follows that

$$z_t = f'(u_n)z_n + q(z_n) + g_n, \;\; z_0 = 0, \tag{23}$$

where $q(z_n) = O\left(\|z_n\|^r\right), \;\; r > 1$.

From (23) it follows that

$$z_{n+1} = S_n z_n + \tau q(z_n) + \tau g_n, \;\; z_0 = 0, \tag{24}$$

where $S_n = 1 + \tau f'(u_n)$.

By Theorem 2, scheme (24) is stable, if the solution of its first approximation is uniformly stable

$$z_{n+1} = S_n z_n, \;\; z_0 = 0. \tag{25}$$

The condition for uniform stability of solution (25) is $\|S_n\| \le M$. If this condition is met, we obtain estimate

$$\|z_{n+1}\| \le M_1 \|z_n\|$$

for all $n > 0$. Since $f'(u)$ is the bounded linear functional, estimate $\|1 + \tau f'(u_n)\| \le M_1$ holds, and the remaining conditions of Theorem 2 are satisfied.

Now we prove the convergence of the scheme. Since (21) is uniformly stable according to the initial data (the first condition of Theorem 1), then from (24) it follows that

$$\|z_{n+1}\| \le \|S_n\| \|z_n\| + \tau \|q(z_n)\| + \tau \|g_n\|.$$

Hence

$$\|z_{n+1}\| \le M_1 \left(\|z_0\| + \tau \sum_{m=0}^{n-1} K_m \|z_n\| + \tau \sum_{m=0}^{n-1} \|g_m\| \right),$$

by Theorem 2, the following estimate holds:

$$\|z_{n+1}\| \le \widetilde{M}_1 \left(\|z_0\| + \tau \sum_{m=0}^{n-1} \|g_m\| \right). \tag{26}$$

From $\|S_n\| \le 1$ we obtain condition $|1 + \tau f'(u_n)| \le 1$ or $-1 \le 1 + \tau f'(u_n) \le 1$, i.e.

a) inequality

$$1 + \tau f'(u_n) \le 1$$

fulfilled for $f'(u_n) \le 0$;

b) inequalities $-1 \le 1 + \tau f'(u_n)$, $\tau \left| f'(u_n) \right| \le 2$, $\tau \le 2/\left| f'(u_n) \right|$ are the conditions for uniform stability of scheme (21). Thus, the following theorem is proven.

Theorem 3. Let conditions $f'(u_n) \le 0$, $\tau \le 2/\left| f'(u_n) \right|$ be satisfied. Then the solution of the explicit difference scheme (21) is stable with respect to the initial data and the right-hand side, and for its solution, there is an a priori estimate (26).

2°. Let us approximate problem (20) with the following implicit difference scheme

$$y_t = f(\hat{y}), \tag{27}$$

where

$$\hat{y} = y^{n+1}.$$

Then for the error we get problem ($z = y - u,\ \ y = z + u$):

$$z_t = f(\hat{y}) - f(\hat{u}) + g_n.$$

Using the Frechet derivative, we get

$$z_t = f'(u_{n+1})z_{n+1} + q(z_{n+1}) + g_n, \tag{28}$$

where $\|q(z_{n+1})\| = K_{n+1}\|z_{n+1}\|^r,\ r > 1\,(n = 0, 1, ...),\ f'(u_{n+1})$ is a bounded linear operator (functional). To study the convergence of scheme (28), we obtain the first approximation equation

$$z_{n+1} = S_n z_n, \quad S_n = \left(1 - \tau f'(u_{n+1})\right)^{-1}.$$

Let the solution to this equation be uniformly stable, i.e. $\|S_n\| \le 1$. Then we obtain the condition for uniform stability of solution $1/\left(1 - \tau f'(u_{n+1})\right) \le 1.$ This condition is always satisfied, if

$$f'(u_{n+1}) \le 0. \tag{29}$$

Therefore, taking (6) into account, the following estimate holds:

$$\|z_{n+1}\| \le M_1 \left(\|z_0\| + \tau \sum_{m=0}^{n} k_{m+1}\|z_{m+1}\|^r + \tau \sum_{m=0}^{n-1} \|g_m\| \right).$$

From here, we get

$$\|z_{n+1}\| \left(1 - M_1 k_{n+1}\tau\|z_{n+1}\|^{r-1}\right) \le M_1 \left(\|z_0\| + \tau \sum_{m=0}^{n-1} K_{m+1}\|z_{n+1}\|^r + \tau \sum_{m=0}^{n-1} \|g_m\| \right). \tag{30}$$

Let

$$1 - M_1 k_{n+1}\tau\|z_{n+1}\|^{r-1} \ge \delta, \quad 0 < \delta < 1.$$

Then, from (30), we have the following estimate:

$$\|z_{n+1}\| \le \frac{M_1}{\delta} \left(\|z_0\| + \tau \sum_{m=0}^{n-1} K_{m+1}\|z_{m+1}\|^r + \tau \sum_{m=0}^{n-1} \|g_m\| \right).$$

Based on Lemma 2, we obtain the following estimate:

$$\|z_{n+1}\| \le \widetilde{M_1} \left(\|z_0\| + \tau \sum_{m=0}^{n-1} \|g_m\| \right), \tag{31}$$

if

$$\|z_{n+1}\|^{r-1} \le \frac{1-\delta}{M_1 K_{n+1}} \ \text{ or } \ \|z_{n+1}\| \le \left(\frac{1-\delta}{M_1 K_{n+1}}\right)^{\frac{1}{r-1}}.$$

Thus, the following theorem is proven.

Theorem 4. Let condition (29) be satisfied. Then the solution of the implicit difference scheme (27) is stable with respect to the initial data and the right-hand side, and its solution has a priori estimate (31).

3°. Let us approximate problem (20) with the following one-parameter difference scheme

$$y_t = y_n + \tau f^2(y_n) / \left[(1+\alpha) f(y_n) - \alpha f(y_n + \alpha \tau f(y_n))\right]. \tag{32}$$

Here

$$F(y) = y + \tau f^2(y) / \left[(1+\alpha) f(y) - \alpha f(y + \alpha \tau f(y))\right].$$

From (32) for $\alpha = 0$, we obtain difference scheme (21), for $\alpha = -1$ and $\alpha = 1$, we obtain V.V. Bobkov's A-stable difference schemes.

Let us obtain the problem for the scheme error (32)

$$z_t = F(y_n) - F(u_n) + g_n, \quad z_0 = 0.$$

Using the Fréchet derivative for $F(y)$, we obtain

$$z_t = F'(u_n) z_n + q(z_n) + g_n, \quad z_0 = 0,$$

where

$$\begin{gathered} F'(u) = 1 + \tau \widetilde{f}'(u), \\ \widetilde{f}'(u) = \frac{f^2(u)}{\left[(1+\alpha) f(y) - \alpha f(y + \alpha \tau f(y))\right]^2} \left[f'(u) + \alpha f'(u) - \right. \\ \left. - \frac{2\alpha}{f(u)} f(u + \alpha \tau f(u)) f'(u) + \alpha f'(u + \alpha \tau f(u)) + \alpha^2 \tau f'(u) f'(u + \alpha \tau f(u)) \right). \end{gathered} \tag{33}$$

Thus, we obtained the first approximation equations

$$z_{n+1} = (1 + \widetilde{f}'(u_n)) z_n.$$

Let us check, under what terms the uniform stability condition $\widetilde{f}'(u) \le 0$ is satisfied. From (33), it follows that $\widetilde{f}'(u) \le 0$, if

$$\begin{gathered} f'(u) + \alpha f'(u) - \frac{2\alpha}{f(u)} f(u + \alpha \tau f(u)) f'(u) + \\ + \alpha f'(u + \alpha \tau f(u)) + \alpha^2 \tau f'(u) f'(u + \alpha \tau f(u))] \le 0. \end{gathered} \tag{34}$$

Applying the Taylor formula for $f(u + \alpha \tau f(u))$ and $f'(u + \alpha \tau f(u))$, we obtain the condition for the fulfillment of inequality (34)

$$f'(u) - \alpha^2 \tau f'^2(u) + \alpha^2 \tau f(u) f''(u) + \frac{\alpha^3 \tau^2}{2} f^2(u) f'''(u) + O(\tau^3) \le 0. \tag{35}$$

This proves the following theorem.

Theorem 5. Let condition (35) be satisfied. Then the solution of the difference scheme (32) is stable with respect to the initial data and the right-hand side, and for its solution, there is a priori estimate (17).

Let us check condition (35) using a test example. Let $f(u) = -\lambda u,\ \ \lambda > 0$.

Then, substituting $f(u) = -\lambda u, f'(u) = -\lambda, f''(u) = 0$ into (35), we obtain inequality, $-\lambda - \tau\lambda^2 \le 0$, from which it follows that $\widetilde{f}'(u) \le 0$. Let now $f(u) = k[A(1-u) - Bu^2]$, where $k > 0$, $A > 0$, $B > 0$. Hence $f'(u) = -k[A + 2Bu]$, $f''(u) = -2kB$, and the remaining derivatives are zero. Then to satisfy (35), we obtain the following condition:

$$-k[A + 2Bu + \tau kA^2 + 2AB\tau ku + 2B^2\tau ku^2 + 2\tau kAB] \le 0,$$

or

$$A + 2Bu + \tau kA^2 + 2AB\tau ku + 2B^2\tau ku^2 + 2\tau kAB \ge 0,$$

which is valid for $0 \le u \le 1$.

The results of Theorem 5 are also valid for the multi-parameter explicit absolutely A-stable Bobkov difference scheme

$$\hat{y} = y + \tau(A+B)\frac{f^{k+1}(y, t+\alpha\tau)}{Af^k(y + a\tau f(y, t+\alpha\tau), t+\beta\tau) + Bf^k(y, t+\alpha\tau)},$$

where $y \approx u(t)$, $\hat{y} \approx u(t+\tau)$ are approximate solutions, $u(t)$ is the solution of equation $\dot{u} = f(t,u)$, A, B, α, β, a, k are some parameters that control the order of accuracy of the scheme.

Conclusion

Stability conditions for solutions of nonlinear difference equations are obtained. Based on the generalized discrete analogue of Bihari's lemma, an a priori estimate of the stability under permanent disturbances of a nonlinear difference equation is obtained. Theorems on the stability of the solution of nonlinear difference equations are proven. Examples of application of the stability theorem to explicit and implicit difference schemes that approximate nonlinear parabolic equations are considered. Based on the proposed methodology for studying the stability of difference equations, it is possible to study the stability of difference schemes for the above semi-linear equations of Zeldovich, Semenov and Fisher, as well as the stability of difference schemes for nonlinear equations of pseudo-parabolic type [18–21].

Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

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Geometric approach to define a railway plan model

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The construction of new railway lines is based on the railway plan. There are various ways to draw up a railway plan. The basis of all railway plans is a scheme of geometric point locations, the projection of the center of gravity of the carriage is on a horizontal plane and consists of a single flat line. The railway plan consists of linear and curved parts connecting straight sections. However, the curves determining the position of the rails of the railway track in the curved part will be spatial. To extinguish the centrifugal force arising in the curved part of the road, an external rail rises. In this case, the elevated curve representing the outer rail becomes spatial. Therefore, in the work, it is proposed to draw up a plan of a railway track as two curves, one of which is flat, and the other depicts an external spatial rail. In this case, the distance between the ends of the rectilinear parts and the angle between the rectilinear parts are selected as the main parameters. In the work, for the simplest case, when both linear parts belong to the same horizontal plane, it is proved that the curved part is a spatial curve. The curvature of the required curve was determined and a dynamic system was constructed, the solution of which would be a curve that satisfied the technical conditions presented for the railway route. This dynamic system is proposed as a mathematical model of the railway route. In the rectilinear parts, the railway plan is straight on a horizontal plane. The curve of the road should be spatial.

Keywords: railway plan, route, curvature, torsion, osculating plane, radius, osculating circle, angle of rotation, car motion profile, vector equation, Frenier's formula.

2024 Mathematics Subject Classification: 53A35, 53Z30.

Introduction

Railroad plan design has been the main issue of practical experience for many years. The railway plan consists of linear parts and curved parts connecting them [1]. The curved part of the plan has a curvature that is the inverse of the radius of the touching circle. Existing methods consider the curved part of the plan to be a flat curve. For technical reasons, the curved part of the plan must be spatial [2]. The work determines the steepness of the curve, which is the route plan, and provides a mathematical model with which one can determine the equation of the route.

There are methods for designing and reconstructing existing roads using laser technology [3]. Reconstruction of existing roads using modern technologies is expensive [4].

1 Elements of the railroad plan

The railway plan is the projection of the track axis onto a horizontal plane. The railway in plan is a combination of alternating linear and curved sections [5].

In straight sections, the main parameter is its direction, the technical name of which is azimuth. In this case, the railway plan is linear on a horizontal plane. But even on a horizontal plane, due to the terrain, the presence of settlements and other obstacles, there is a need to change the azimuth. The new azimuth defines a new straight section on the horizontal plane.

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The pairing of linear sections with each other is carried out using curves. The presence of curves in the railway plan is due to the need to deviate the route from the linear direction for the purpose of bypassing [6].

The angle α between linear sections is called the angle of rotation. When a railway plan is considered only the horizontal plane $\alpha \in (0, \pi)$. In general, α can take any value. If the linear sections belong to horizontal planes of different levels, the connecting curve has the form of a spiral, then the angle of rotation can take on arbitrary values. A section of a railway plan, on a horizontal plane, with a rotation angle α can be represented in the form of two rays that make up this rotation angle (Fig. 1).

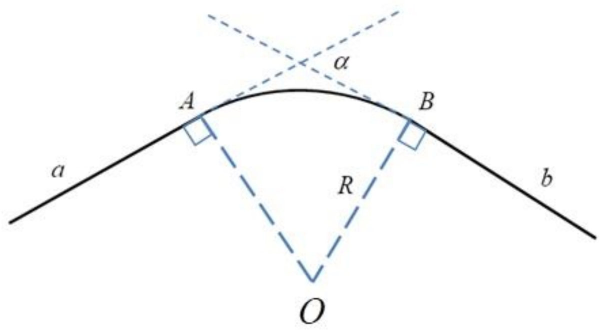

Figure 1. Curved section of the railway plan

The simplest, cost-effective solution for choosing a curved section is a circle of radius R and another connecting points A and B of the straight sections a and b of the turn (Fig. 1). The position of the railway significantly depends on the value of the radius of the circle R and it is called the radius of the curve [7].

In the railway there are special tables that determine the radius of the curve R, taking into account the technical and economic requirements for a given road [8]. In addition, the curved part of the road is considered clothoidal [9, 10]. This table is compiled taking into account the need to limit the speed of trains, removal of the designed line, increased wear of the rails, increased costs for the ongoing maintenance of the upper structure of the track and repair of rolling stock and other factors associated with the operation of the road.

Note that the curve of a railway plan section on a horizontal plane is considered as a flat curve on this plane.

2 Properties of the curve describing the railway plan

To study the curve of the railway plan, it is important to consider the point describing the curve in the profile section by the movement of the wagon.

When the wagon moves in a linear direction on a horizontal plane, the plan of the railway will be a straight line, which is obtained by moving point M in azimuth (Fig. 2).

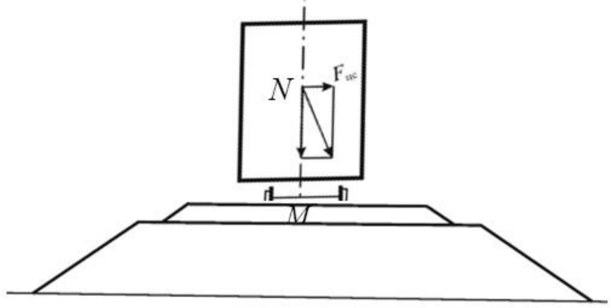

Figure 2. The profile of the movement of the wagon in a straight section

Here N is the center of gravity of the wagon and point M is its projection on the horizontal plane π_0. The plan of a railway track is understood as the geometric location of points M on a horizontal plane. These geometric locations of points generally determine the curve that defines the railroad plan. As stated above, this curve consists of linear and curved parts connecting straight parts.

To clarify the geometry of the curve formed by the point M, we consider the mechanics of the movement of the wagon when turning, that is, when moving from one straight section to the next, when these straight sections both belong to the same horizontal plane.

Theorem 1. With curved sections of the railway plan, the trace of point M will be a spatial curve.

Proof. We will prove the theorem for the simplest case of the railway plan, when both linear sections a and b of the railway plan belong to the same horizontal plane α.

The proof of the theorem in complex conditions, that is, at least in the case of a and b lying on different horizontal planes, becomes obvious, since the connection of points A and B at different levels ensures the spatiality of the curve of the railway plan.

To prove the theorem, let's consider the profile of the movement of the wagon in a curve, which looks like the one shown in Figure 3. To avoid the influence of centrifugal force when the part is curved, an elevation of the outer rail h is arranged in relation to the inner one. The value of h depends on the radius of R and the speed of the train in this section.

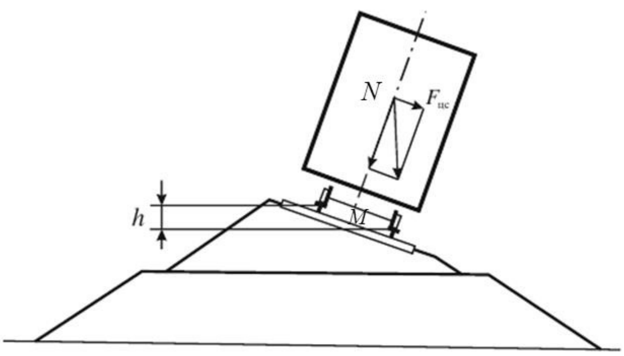

Figure 3. The profile of the movement of the wagon in a curve section

If we establish a Cartesian coordinate system with the origin at point A and the direction of the x-axis along the direction of the segment AB, the y-axis is perpendicular to the x-axis and the z-axis is along the normal of the horizontal plane, then the point M' has three coordinates (x_0, y_0, z_0). Moreover, the value of z_0 depends on the value of h and will be different from zero, if $h \neq 0$. In the curved part of the railway plan there is $h \neq 0$, therefore the curve that is the trace of point M is spatial. The theorem is proved.

To study the movement of point M of the railway plan, the horizontal plane π_0 is taken as the plane $z = 0$. We select the y-axis perpendicular to the x-axis with a positive direction to the corresponding direction of the linear part starting from point B.

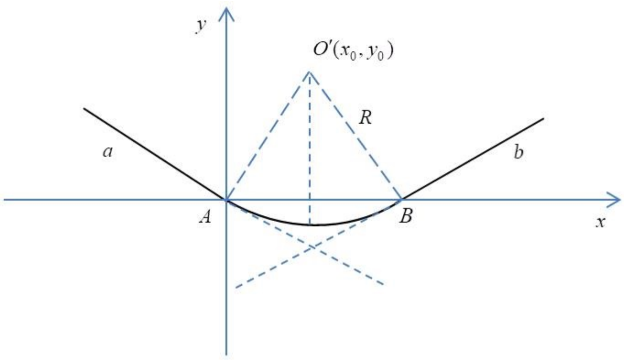

Figure 4. The inner path of the curve section

Let's assume that points A and B are connected to an arc of a circle with radius R and center at point $O'(x_0, y_0)$.

With the current selection of the coordinate system, the points A and B have the following coordinates $A(0, 0)$ and $B(2x_0, 0)$. The equation of a circle with center at point $O'(x_0, y_0)$ and radius R has the form

$$(x - x_0)^2 + (y - y_0)^2 = x_0^2 + y_0^2, \tag{1}$$

since

$$R^2 = x_0^2 + y_0^2.$$

The same equation can be written in parametric form:

$$\begin{cases} x = x_0 + \left(x_0^2 + y_0^2\right)^{\frac{1}{2}} \cos t, \\ y = y_0 + \left(x_0^2 + y_0^2\right)^{\frac{1}{2}} \sin t, \end{cases}$$

where the parameter $t = \frac{S}{\left(x_0^2+y_0^2\right)^{\frac{1}{2}}}$ is proportional to the length of the circular arc.

When the train moves along a curved part of the road, in order to extinguish the centrifugal force that appears when turning, the outer country of the rail is raised to a certain height H. The value of H depends on the radius of the curve R and on the speed of the train.

The position of the base of the wagon when turning is shown schematically in Figure 5.

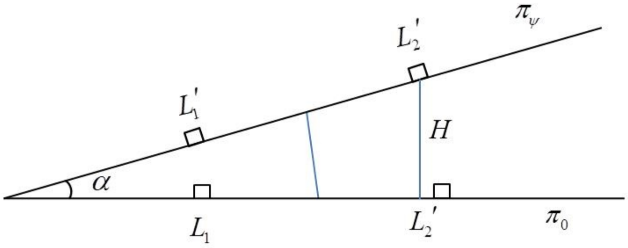

Figure 5. Deviation from the horizontal plane

The horizontal plane π_0 takes on another position π_ψ, where ψ is the angle between these planes. But the size of the angle ψ depends on the value of H. In this case, point M of the railway plan goes into point M' on the plane π_ψ. Since the point M' differs from the M belonging to the horizontal plane, it is spatial. Therefore, the curve describing the point M will also be spatial.

The technical and economic requirement for railways prefers not to change the internal part of the track rail. Therefore, it is advisable to select plane π_0 so that the inner part of the road rail remains on a horizontal plane. Then the road traffic pattern takes the following form.

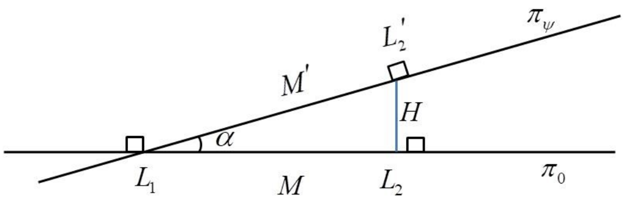

Figure 6. Deviation from the horizontal plane while maintaining the internal path on the horizontal plane

At the same time, the spatiality of the curve formed by the point M' remains.

In Figures 5 and 6, L_1 and L_2 indicate the track rails. It is obvious in Figure 6 that the inner part of the track rail is left unchanged and the outer part, that is, the point L_2 goes into the point L_2'.

Note that changes in point L_2 linearly depend on point M and on the track width, which is constant. The position of the point L_2' is completely determined by the position of the point M'. Therefore, we can reason only with respect to the point M'.

Let us assume that, relative to the section AB, the radius R is selected and the speed V is the passage of the train. Then the lifting height of the outer rail H_{AB} can be accurately determined, so that the inner side of the track rail remains on a horizontal plane.

Then the plane π_0 to which the circular arc belongs can be accurately determined by the equation. This plane will be the plane passing through the points $(x_0, y_0, 0)$, $(0, 0, H)$ and $(2x_0, 0, H)$ of the equation of this plane.

$$z = H\left(\frac{x}{x_0} + \frac{y}{y_0} - 2\right). \tag{2}$$

The curve M' described by the point M is a spatial curve relative to the railway plan described by the point M, but belongs to the plane (2). The equation of this curve can be written in metric form:

$$\begin{cases} x = x_0 + \left(x_0^2 + y_0^2\right)^{\frac{1}{2}} \cos t, \\ y = y_0 + \left(x_0^2 + y_0^2\right)^{\frac{1}{2}} \sin t, \\ z = \frac{\left(x_0^2+y_0^2\right)^{\frac{1}{2}}}{x_0} \cos t + \frac{\left(x_0^2+y_0^2\right)^{\frac{1}{2}}}{y_0} \sin t - 2. \end{cases}$$

But curve γ will not be a continuous continuation of the path; it lies on a different plane relative to the horizontal plane. A connection should be established using an additional curve connecting the curve γ with the linear part a and b. For this purpose, we divide $\widehat{AB}$ into three parts $\widehat{AC}, \widehat{CD}$ and $\widehat{DB}$ with the condition that the length of the arc $\widehat{AC}$ and $\widehat{DB}$ in duration is greater than the length of the two wagons. This is a general requirement for a curved part to ensure smooth movement of the train along that part. Let's assume that points C' and D' are the images of points C and D on the curve γ. Then we take part of the curve γ with ends at points C' and D' as a part of the route.

It is required to construct a part of the railway plan connecting points A and C', also points D' and B, so that when crossing the curve $AC'D'B$, the smooth movement of the train is ensured.

We denote by γ_1 and γ_2 the curved parts of the curve connecting points A and B consisting of the arc $\widehat{AC'}$ and $\widehat{D'B}$, respectively. For convenience, curves γ_1 and γ_2 can be considered symmetrical with respect to the bisector x_0 of the angle formed by the straight part a and b of the railway plan (Fig. 7).

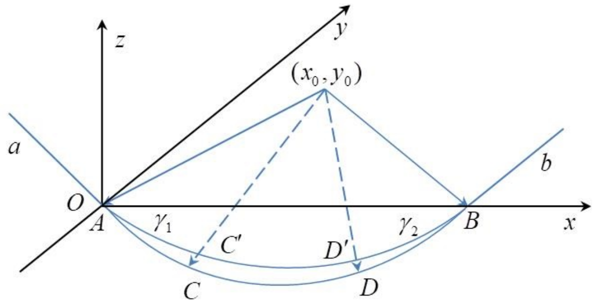

Figure 7. The curve of the outer rail

The curve equation of part γ_1 can be thought of as a spatial curve connecting points $A(0,0,0)$ and $C'(x_1, y_1, z_1) \in \gamma$ with curvature $k = \frac{1}{\left(x_0^2+y_0^2\right)^{\frac{1}{2}}}$ and torsion σ.

But torsion σ can be considered a linearly increasing function of the length of the road in the form $\sigma = m \cdot s + e$. Moreover, given R and V, the values of m and e can be determined. Thus, curvature k and torsion σ are functions of R radius of curvature and speed of movement. Then, using the Frenier formula [11] for the curve γ_1 , one can determine the equation of the curve. Having obtained the equation of the curve, we can calculate the size of the railway track with the necessary accuracy, which ensures the safe movement of the train along this track.

3 Dynamic system for determining the route schedule

The railway with the curved part is called the railway route. It has been proven that the route is a spatial curve.

In the previous section it was shown that the radius of the curved part of the railway plan completely determines the curvature of the curved part of the route [12].

Torsion of the curved part of the road is defined as a change in the angle of the contacting plane of the curve representing the road route.

If the curve is given by the vector equation

$$\vec{r}(s) = \vec{x}(s)i + \vec{y}(s)j + \vec{z}(s)k,$$

where s is mastiff curve length, $\{i, j, k\}$ is basis vectors and $x(s), y(s), z(s) \in C^2$.

Then the touching plane at point (x_0, y_0, z_0) is determined by the formula:

$$\begin{vmatrix} x - x_0 & y - y_0 & z - z_0 \\ x'(s_0) & y'(s_0) & z'(s_0) \\ x''(s_0) & y''(s_0) & z''(s_0) \end{vmatrix} = 0.$$

Consider at two points $M(x_0, y_0, z_0)$ and $N(x_0 + \Delta x, y_0 + \Delta y, z_0 + \Delta z)$ the osculating plane of the curve γ. Let us determine the angle $\Delta\psi$ between these planes. Speed change of angle ψ between osculating planes

$$\lim_{\Delta s \to 0} \frac{\Delta\varphi}{\Delta s} = \varphi'_s = \sigma$$

called torsion of a curve into points $M(x_0, y_0, z_0)$.

The torsion of plane curves is zero everywhere.

If we have a vector equation of a curve given by formula (1), then the curvature and torsion of the curve are calculated by the formulas [13, 14]:

$$k = \left|r''(s)\right|$$

and

$$\sigma = \frac{|(r'r''r''')|}{k^2}.$$

It should be noted that the curvature depends on the coordinates of the center of the osculating circle (x_0, y_0). The radius of the osculating circle is calculated using the formula $R = \sqrt{x_0^2 + y_0^2}$. The torsion of the curve is determined depending on the radius R and the rise h- the outer part of the track rail $\sigma = f(R, h)$.

But the parameters R, h can be selected depending on the requirements for the road, which are determined by technical and economic conditions. Therefore, Frenier's formula

$$\begin{cases} \dot{\tau} = k\nu, \\ \dot{\nu} = -k\tau - \sigma\beta \\ \dot{\beta} = \sigma\nu \end{cases} \tag{3}$$

is a dynamic system of differential equations overestimated from parameters R and V. The path equation is a solution to the dynamic system (3). Setting parameters R, h is completely determined by the solution of the system. Therefore, system (3) can be taken as a mathematical model of the railway route.

Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

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https://doi.org/10.31489/2024M3/34-45 *Research article*

On a non-local problem for a fractional differential equation of the Boussinesq type

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In recent years, the fractional partial differential equation of the Boussinesq type has attracted much attention from researchers due to its practical importance. In this paper, we study a non-local problem for the Boussinesq type equation $D_t^\alpha u(t) + AD_t^\alpha u(t) + \nu^2 Au(t) = 0$, $0 < t < T$, $1 < \alpha < 3/2$, where D_t^α is the Caputo fractional derivative, and A is an abstract operator. In the classical case, i.e., when $\alpha = 2$, this problem has been studied previously, and an interesting effect has been discovered: the existence and uniqueness of a solution depend significantly on the length of the time interval and the parameter ν. In this note, we show that in the case of a fractional equation, there is no such effect: a solution of the problem exists and is unique for any T and ν.

Keywords: fractional equation, Caputo derivative, forward and inverse problems, Fourier method.

2020 Mathematics Subject Classification: 35A01, 35A02.

Introduction

Let H be a separable Hilbert space, and let $A : D(A) \subset H \to H$ be an arbitrary unbounded, positive self-adjoint operator, and we assume that A has a compact inverse A^{-1}, where $D(A)$ is the domain of A. Let λ_k and $\{v_k\}$ be the eigenvalues and corresponding eigenfunctions of A.

Let us introduce the Caputo fractional derivative D_t^α of order $\alpha \in (1, 2)$ of a vector-valued function $h(t) \in H$ (see, for example [1])

$$D_t^\alpha h(t) = \frac{1}{\Gamma(2-\alpha)} \int_0^t \frac{h''(\xi)}{(t-\xi)^{\alpha-1}} d\xi, \quad t > 0,$$

provided the right-hand side exists. Here $\Gamma(\alpha)$ is Euler's gamma function.

Let $1 < \alpha < 3/2$. The object of study of this work is the following fractional differential equation

$$D_t^\alpha u(t) + AD_t^\alpha u(t) + \nu^2 Au(t) = 0, \quad 0 < t < T \tag{1}$$

with non-local conditions

$$u(0) = u(T), \tag{2}$$

and

$$\int_0^T u(t)dt = \varphi, \tag{3}$$

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where $\varphi \in H$ is a given vector and $\nu > 0$ is a fixed number.

Note that since the abstract operator A is only required to have a complete orthonormal system of eigenfunctions, any elliptic operator can be considered as A. For example, if we take $L_2(\Omega)$, $\Omega \subset \mathbb{R}^N$, as the Hilbert space H, then we can take the Laplace operator $(-\Delta)$ with the Dirichlet condition as A.

The equation (1) has different names for different values of the parameter α. Thus, if $\alpha = 1$, it is called a differential equation of the Barenblatt-Zheltov-Kochina type (see [2]), and if $\alpha = 2$, it is called a differential equation of the Boussinesq type (see [3]). If $0 < \alpha < 1$, it is called a fractional differential equation of the Barenblatt-Zheltov-Kochina type, in the case $1 < \alpha < 2$, it is called a fractional differential equation of the Boussinesq type. Differential equations of the Boussinesq type were introduced by Joseph Boussinesq in 1872 (see [3], eq. 26). The Boussinesq equations are widely used in numerical modeling in coastal engineering for modeling waves in shallow water and harbors. Although wave modeling in such cases is well described by the Navier-Stokes equations, it is currently extremely difficult to solve three-dimensional equations in complex models. Therefore, approximate models, such as the Boussinesq equations can be used to reduce three-dimensional problems to two-dimensional states (see, e.g., [4]).

There is a number of works (see, for example, [2], [5]–[7]) in which specialists consider various initial-boundary value problems for differential and fractional differential equations of the Barenblatt-Zheltov-Kochina type. Since our study relates to the Boussinesq type differential equation, we present some results related specifically to these equations.

Due to the mathematical and physical importance, over the last couple of decades, existence and nonexistence of solutions of the Boussinesq type equations have been extensively studied by many mathematicians and physicists (see, for example [8]–[12] with fractional order, and literature therein). Nonlinear Boussinesq type equations arise in a number of mathematical models of physical processes, for example, in the modeling of surface waves in shallow waters or considering the possibility of energy exchange through the lateral surfaces of the wave guide in the physical study of nonlinear wave propagation in wave guide (see, for example, [13] and [14], and literature therein). In [13], the authors consider the Cauchy problem of the two-dimensional generalized Boussinesq type equation $u_{tt} - \Delta u - \Delta u_{tt} + \Delta^2 u + \Delta f(u) = 0$. Under the assumption that $f(u)$ is a function with exponential growth at infinity and under some assumptions on the initial data, the authors prove the existence and, in some cases the nonexistence of a global weak solution.

Model equations of the Boussinesq type (the problem (1)–(3) with $\alpha = 2$, $\nu = 1$ and $A = -\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}$, $x, y \in (0, l)$) and equations of mixed type and nonlinear equations containing equations of the Boussinesq type are systematically studied in a series of works [15]–[17]. In these works, the existence and uniqueness of the classical solution of initial-boundary value problems were proved and some inverse problems were studied. In the work [18], problems for the Boussinesq equation with a spectral parameter were investigated.

Let us cite two more works [19] and [20] that motivated the appearance of our research. In these works, the above non-local problem (1)–(3) was studied for a classical partial differential equation in which A is the Laplace operator with the Dirichlet condition. So in the fundamental work [19], Alimov and Khalmukhamedov studied the following non-local problem in the cylinder $\Omega \times (0, T)$:

$$\begin{cases} u_{tt} - \Delta u_{tt} - \nu^2 \Delta u = 0, \quad x \in \Omega, \quad 0 < t < T, \\ u(x, 0) = u(x, T), \quad x \in \Omega, \\ \int\limits_0^T u(x, t)dt = \varphi(x), \end{cases} \tag{4}$$

where $\varphi(x)$ is a given function. The authors discovered an interesting effect: it turns out that the existence and uniqueness of the solution of this problem significantly depend on the length of the time interval and the parameter ν. If $\frac{\nu T}{2\pi} \in (0, 1)$, then the solution exists and is unique for all $\varphi \in D(A)$. The case $\frac{\nu T}{2\pi} \geq 1$ is more complicated: if $\frac{\nu T}{2\pi} > 1$, and this number is not a natural number, then for the existence of a solution, it is necessary that the function φ is orthogonal to some eigenfunctions of the Laplace operator, and in this case, the solution is not unique. If the number $\frac{\nu T}{2\pi}$ is a natural number, then only orthogonality is not enough; it is necessary that the function φ is smoother: $\varphi \in D(A^2)$.

Since the parameter ν in the equation is fixed, this result means that if the process under study lasts "not so long", then a solution to the problem exists for any measurements φ. However, if the process lasts "a little" longer, then the solution does not exist for all data φ.

In the recent work [20], problem (4) was studied with the kernel $tu(x,t)$ in the integral condition. Similar to the paper [19], conditions have been found for the time interval $(0,T]$, function φ and parameter ν, which guarantees the existence of a solution to the problem.

A natural question arises: will the effect found in [19] be preserved, if instead of the second time derivative in equation (4) we take the fractional derivative of order $\alpha \in (1, 3/2)$, in other words, instead of equation in (4), consider equation (1)? In this paper it will be shown that the above parameter $\frac{\nu T}{2\pi}$ does not play a significant role in solving the non-local problem (1)–(3) and the solution to this problem exists and is unique for any function $\varphi \in D(A)$, regardless of the value of the number $\frac{\nu T}{2\pi}$.

The article is organized as follows: Section 2 provides some information about the domain of definition of the operator A and proves the necessary estimates for the Mittag-Leffler functions. In Section 3, we will formulate the main result of the work and construct a formal solution to the problem (1)–(3). Section 4 is devoted to the proof of Theorem 1. In the "Conclusions" section discusses possible further developments of the obtained results.

1 Preliminaries

In this section, we provide some information about the operator A and present new bounds for the Mittag-Leffler function in the case $1 < \rho < 3/2$, based on the findings of the study conducted by [21].

The action of the abstract operator A under consideration on the element $h \in H$ can be written as

$$Ah = \sum_{k=1}^{\infty} \lambda_k h_k v_k,$$

where h_k is the Fourier coefficient of the element h: $h_k = (h, v_k)$. Obviously, the domain of this operator has the form

$$D(A) = \{h \in H : \sum_{k=1}^{\infty} \lambda_k^2 |h_k|^2 < \infty\}.$$

For elements h and g of $D(A)$ we introduce the norm and inner product as

$$||h||_1^2 = \sum_{k=1}^{\infty} \lambda_k^2 |h_k|^2 = ||Ah||^2,$$

$$(h,g)_1 = \sum_{k=1}^{\infty} \lambda_k^2 h_k \overline{g}_k,$$

respectively. Together with this norm $D(A)$ turns into a Hilbert space.

Let us denote by $C((a,b);H)$ the sets of continuous vector functions $u(t)$ on the interval $t \in (a,b)$, whose values lie in H, and by $AC^1((a,b);H)$ the sets of vector functions whose derivatives are absolutely continuous with respect to $t \in (a,b)$.

Recall, the Mittag-Leffler function $E_{\rho,\mu}(t)$ has the form (see e.g. [22], p. 56):

$$E_{\rho,\mu}(t) = \sum_{n=0}^{\infty} \frac{t^n}{\Gamma(\rho n + \mu)},$$

where $\rho > 0$ and μ complex number.

Next, we establish some two-sided estimates for the Mittag-Leffler function $E_{\rho,\mu}(-t)$, $1 < \rho < 3/2$, $t \geq 0$, $\mu = 1, 2, 3, \rho$. The following simple method for obtaining these estimates was suggested to the authors by Professor A.V. Pskhu (see, [21]).

Let $\phi(\delta, \beta; z)$ stand for the Wright function, defined as

$$\phi(\delta, \beta; z) = \sum_{k=0}^{\infty} \frac{z^k}{k!\Gamma(\beta + \delta k)}, \ \delta > -1, \ \beta \in \mathbb{R}, \ z \in \mathbb{C}.$$

Let $0 < \xi < 1$. In the work of A.V. Pskhu [21] for functions $h(t)$ defined at $t \geq 0$, the following integral transform is introduced and studied:

$$P^{\xi,\eta} h(t) = t^{\eta-1} \int_0^{\infty} h(s) \phi\Big(-\xi, \eta; -\frac{s}{t^{\xi}} \Big) ds.$$

Note that $P^{\xi,\eta}h(t)$ is some modification of the integral transform introduced by B. Stankovič in 1955 (see [23]).

Let us present the following statement from [21].

Lemma 1. Let $\gamma > 0$. Then

$$P^{\xi,\eta} t^{\gamma-1} = t^{\xi\gamma+\eta-1} \frac{\Gamma(\gamma)}{\Gamma(\xi\gamma + \eta)}.$$

From Lemma 1, by the definition of the Mittag-Leffler function, we get

$$P^{\xi,\eta}[t^{\mu-1} E_{\rho,\mu}(\lambda t^{\rho})] = t^{\mu\xi+\eta-1} E_{\rho\xi,\mu\xi+\eta}(\lambda t^{\rho\xi}). \tag{5}$$

Lemma 2. (see [24], p. 372, 373) There is a function $f(\alpha)$ decreasing on the interval $(1, 3/2)$ such that for any $\alpha \in (1, 3/2)$ and $\beta > f(\alpha)$ function $E_{\alpha,\beta}(z)$ does not vanish, where $f(\alpha)$ satisfies the following inequality:

$$\alpha + h(\alpha) < f(\alpha) < \frac{4}{3}\alpha, \quad 1 < \alpha < 3/2,$$

where

$$h(\alpha) = \exp\left[-\pi(1 - 1/\alpha)\right].$$

Lemma 3. Let $\alpha \in (1, 2)$. Then the following estimate holds:

$$E_{\alpha,1}(-t^{\alpha}) \leq 1 \quad t > 0.$$

Proof. Let $\mu = 1, \rho = 2, \xi = \frac{\alpha}{2}, \eta = 1 - \frac{\alpha}{2}$ and $\lambda = -1$ in equality (5). Then, we have:

$$E_{\alpha,1}(-t^{\alpha}) = P^{\frac{\alpha}{2},1-\frac{\alpha}{2}}(E_{2,1}(-t^2)) = P^{\frac{\alpha}{2},1-\frac{\alpha}{2}} \cos t.$$

Using the inequality $|\cos t| \leq 1$ and Lemma 1, we get

$$|E_{\alpha,1}(-t^{\alpha})| \leq P^{\frac{\alpha}{2},1-\frac{\alpha}{2}} 1 = \frac{\Gamma(1)}{\Gamma(1)} = 1.$$

Lemma 3 is proved.

Lemma 4. Let $\alpha \in (1, 3/2)$ and $0 < a < \infty$. Then there exists a number $\varepsilon_1 = \varepsilon_1(a) > 0$, depending on a such that the following estimate holds

$$0 < \varepsilon_1 < E_{\alpha,2}(-t^\alpha) \leq 1, \quad 0 < t \leq a.$$

Proof. Let $\mu = 1, \rho = 2, \xi = \frac{\alpha}{2}, \eta = 2 - \frac{\alpha}{2}$ and $\lambda = -1$. Then from (5) it follows the following equality

$$tE_{\alpha,2}(-t^\alpha) = P^{\frac{\alpha}{2},2-\frac{\alpha}{2}}(E_{2,1}(-t^2)) = P^{\frac{\alpha}{2},2-\frac{\alpha}{2}} \cos t.$$

Using the inequality $|\cos t| \leq 1$ and Lemma 1, we get

$$|tE_{\alpha,2}(-t^\alpha)| \leq P^{\frac{\alpha}{2},2-\frac{\alpha}{2}} 1 = t\frac{\Gamma(1)}{\Gamma(2)} = t.$$

Therefore, for $t > 0$, we have that

$$|E_{\alpha,2}(-t^\alpha)| \leq 1.$$

Let $0 < a < \infty$. First we show that $E_{\alpha,2}(-t^\alpha) > 0$. Since $\beta = 2 > f(\alpha)$, then from Lemma 2 it follows $E_{\alpha,2}(-t^\alpha) \neq 0$, and therefore $E_{\alpha,2}(-t^\alpha)$ function keeps its sign for all $t \geq 0$. On the other hand, we know that $E_{\alpha,2}(0) = 1 > 0$ and therefore $E_{\alpha,2}(-t^\alpha) > 0$ for all $t \geq 0$. Further it is well known that $E_{\alpha,2}(-t^\alpha) \in C[0, \infty)$. Since function $E_{\alpha,2}(-t^\alpha)$, continuous in a closed domain $[0, a]$, reaches its minimum and this minimum is obviously positive, denoting it by $\varepsilon_1 = \varepsilon_1(a) > 0$, we obtain the statement of the lemma. Lemma 4 is proved.

Lemma 5. Let $\alpha \in (1, 3/2)$. Then the following estimate holds

$$0 < E_{\alpha,3}(-t^\alpha) \leq \frac{1}{2}, \quad 0 < t \leq b.$$

Proof. Let $\mu = 1, \rho = 2, \xi = \frac{\alpha}{2}, \eta = 3 - \frac{\alpha}{2}$ and $\lambda = -1$. Then from (5) it follows the following equality

$$t^2E_{\alpha,3}(-t^\alpha) = P^{\frac{\alpha}{2},3-\frac{\alpha}{2}}(E_{2,1}(-t^2)) = P^{\frac{\alpha}{2},3-\frac{\alpha}{2}} \cos t.$$

Using the inequality $|\cos t| \leq 1$ and Lemma 1, we get

$$|t^2E_{\alpha,3}(-t^\alpha)| \leq P^{\frac{\alpha}{2},3-\frac{\alpha}{2}} 1 = t^2\frac{\Gamma(1)}{\Gamma(3)} = \frac{t^2}{2}.$$

Therefore

$$|E_{\alpha,3}(-t^\alpha)| \leq \frac{1}{2}, \quad t > 0.$$

Now we show that $E_{\alpha,3}(-t^\alpha) > 0$. Since $\beta = 3 > f(\alpha)$, then from Lemma 2 it follows $E_{\alpha,3}(-t^\alpha) \neq 0$, and therefore $E_{\alpha,3}(-t^\alpha)$ function keeps its sign for all $t \geq 0$. Also, we know that $E_{\alpha,3}(0) = \frac{1}{2} > 0$ and therefore $E_{\alpha,3}(-t^\alpha) > 0$ for all $t \geq 0$. Lemma 5 is proved.

Lemma 6. Let $\alpha \in (1, 3/2)$. Then there exists a number $C_0 > 0$, such that the following estimate holds:

$$(E_{\alpha,2}(-T^\alpha))^2 + E_{\alpha,3}(-T^\alpha)(1 - E_{\alpha,1}(-T^\alpha)) > C_0.$$

Proof. We have that

$$(E_{\alpha,2}(-T^\alpha))^2 + E_{\alpha,3}(-T^\alpha)(1 - E_{\alpha,1}(-T^\alpha)) \geq (E_{\alpha,2}(-T^\alpha))^2.$$

According to Lemma 4, there exists a positive number C_0, such that

$$(E_{\alpha,2}(-T^\alpha))^2 > C_0,$$

where $C_0 = \varepsilon_1^2$. Lemma 6 is proved.

Lemma 7. Let $\alpha \in (1,2)$. Then, the following estimate holds

$$|E_{\alpha,\alpha}(-t^\alpha)| \leq \frac{1}{\Gamma(\alpha)}, \quad t \geq 0.$$

Proof. Let $\mu = 1, \rho = 2, \xi = \frac{\alpha}{2}, \eta = 0$ and $\lambda = -1$. Then from (5) it follows the equality

$$t^{\alpha-1}E_{\alpha,\alpha}(-t^\alpha) = P^{\frac{\alpha}{2},0}(E_{2,1}(-t^2)) = P^{\frac{\alpha}{2},0}\cos t.$$

Using the inequality $|\cos t| \leq 1$ and Lemma 1, we get

$$|t^{\alpha-1}E_{\alpha,\alpha}(-t^\alpha)| \leq P^{\frac{\alpha}{2},0}1 = t^{\alpha-1}\frac{\Gamma(1)}{\Gamma(\alpha)} = t^{\alpha-1}\frac{1}{\Gamma(\alpha)}.$$

Therefore, for $t > 0$, we have that

$$|E_{\alpha,\alpha}(-t^\alpha)| \leq \frac{1}{\Gamma(\alpha)}.$$

Lemma 7 is proved.

2 Formulation of the main result and formal solution of the problem (1)–(3)

The solution of problem (1)–(3) will be understood in the sense of the following definition:

Definition 1. If a function $u(t) \in AC^1([0,T];H)$, $D_t^\alpha u(t)$, $Au(t)$, $AD_t^\alpha u(t) \in C((0,T);H)$ and satisfies all the conditions of problem (1)–(3), then it is called the solution of problem (1)–(3).

Note that here the absolute continuity of the derivative $u'(t)$ is necessary to avoid non-uniqueness of solutions due to singular functions.

Here is the main result of this paper.

Theorem 1. Let $\varphi \in D(A)$. Then, there is a unique solution of problem (1)–(3) and it has the form:

$$u(t) = \sum_{k=1}^{\infty}\left(\frac{\varphi_k E_{\alpha,2}(-\nu_k^2T^\alpha)E_{\alpha,1}(-\nu_k^2t^\alpha)}{T((E_{\alpha,2}(-\nu_k^2T^\alpha))^2 + E_{\alpha,3}(-\nu_k^2T^\alpha)(1 - E_{\alpha,1}(-\nu_k^2T^\alpha)))} + \right.$$

$$\left. + \frac{\varphi_k t(1 - E_{\alpha,1}(-\nu_k^2T^\alpha))E_{\alpha,2}(-\nu_k^2t^\alpha)}{T^2((E_{\alpha,2}(-\nu_k^2T^\alpha))^2 + E_{\alpha,3}(-\nu_k^2T^\alpha)(1 - E_{\alpha,1}(-\nu_k^2T^\alpha)))}\right)v_k, \tag{6}$$

where $\nu_k = \nu\sqrt{\frac{\lambda_k}{1+\lambda_k}}$ and $\varphi_k = (\varphi, v_k)$ are the Fourier coefficients of function φ.

In this section we will construct a formal solution of problem (1)–(3) and prove the uniqueness of the solution.

Let $u(t)$ be any solution of the non-local problem (1)–(3). Then since the system $\{v_k\}$ is complete in H, the solution has the form:

$$u(t) = \sum_{k=1}^{\infty} T_k(t)v_k. \tag{7}$$

If we multiply both sides of this equality scalarly by v_j, then from the orthonormality of the system of eigenfunctions $\{v_k\}$, we obtain the equalities $T_j(t) = (u(t), v_j)$.

Substituting (7) into equation (1), we get

$$D_t^\alpha T_k(t) + \lambda_k D_t^\alpha T_k(t) + \nu^2\lambda_k T_k(t) = 0.$$

Then, we have that

$$(1+\lambda_k)D_t^\alpha T_k(t)+\nu^2\lambda_k T_k(t)=0.$$

If we divide above equation to $1+\lambda_k$ and by ν_k we denote $\nu\sqrt{\frac{\lambda_k}{1+\lambda_k}}$, then we obtain:

$$D_t^\alpha T_k(t)+\nu_k^2 T_k(t)=0, \tag{8}$$

and using the conditions (2) and (3), we have:

$$T_k(0)=T_k(T), \tag{9}$$

and

$$\int\limits_0^T T_k(t)dt=\varphi_k. \tag{10}$$

The solution of the equation (8) has the form (see, for example [25], p. 231.)

$$T_k(t)=a_k E_{\alpha,1}(-\nu_k^2 t^\alpha)+b_k t E_{\alpha,2}(-\nu_k^2 t^\alpha). \tag{11}$$

To find the unknown coefficients a_k and b_k, we use the non-local conditions (9) and (10).

Apply conditions (9) and (10) to (11), we get:

$$\begin{cases} a_k=a_k E_{\alpha,1}(-\nu_k^2 T^\alpha)+b_k T E_{\alpha,2}(-\nu_k^2 T^\alpha),\\ \int\limits_0^T (a_k E_{\alpha,1}(-\nu_k^2 t^\alpha)+b_k t E_{\alpha,2}(-\nu_k^2 t^\alpha))dt=\varphi_k. \end{cases}$$

Solving this system of equations, we will have

$$a_k=\frac{\varphi_k E_{\alpha,2}(-\nu_k^2 T^\alpha)}{T((E_{\alpha,2}(-\nu_k^2 T^\alpha))^2+E_{\alpha,3}(-\nu_k^2 T^\alpha)(1-E_{\alpha,1}(-\nu_k^2 T^\alpha)))}, \tag{12}$$

$$b_k=\frac{\varphi_k(1-E_{\alpha,1}(-\nu_k^2 T^\alpha))}{T^2((E_{\alpha,2}(-\nu_k^2 T^\alpha))^2+E_{\alpha,3}(-\nu_k^2 T^\alpha)(1-E_{\alpha,1}(-\nu_k^2 T^\alpha)))}. \tag{13}$$

Using the equalities (7), (11), (12) and (13) we get the formal solution (6) for the problem (1)–(3). It remains to prove that the constructed formal solution satisfies all the requirements of Definition 1, i.e. is indeed a solution to problem (1)–(3). We will do this in the next section.

On the other hand, the uniqueness of the solution follows from the already established equalities (12) and (13). Indeed, let us show that the solution to the homogeneous problem (1)–(3) with function $\varphi=0$ is identically zero. From equalities (12) and (13) it follows that $a_k=b_k=0$, and then all coefficients $T_k(t)$ of series (7) are equal to zero. Due to the completeness of system $\{v_k\}$, it follows that $u(t)\equiv 0$.

3 Proof of Theorem 1

Let $S_j(t)$ be the partial sums of (6). Then

$$AS_j(t)=\sum_{k=1}^{j}\lambda_k(a_k E_{\alpha,1}(-\nu_k^2 t^\alpha)+b_k t E_{\alpha,2}(-\nu_k^2 t^\alpha))v_k.$$

By Parseval equality, we obtain

$$||AS_j(t)||^2 = \sum_{k=1}^{j} \lambda_k^2 |a_k E_{\alpha,1}(-\nu_k^2 t^\alpha) + b_k t E_{\alpha,2}(-\nu_k^2 t^\alpha)|^2 \leq$$

$$\leq C \sum_{k=1}^{j} \lambda_k^2 |a_k E_{\alpha,1}(-\nu_k^2 t^\alpha)|^2 + C \sum_{k=1}^{j} \lambda_k^2 |b_k t E_{\alpha,2}(-\nu_k^2 t^\alpha)|^2.$$

Let us estimate the following two terms, separately

$$I_1 = |a_k E_{\alpha,1}(-\nu_k^2 t^\alpha)| = \left| \frac{\varphi_k E_{\alpha,2}(-\nu_k^2 T^\alpha) E_{\alpha,1}(-\nu_k^2 t^\alpha)}{T((E_{\alpha,2}(-\nu_k^2 T^\alpha))^2 + E_{\alpha,3}(-\nu_k^2 T^\alpha)(1 - E_{\alpha,1}(-\nu_k^2 T^\alpha)))} \right|$$

and

$$I_2 = |b_k t E_{\alpha,2}(-\nu_k^2 t^\alpha)| = \left| \frac{\varphi_k t (1 - E_{\alpha,1}(-\nu_k^2 T^\alpha)) E_{\alpha,2}(-\nu_k^2 t^\alpha)}{T^2((E_{\alpha,2}(-\nu_k^2 T^\alpha))^2 + E_{\alpha,3}(-\nu_k^2 T^\alpha)(1 - E_{\alpha,1}(-\nu_k^2 T^\alpha)))} \right|.$$

To estimate I_1, we apply Lemma 3, Lemma 4 and Lemma 6. Then

$$I_1 \leq \frac{|\varphi_k|}{T} \frac{1}{C_0} \leq C T^{-1} |\varphi_k|. \tag{14}$$

Similarly

$$I_2 \leq \frac{t|\varphi_k|}{T^2} \frac{1}{C_0} \leq C T^{-2} t |\varphi_k|. \tag{15}$$

Using estimates (14) and (15), we obtain:

$$||AS_j(t)||^2 \leq C^2 T^{-2} \sum_{k=1}^{j} \lambda_k^2 |\varphi_k|^2 + C^2 T^{-4} t^2 \sum_{k=1}^{j} \lambda_k^2 |\varphi_k|^2.$$

Therefore, if $\varphi \in D(A)$, then

$$C^2 T^{-2} \sum_{k=1}^{j} \lambda_k^2 |\varphi_k|^2 + C^2 T^{-4} t^2 \sum_{k=1}^{j} \lambda_k^2 |\varphi_k|^2 \leq const.$$

Thus $Au(t) \in C([0,T]; D(A))$.

Now we will show that the termwise differentiated series (6) converges uniformly on $[0,T]$, which will mean that $u'(t) \in C([0,T], H)$. We have that

$$S_j'(t) = \sum_{k=1}^{j} (a_k t^{\alpha-1} E_{\alpha,\alpha}(-\nu_k^2 t^\alpha) + b_k E_{\alpha,1}(-\nu_k^2 t^\alpha)) v_k.$$

By Parseval equality, we obtain that

$$||S_j'(t)||^2 = \sum_{k=1}^{j} |a_k t^{\alpha-1} E_{\alpha,\alpha}(-\nu_k^2 t^\alpha) + b_k E_{\alpha,1}(-\nu_k^2 t^\alpha)|^2 \leq$$

$$\leq C \sum_{k=1}^{j} |a_k t^{\alpha-1} E_{\alpha,\alpha}(-\nu_k^2 t^\alpha)|^2 + C \sum_{k=1}^{j} |b_k E_{\alpha,1}(-\nu_k^2 t^\alpha)|^2.$$

Let us estimate the following two terms, separately

$$I_1 = |a_k t^{\alpha-1} E_{\alpha,\alpha}(-\nu_k^2 t^\alpha)| = \left| \frac{\varphi_k t^{\alpha-1} E_{\alpha,2}(-\nu_k^2 T^\alpha) E_{\alpha,\alpha}(-\nu_k^2 t^\alpha)}{T((E_{\alpha,2}(-\nu_k^2 T^\alpha))^2 + E_{\alpha,3}(-\nu_k^2 T^\alpha)(1 - E_{\alpha,1}(-\nu_k^2 T^\alpha)))} \right|,$$

and

$$I_2 = |b_k E_{\alpha,1}(-\nu_k^2 t^\alpha)| = \left| \frac{\varphi_k (1 - E_{\alpha,1}(-\nu_k^2 T^\alpha)) E_{\alpha,1}(-\nu_k^2 t^\alpha)}{T^2((E_{\alpha,2}(-\nu_k^2 T^\alpha))^2 + E_{\alpha,3}(-\nu_k^2 T^\alpha)(1 - E_{\alpha,1}(-\nu_k^2 T^\alpha)))} \right|.$$

To estimate I_1, we apply Lemmas 3–7. Then

$$I_1 \le t^{\alpha-1} \frac{|\varphi_k|}{T} \frac{1}{C_0 \Gamma(\alpha)} \le C t^{\alpha-1} T^{-1} |\varphi_k|. \tag{16}$$

Similarly

$$I_2 \le \frac{|\varphi_k|}{T^2} \frac{1}{C_0} \le C T^{-2} |\varphi_k|. \tag{17}$$

Apply estimates (16) and (17), we get

$$||S_j'(t)||^2 \le C^2 t^{2(\alpha-1)} T^{-2} \sum_{k=1}^{j} |\varphi_k|^2 + C^2 T^{-4} \sum_{k=1}^{j} |\varphi_k|^2.$$

Hence

$$||S_j'(t)||^2 \le C||\varphi||, \ t \ge 0.$$

Further let us show that $u'(t)$ is absolutely continuous. For this, we take the first-order derivative with respect to t from the partial sums $S_j'(t)$:

$$S_j''(t) = \sum_{k=1}^{j} (a_k t^{\alpha-2} E_{\alpha,\alpha-1}(-\nu_k^2 t^\alpha) + b_k t^{\alpha-1} E_{\alpha,\alpha}(-\nu_k^2 t^\alpha)) v_k.$$

From this it is easy to see that $S_j''(t) \in L((0,T), H)$. Therefore, we get $u(t) \in AC^1([0,T]; H)$.

Now we show that the following sum $D_t^\alpha S_j(t)$ converge uniformly in $t \in (0,T)$. To do this, first consider the sums

$$(I + A)^{-1} A S_j(t) = \sum_{k=1}^{j} \frac{\lambda_k}{1 + \lambda_k} (a_k E_{\alpha,1}(-\nu_k^2 t^\alpha) + b_k t E_{\alpha,2}(-\nu_k^2 t^\alpha)) v_k.$$

By Parseval equality, we get

$$||(I + A)^{-1} A S_j(t)||^2 = \sum_{k=1}^{j} \frac{\lambda_k^2}{(1 + \lambda_k)^2} |a_k E_{\alpha,1}(-\nu_k^2 t^\alpha) + b_k t E_{\alpha,2}(-\nu_k^2 t^\alpha)|^2 \le$$

$$\le C \sum_{k=1}^{j} \frac{\lambda_k^2}{(1 + \lambda_k)^2} |a_k E_{\alpha,1}(-\nu_k^2 t^\alpha)|^2 + C \sum_{k=1}^{j} \frac{\lambda_k^2}{(1 + \lambda_k)^2} |b_k t E_{\alpha,2}(-\nu_k^2 t^\alpha)|^2.$$

By estimates (14), (15) and $\frac{\lambda_k}{1+\lambda_k} \le 1$ we have that

$$||(I + A)^{-1} A S_j(t)||^2 \le C^2 T^{-2} \sum_{k=1}^{j} |\varphi_k|^2 + C^2 T^{-4} t^2 \sum_{k=1}^{j} |\varphi_k|^2.$$

From this, since $\varphi \in H$, we have that

$$C^2 T^{-2} \sum_{k=1}^{j} |\varphi_k|^2 + C^2 T^{-4} t^2 \sum_{k=1}^{j} |\varphi_k|^2 \le const.$$

Therefore $(I + A)^{-1} Au(t) \in C((0, T); H)$. Now applying the obvious equality
$D_t^\alpha u(t) = -\nu^2 (I + A)^{-1} Au(t)$, which follows from the commutativity of the corresponding operators, we obtain $D_t^\alpha u(t) \in C((0, T), H)$.

It remains to prove the continuity of $AD_t^\alpha u(t)$. From equality $AD_t^\alpha u(t) = -D_t^\alpha u(t) - \nu^2 Au(t)$ and continuity of $D_t^\alpha u(t)$ and $Au(t)$, it follows $AD_t^\alpha u(t) \in C((0, T), D(A))$. Theorem 1 is proved.

4 Conclusions

The work is devoted to the study of the correctness of a certain non-local problem (1)–(3) for equations of Busineski type. Namely, the question of the existence and uniqueness of a solution to the corresponding non-local problem is analyzed. In recent years, a number of works have appeared where initial boundary value problems for various types of equations of Busineski type have been studied. The motivation for this was primarily the numerous applications of such problems in the modeling of various processes in physics and mechanics.

Recently, a fundamental work [19] (see also [20]) appeared, where the correctness of a similar non-local problem was studied in the case when $\alpha = 2$. Here the authors discovered an interesting phenomenon: the correctness of the problem significantly depends on the duration of the process T and the parameter ν. It turned out that the most optimal case is when the process does not last that long, i.e. $\frac{\nu T}{2\pi} \in (0, 1)$: here the problem is correct for any $\varphi \in D(A)$. If the process lasts longer, i.e. $\frac{\nu T}{2\pi} \ge 1$, then additional conditions will appear on the function φ and these conditions depend on whether the number $\frac{\nu T}{2\pi}$ is a natural number or not.

The question naturally arises: does this phenomenon persist in the case when, instead of the second derivative with respect to time, we take a derivative in the sense of Caputo D_t^α of order $1 < \alpha < 3/2$. In this paper it is shown that there is no such effect and the corresponding non-local problem has a unique solution for any $\varphi \in D(A)$.

In the future, it would be interesting to consider other fractional derivatives instead of Caputo derivatives, to see if the corresponding effect would take place. Also interesting is the study of inverse problems to determine the right-hand side of the equation for such non-local problems.

These tasks are the subject of further research.

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Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

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A stable difference scheme for the solution of a source identification problem for telegraph-parabolic equations

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In the present paper, we construct a first order of accuracy difference scheme for the approximate solution of the inverse problem for telegraph-parabolic equations with an unknown spacewise dependent source term. The unique solvability of constructed difference scheme and the stability estimates for its solution were obtained. The proofs are based on the spectral representation of the self-adjoint positive definite operator in a Hilbert space.

Keywords: Difference scheme, source identification problem, telegraph-parabolic equation, stability estimates.

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Introduction

Differential equations with unknown source terms are widely used in the mathematical modelling of real-life phenomena in many different fields of science and have been broadly investigated over the years (see, e.g., [1]–[9] and the references therein).

The problems for differential equations containing a time- and/or space-dependent parameter (source term) are called source identification problems. These types of problems are inverse and their solutions cannot be determined uniquely from imposed initial and/or boundary conditions. To achieve a well-posedness of a source identification problem, one needs to provide some additional condition(s). Source identification problems for mixed type differential equations have been receiving a great deal of attention recently (see, e.g., [10]–[19] for hyperbolic-parabolic, [20]–[22] for elliptic-hyperbolic, and [23] for parabolic-elliptic source identification problems).

Numerous local and nonlocal boundary value problems for telegraph-parabolic equations with unknown source terms can be reduced to the following abstract problem for the differential equation with a spacewise dependent parameter p

$$\begin{cases} u''(t) + \alpha u'(t) + Au(t) = p + f(t), \ 0 < t < 1, \\ u'(t) + Au(t) = p + g(t), \ -1 < t < 0, \\ u(0+) = u(0-), \ u'(0+) = u'(0-), \\ u(-1) = \varphi, \ u(\lambda) = \psi, \ -1 < \lambda \leq 1 \end{cases} \tag{1}$$

in a Hilbert space H with a self-adjoint positive definite operator A satisfying $A \geq \delta I$, where $\delta > \frac{\alpha^2}{4}$ and $\alpha \geq 0$. The last condition in (1) is considered in order to compensate the uncertainty in the problem due to unknown term p.

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The unique solvability of problem (1) in the space $C(H)$ of the continuous H-valued functions $u(t)$ defined on $[-1,1]$, equipped with the norm

$$\|u\|_{C(H)} = \max_{-1\le t\le 1} \|u(t)\|_H$$

was established in [24], and the following theorem on the continuous dependence of the solution on the given data was proven.

Theorem 1 ([24]). Assume that $\varphi,\ \psi \in D(A)$. Let $f(t)$ and $g(t)$ be continuously differentiable functions on $[0,1]$ and $[-1,0]$, respectively. Then, for the solution $\{u(t),p\}$ of problem (1) in $C(H)\times H$ the following stability inequalities

$$\|u\|_{C(H)} + \|A^{-1}p\|_H \le M(\delta,\lambda)\Big[\|\varphi\|_H + \|\psi\|_H + \max_{0\le t\le 1}\|f(t)\|_H + \max_{-1\le t\le 0}\|g(t)\|_H\Big],$$

$$\max_{0\le t\le 1}\|u''(t)\|_H + \max_{0\le t\le 1}\|\alpha u'(t)\|_H + \max_{-1\le t\le 0}\|u'(t)\|_H + \|Au\|_{C(H)} + \|p\|_H$$
$$\le M(\delta,\lambda)\Big[\|A\varphi\|_H + \|A\psi\|_H + \max_{0\le t\le 1}\|f'(t)\|_H + \|f(0)\|_H + \max_{-1\le t\le 0}\|g'(t)\|_H + \|g(0)\|_H\Big]$$

hold, where $M(\delta,\lambda)$ does not depend on φ, ψ, $f(t)$ and $g(t)$.

In general, the differential equations with unknown parameters are not solvable analytically and therefore one needs to use numerical methods to approximate their solutions. The main goal of this study is to construct and investigate a first order of accuracy stable difference scheme for the approximate solution of abstract problem (1). We prove the unique solvability of the constructed difference scheme and obtain the stability estimates for its solution. The analysis is based on the operator approach and the proofs of the stability estimates are based on the spectral representation of the self-adjoint positive definite operator in a Hilbert space.

1 *First order of accuracy stable difference scheme*

Let $\tau = 1/N$ be sufficiently small positive number satisfying $\lambda \ge -1+\tau$. Let us define the grid points $t_k = k\tau,\ -N\le k\le N$. For the approximate solution of problem (1), we construct the first order of accuracy stable difference scheme

$$\begin{cases} \frac{u_{k+1}-2u_k+u_{k-1}}{\tau^2} + \alpha\frac{u_{k+1}-u_k}{\tau} + Au_{k+1} = p + f_k, \quad 1\le k\le N-1, \\ \frac{u_k-u_{k-1}}{\tau} + Au_k = p + g_k, \quad -N+1\le k\le 0, \\ \frac{u_1-u_0}{\tau} = p - Au_0 + g_0, \quad u_{-N}=\varphi, \quad u_\ell = \psi, \end{cases} \tag{2}$$

where $\ell = \lfloor \lambda/\tau \rfloor$, $f_k = f(t_k)$, $1\le k\le N-1$ and $g_k = g(t_k)$, $-N+1\le k\le 0$.

We first present some lemmas, which we will need in the remaining part of this paper. Here and everywhere else, we denote

$$R = \left(\left(1+\frac{\alpha\tau}{2}\right)I + i\tau\left(A - \frac{\alpha^2}{4}I\right)^{1/2}\right)^{-1}, \quad \tilde{R} = \left(\left(1+\frac{\alpha\tau}{2}\right)I - i\tau\left(A - \frac{\alpha^2}{4}I\right)^{1/2}\right)^{-1}$$

and

$$Q = (I+\tau A)^{-1}.$$

Lemma 1 ([25]). The following estimates hold

$$\|R\|_{H\to H}\le 1, \qquad \|\tilde{R}\|_{H\to H}\le 1, \qquad \|\tilde{R}^{-1}R\|_{H\to H}\le 1, \qquad \left\|R^{-1}\tilde{R}\right\|_{H\to H}\le 1. \tag{3}$$

Lemma 2 ([25]). The following estimates hold

$$\|Q^m\|_{H\to H} \le \frac{1}{1+m\tau\delta} < 1, \quad m \ge 1, \tag{4}$$

$$\|A^{1/2}Q^m\|_{H\to H} \le \frac{1}{2\sqrt{m\tau}}, \quad m \ge 1. \tag{5}$$

Lemma 3. If $-1+\tau \le \lambda < \tau$, then $-N+1 \le \ell \le 0$, and the following estimate holds

$$\left\| \left(I - Q^{N+\ell}\right)^{-1} \right\|_{H\to H} \le M_1(\delta,\lambda). \tag{6}$$

Proof. The proof of estimate (6) is based on estimate (4).

Lemma 4. The following estimates hold for $m \ge 1$

$$\left\| \left[\frac{R^{m-1}}{2} \left(I - \frac{\alpha}{2i}\left(A - \frac{\alpha^2}{4}I\right)^{-1/2} \right) + \frac{\tilde{R}^{m-1}}{2} \left(I + \frac{\alpha}{2i}\left(A - \frac{\alpha^2}{4}I\right)^{-1/2} \right) \right.\right.$$
$$\left.\left. + \frac{A}{2i}\left(A - \frac{\alpha^2}{4}I\right)^{-1/2} \left(\tilde{R}^{-1}R^{m-1} - R^{-1}\tilde{R}^{m-1}\right) \right] Q^N \right\|_{H\to H} < 1. \tag{7}$$

Proof. Since

$$\frac{R^{m-1}}{2} \left(I - \frac{\alpha}{2i}\left(A - \frac{\alpha^2}{4}I\right)^{-1/2} \right) + \frac{\tilde{R}^{m-1}}{2} \left(I + \frac{\alpha}{2i}\left(A - \frac{\alpha^2}{4}I\right)^{-1/2} \right)$$
$$+ \frac{A}{2i}\left(A - \frac{\alpha^2}{4}I\right)^{-1/2} \left(\tilde{R}^{-1}R^{m-1} - R^{-1}\tilde{R}^{m-1}\right)$$
$$= \left[I - \tau A + i\left\{ \frac{\alpha}{2}I - \left(1 + \frac{\alpha\tau}{2}\right)A \right\} \left(A - \frac{\alpha^2}{4}I\right)^{-1/2} \right] \frac{R^{m-1}}{2}$$
$$+ \left[I - \tau A - i\left\{ \frac{\alpha}{2}I - \left(1 + \frac{\alpha\tau}{2}\right)A \right\} \left(A - \frac{\alpha^2}{4}I\right)^{-1/2} \right] \frac{\tilde{R}^{m-1}}{2},$$

using (3) and the following estimates

$$\left\| \left[I - \tau A \pm i\left\{ \frac{\alpha}{2}I - \left(1 + \frac{\alpha\tau}{2}\right)A \right\} \left(A - \frac{\alpha^2}{4}I\right)^{-1/2} \right] Q^N \right\|_{H\to H} < 1, \tag{8}$$

we obtain (7). The proof of estimates (8) is based on the spectral representation of the self-adjoint positive definite operator A in a Hilbert space H [25].

Lemma 5. If $\tau \le \lambda \le 1$, then $1 \le \ell \le N$ and the following estimate holds

$$\left\| \left(I - \left[\frac{R^{\ell-1}}{2} \left(I - \frac{\alpha}{2i}\left(A - \frac{\alpha^2}{4}I\right)^{-1/2} \right) + \frac{\tilde{R}^{\ell-1}}{2} \left(I + \frac{\alpha}{2i}\left(A - \frac{\alpha^2}{4}I\right)^{-1/2} \right) \right.\right.\right.$$
$$\left.\left.\left. + \frac{A}{2i}\left(A - \frac{\alpha^2}{4}I\right)^{-1/2} \left(\tilde{R}^{-1}R^{\ell-1} - R^{-1}\tilde{R}^{\ell-1}\right) \right] Q^N \right)^{-1} \right\|_{H\to H} \le M_2(\delta,\lambda,\alpha). \tag{9}$$

Proof. The proof of estimate (9) is based on estimate (7).

We now present the main theorem for the solution of the first order of accuracy difference scheme (2).

Theorem 2. The difference scheme (2) has a unique solution and the following stability estimate holds

$$\max_{-N\le k\le N}\|u_k\|_H+\left\|A^{-1}p\right\|_H$$
$$\le M^*(\delta,\lambda,\alpha)\Big[\|\varphi\|_H+\|\psi\|_H+\max_{1\le k\le N-1}\|f_k\|_H+\max_{-N+1\le k\le 0}\|g_k\|_H\Big], \tag{10}$$

where $M^*(\delta,\lambda,\alpha)$ is independent of φ, ψ, τ, f_k and g_k.

Proof. Let us denote

$$u_k=v_k+A^{-1}p,\quad -N\le k\le N. \tag{11}$$

Then, the difference scheme (2) results in the following auxiliary difference scheme

$$\begin{cases}\frac{v_{k+1}-2v_k+v_{k-1}}{\tau^2}+\alpha\frac{v_{k+1}-v_k}{\tau}+Av_{k+1}=f_k,\quad 1\le k\le N-1,\\ \frac{v_k-v_{k-1}}{\tau}+Av_k=g_k,\quad -N+1\le k\le 0,\\ \frac{v_1-v_0}{\tau}=-Av_0+g_0,\quad v_\ell=v_{-N}+\psi-\varphi.\end{cases} \tag{12}$$

First, we obtain the formulas for solution of scheme (12). For the given v_0 the following difference scheme

$$\begin{cases}\frac{v_{k+1}-2v_k+v_{k-1}}{\tau^2}+\alpha\frac{v_{k+1}-v_k}{\tau}+Av_{k+1}=f_k,\quad 1\le k\le N-1,\\ \frac{v_1-v_0}{\tau}=-Av_0+g_0\end{cases}$$

has a solution

$$v_k=\left[\frac{R^{k-1}}{2}\left(I-\frac{\alpha}{2i}\left(A-\frac{\alpha^2}{4}I\right)^{-1/2}\right)+\frac{\tilde{R}^{k-1}}{2}\left(I+\frac{\alpha}{2i}\left(A-\frac{\alpha^2}{4}I\right)^{-1/2}\right)\right]v_0$$
$$+\left(R-\tilde{R}\right)^{-1}\tau\left(R^k-\tilde{R}^k\right)(-Av_0+g_0)$$
$$-\frac{1}{2i}\sum_{j=1}^{k}\left(A-\frac{\alpha^2}{4}I\right)^{-1/2}\left(R^{k-j}-\tilde{R}^{k-j}\right)f_j\tau,\quad 1\le k\le N. \tag{13}$$

Furthermore, for the given v_{-N}, the following difference scheme

$$\frac{v_k-v_{k-1}}{\tau}+Av_k=g_k,\ -N+1\le k\le 0$$

has a solution

$$v_k=Q^{N+k}v_{-N}+\sum_{j=-N+1}^{k}Q^{k-j+1}g_j\tau,\quad -N+1\le k\le 0. \tag{14}$$

In particular, putting $k=0$ in (14), we get

$$v_0=Q^Nv_{-N}+\sum_{j=-N+1}^{0}Q^{-j+1}g_j\tau.$$

Then, by putting this expression for v_0 in (13), we obtain

$$v_k = \left[\frac{R^{k-1}}{2} \left(I - \frac{\alpha}{2i} \left(A - \frac{\alpha^2}{4} I \right)^{-1/2} \right) + \frac{\tilde{R}^{k-1}}{2} \left(I + \frac{\alpha}{2i} \left(A - \frac{\alpha^2}{4} I \right)^{-1/2} \right) - \tau A \left(R - \tilde{R} \right)^{-1} \left(R^k - \tilde{R}^k \right) \right] \left(Q^N v_{-N} + \sum_{j=-N+1}^{0} Q^{-j+1} g_j \tau \right) + \left(R - \tilde{R} \right)^{-1} \left(R^k - \tilde{R}^k \right) \tau g_0 - \frac{1}{2i} \sum_{j=1}^{k} \left(A - \frac{\alpha^2}{4} I \right)^{-1/2} \left(R^{k-j} - \tilde{R}^{k-j} \right) f_j \tau, \quad 1 \le k \le N.$$

Using $R - \tilde{R} = -2i\tau \left(A - \frac{\alpha^2}{4} I \right)^{1/2} R\tilde{R}$, we have

$$v_k = \left[\frac{R^{k-1}}{2} \left(I - \frac{\alpha}{2i} \left(A - \frac{\alpha^2}{4} I \right)^{-1/2} \right) + \frac{\tilde{R}^{k-1}}{2} \left(I + \frac{\alpha}{2i} \left(A - \frac{\alpha^2}{4} I \right)^{-1/2} \right) + \frac{A}{2i} \left(A - \frac{\alpha^2}{4} I \right)^{-1/2} \left(\tilde{R}^{-1} R^{k-1} - R^{-1} \tilde{R}^{k-1} \right) \right] \left(Q^N v_{-N} + \sum_{j=-N+1}^{0} Q^{-j+1} g_j \tau \right) - \frac{1}{2i} \left(A - \frac{\alpha^2}{4} I \right)^{-1/2} \left(\tilde{R}^{-1} R^{k-1} - R^{-1} \tilde{R}^{k-1} \right) g_0 - \frac{1}{2i} \sum_{j=1}^{k} \left(A - \frac{\alpha^2}{4} I \right)^{-1/2} \left(R^{k-j} - \tilde{R}^{k-j} \right) f_j \tau, \quad 1 \le k \le N. \tag{15}$$

If $-1 + \tau \le \lambda < \tau$, then $-N + 1 \le \ell \le 0$, and therefore from (12) and (14) it follows

$$v_\ell = v_{-N} + \psi - \varphi = Q^{N+\ell} v_{-N} + \sum_{j=-N+1}^{\ell} Q^{\ell-j+1} g_j \tau,$$

so that

$$v_{-N} = \left(I - Q^{N+\ell} \right)^{-1} \left(\sum_{j=-N+1}^{\ell} Q^{\ell-j+1} g_j \tau + \varphi - \psi \right). \tag{16}$$

If $\tau \le \lambda \le 1$, then $1 \le \ell \le N$, and therefore from (12) and (15) it follows

$$v_\ell = v_{-N} + \psi - \varphi = \left[\frac{R^{\ell-1}}{2} \left(I - \frac{\alpha}{2i} \left(A - \frac{\alpha^2}{4} I \right)^{-1/2} \right) + \frac{\tilde{R}^{\ell-1}}{2} \left(I + \frac{\alpha}{2i} \left(A - \frac{\alpha^2}{4} I \right)^{-1/2} \right) + \frac{A}{2i} \left(A - \frac{\alpha^2}{4} I \right)^{-1/2} \left(\tilde{R}^{-1} R^{\ell-1} - R^{-1} \tilde{R}^{\ell-1} \right) \right] \left(Q^N v_{-N} + \sum_{j=-N+1}^{0} Q^{-j+1} g_j \tau \right) - \frac{1}{2i} \left(A - \frac{\alpha^2}{4} I \right)^{-1/2} \left(\tilde{R}^{-1} R^{\ell-1} - R^{-1} \tilde{R}^{\ell-1} \right) g_0 - \frac{1}{2i} \sum_{j=1}^{\ell} \left(A - \frac{\alpha^2}{4} I \right)^{-1/2} \left(R^{\ell-j} - \tilde{R}^{\ell-j} \right) f_j \tau,$$

so that

$$v_{-N} = \Bigg(I - \Bigg[\frac{R^{\ell-1}}{2}\left(I - \frac{\alpha}{2i}\left(A - \frac{\alpha^2}{4}I\right)^{-1/2}\right) + \frac{\tilde{R}^{\ell-1}}{2}\left(I + \frac{\alpha}{2i}\left(A - \frac{\alpha^2}{4}I\right)^{-1/2}\right)$$

$$+\frac{A}{2i}\left(A - \frac{\alpha^2}{4}I\right)^{-1/2}\left(\tilde{R}^{-1}R^{\ell-1} - R^{-1}\tilde{R}^{\ell-1}\right)\Bigg] Q^N\Bigg)^{-1}$$

$$\times\Bigg[\Bigg\{\frac{R^{\ell-1}}{2}\left(I - \frac{\alpha}{2i}\left(A - \frac{\alpha^2}{4}I\right)^{-1/2}\right) + \frac{\tilde{R}^{\ell-1}}{2}\left(I + \frac{\alpha}{2i}\left(A - \frac{\alpha^2}{4}I\right)^{-1/2}\right)$$

$$+\frac{A}{2i}\left(A - \frac{\alpha^2}{4}I\right)^{-1/2}\left(\tilde{R}^{-1}R^{\ell-1} - R^{-1}\tilde{R}^{\ell-1}\right)\Bigg\}\sum_{j=-N+1}^{0} Q^{-j+1}g_j\tau$$

$$-\frac{1}{2i}\left(A - \frac{\alpha^2}{4}I\right)^{-1/2}\left(\tilde{R}^{-1}R^{\ell-1} - R^{-1}\tilde{R}^{\ell-1}\right)g_0$$

$$-\frac{1}{2i}\sum_{j=1}^{\ell}\left(A - \frac{\alpha^2}{4}I\right)^{-1/2}\left(R^{\ell-j} - \tilde{R}^{\ell-j}\right)f_j\tau + \varphi - \psi\Bigg]. \tag{17}$$

Thus, for the solution of auxiliary difference scheme (12), we have formulas (14) and (15), with v_{-N} being found by formula (16) if $-1+\tau \le \lambda < \tau$ and formula (17), if $\tau \le \lambda \le 1$. Now, taking into account that $u_{-N} = \varphi$, we have $A^{-1}p = \varphi - v_{-N}$. Then, using (11), we obtain the solution of difference scheme (2).

Now, let us obtain the estimate (10). Using (16) and estimates (4) and (6), we obtain

$$\|v_{-N}\|_H \le M_1(\delta,\lambda)\Big[\|\varphi\|_H + \|\psi\|_H + \max_{-N+1\le k\le 0}\|g_k\|_H\Big]. \tag{18}$$

Next, using (17) and the estimates (3), (4), (5), and (9), we obtain

$$\|v_{-N}\|_H \le M_2(\delta,\lambda,\alpha)\Big[\|\varphi\|_H + \|\psi\|_H + \max_{1\le k\le N-1}\|f_k\|_H + \max_{-N+1\le k\le 0}\|g_k\|_H\Big]. \tag{19}$$

Then, using (14) and the estimates (4), (18), and (19), we get

$$\begin{aligned}\|v_k\|_H &\le \|v_{-N}\|_H + \max_{-N+1\le k\le 0}\|g_k\|_H \\ &\le M_3(\delta,\lambda,\alpha)\Big[\|\varphi\|_H + \|\psi\|_H + \max_{1\le k\le N-1}\|f_k\|_H + \max_{-N+1\le k\le 0}\|g_k\|_H\Big]\end{aligned} \tag{20}$$

for $k = -N+1,\dots,0$. Using (15) and the estimates (3), (4), (5), (7), (18), and (19), we obtain

$$\begin{aligned}\|v_k\|_H &\le \|v_{-N}\|_H + M_4(\delta,\alpha)\Big(\max_{1\le k\le N-1}\|f_k\|_H + \max_{-N+1\le k\le 0}\|g_k\|_H\Big) \\ &\le M_5(\delta,\lambda,\alpha)\Big[\|\varphi\|_H + \|\psi\|_H + \max_{1\le k\le N-1}\|f_k\|_H + \max_{-N+1\le k\le 0}\|g_k\|_H\Big]\end{aligned} \tag{21}$$

for $k = 1,\dots,N$. Since $A^{-1}p = \varphi - v_{-N}$, using (18), (19), and the triangle inequality, we have

$$\begin{aligned}\left\|A^{-1}p\right\|_H &\le \|\varphi\|_H + \|v_{-N}\|_H \\ &\le M_6(\delta,\lambda,\alpha)\Big[\|\varphi\|_H + \|\psi\|_H + \max_{1\le k\le N-1}\|f_k\|_H + \max_{-N+1\le k\le 0}\|g_k\|_H\Big].\end{aligned} \tag{22}$$

Finally, using (11), (20), (21), and (22), we prove the estimate

$$\begin{aligned} \|u_k\|_H \leq & \left\|A^{-1}p\right\|_H + \|v_k\|_H \\ \leq & M_7(\delta,\lambda,\alpha)\left[\|\varphi\|_H + \|\psi\|_H + \max_{1\leq k\leq N-1}\|f_k\|_H + \max_{-N+1\leq k\leq 0}\|g_k\|_H\right] \end{aligned} \tag{23}$$

for $k=-N,\dots,N$. Estimate (10) follows from (22) and (23).

Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

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Absolutely stable difference scheme for the delay partial differential equation with involution and Robin boundary condition

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This paper examines the initial value problem for a third-order delay partial differential equation with involution and Robin boundary condition. We construct a first-order accurate difference scheme to obtain the numerical solution for this equation. Illustrative numerical results are provided.

Keywords: numerical algorithm, involution, Robin boundary condition, third order partial differential equations, delay.

2020 Mathematics Subject Classification: 35G10, 35L90, 58D25.

Introduction

Over the years, nonlocal and local boundary value problems (BVPs) for third-order partial differential equations (PDEs) have gone through extensive investigations (see, for instance, [1–9]). Time delay (TD) is a common phenomenon in various engineering projects. The theory as well as applications of delay nonlinear and linear third-order ordinary differential and difference equations having delay terms have been explored in numerous works (see, for instance, [10–16]).

The stability of the third order partial delay differential equation (PDDE) having involution and Dirichlet condition was investigated in [17]. Nevertheless, the third order PDDE with involution and Robin condition (IRC) is not studied before. Therefore, the main motivation for this paper is to study the stability of the third order partial delay differential and difference equations with IRC.

1 Differential problem stability

In $[0,\infty)\times(-\rho,\rho)$, the initial BVP for the TD third order PDE with IRC.

$$\begin{cases} \frac{\partial^3 u(\zeta,y)}{\partial\zeta^3} - \left(\delta(y)u_{\zeta y}(\zeta,y)\right)_y + \beta\left(\delta(-y)u_{\zeta,-y}(\zeta,-y)\right)_{-y} \\ = -b\left(-\delta(y)u_y(\zeta-w,y)\right)_y + \beta\left(\delta(-y)u_{-y}(\zeta,-y)\right)_{-y} \\ +\Phi(\zeta,y),\ 0<t<\infty, (-\rho,\rho), \\ \\ u(\zeta,y)=g(\zeta,y), -w\le\zeta\le 0, y\in[-\rho,\rho], \\ \\ \alpha_1 u(\zeta,-\rho)-\gamma_1 u_y(\zeta,-\rho)=0, \alpha_2 u(\zeta,\rho)+\gamma_2 u_y(\zeta,\rho)=0,\ \ 0\le\zeta<\infty \end{cases} \quad (1)$$

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is studied. In this study, we make the assumption that $w > 0$, $\overline{\delta} \geq \delta(y) = \delta(-y) \geq \underline{\delta} > 0$, $y \in (-t, t)$ and $\underline{\delta} - \overline{\delta}|\beta| \geq 0$, α_1, $\alpha_2, \gamma_1, \gamma_2$ are non negative constants.

We examine the Hilbert space $L_2[-\rho, \rho]$ consisting of all square integrable functions defined on $[-\rho, \rho]$, equipped with the norm

$$\| \Phi \|_{L_2[-\rho,\rho]} = \left(\int_{-\rho}^{\rho} |\Phi(y)|^2 dy \right)^{\frac{1}{2}}.$$

A unique solution $u(\zeta, y)$ is possessed by problem (1) for the smooth functions $\delta(y)$, $y \in (-t, t)$, $g(\zeta, y), -w \leq \zeta \leq 0$, $y \in [-\rho, \rho]$, $\Phi(\zeta, y)$, $0 < \zeta < \infty$, $y \in (-\rho, \rho)$, and $b \in R^1$, provided the compatibility conditions are met.

Theorem 1. The following stability estimates hold for the solutions of problem (1):

$$\max_{0\leq\zeta\leq nw} \|v_{\zeta\zeta}(\zeta, \cdot)\|_{W_2^1(-\rho,\rho)}, \max_{0\leq\zeta\leq nw} \|v_{\zeta}(\zeta, \cdot)\|_{W_2^2(-\rho,\rho)}, \max_{0\leq\zeta\leq nw} \|v(\zeta, \cdot)\|_{W_2^3(-\rho,\rho)}$$

$$\leq M_2 \left[(2 + |b|\, w)^n a_0 + \sum_{i=1}^{n} (2 + |b|\, w)^{n-i} \int_{(i-1)\omega}^{iw} \|\Phi(s, \cdot)\|_{W_2^1(-\rho,\rho)} ds \right],$$

$$a_0 = \max \left\{ \max_{-w\leq\zeta\leq 0} \|g_{\zeta\zeta}(\zeta, \cdot)\|_{W_2^1(-\rho,\rho)}, \right.$$

$$\left. \max_{-w\leq\zeta\leq 0} \|g(\zeta, \cdot)\|_{W_2^3(-\rho,\rho)} \right\}.$$

Here, the Sobolev spaces W_2^k for $k = 1, 2, 3$ consist of all square integrable functions $\psi(y)$ defined on $[-\rho, \rho]$, each equipped with their respective norms

$$\|\psi\|_{W_2^k(-\rho,\rho)} = \left(\int_{-\rho}^{\rho} \sum_{i=0}^{k} \left(\psi_{\underbrace{y \cdots y}_{i \text{ time}}}(y) \right)^2 dy \right)^{\frac{1}{2}}.$$

Note that M_2 does not depend on $g(t, y)$ and $\Phi(\zeta, y)$.

Proof. With this we are able to change problem (1) to the following initial value problem

$$\begin{cases} \frac{d^3 v(\zeta)}{d\zeta^3} + A \frac{dv(\zeta)}{d\zeta} = bAv(\zeta - w) + \Phi(\zeta), \ 0 < \zeta < \infty, \\ v(\zeta) = g(\zeta), -w \leq \zeta \leq 0 \end{cases} \tag{2}$$

in $H = L_2[-\rho, \rho]$ which happens to be Hilbert space having a self-adjoint positive definite operator (SAPDO) A that is given by the formula below:

$$Au(y) = -(\delta(y)u_y(y))_y + \beta(\delta(-y)u_{-y}(-y))_{-y}, \tag{3}$$

having domain

$$D(A) = \{u(y) : u(y),\ u_y(y),\ (\delta(y)u_y)_y \in L_2[-\rho, \rho], \alpha_1 u(-\rho) - \gamma_1 u_y(-\rho) = 0, \alpha_2 u(\rho) + \gamma_2 u_y(\rho) = 0\}.$$

Theorem 1's proof relies on the positive definiteness as well as the self-adjointness of the space operator A as specified by equation (3), as well as the results presented in paper [18]. Additionally, the proof incorporates the theorem on the stability of the solution to problem (2).

Theorem 2. [19] The following estimate applies to the solution of problem (2):

$$\max_{0\leq\zeta\leq nw}\left\|A^{\frac{1}{2}}\frac{d^2v(\zeta)}{d\zeta^2}\right\|_H, \max_{0\leq\zeta\leq nw}\left\|A\frac{d\zeta(\zeta)}{d\zeta}\right\|_H, \frac{1}{2}\max_{0\leq\zeta\leq nw}\left\|A^{\frac{3}{2}}v(\zeta)\right\|_H$$

$$\leq (2+|b|\,w)^n\,a_0+\int_0^{nw}\left\|A^{\frac{1}{2}}\Phi(s)\right\|_H ds, n=1,2,...,$$

where

$$a_0=\max\left\{\max_{-w\leq\zeta\leq 0}\left\|A^{\frac{1}{2}}\frac{d^2g(\zeta)}{d\zeta^2}\right\|_H, \max_{-w\leq\zeta\leq 0}\left\|A\frac{dg(\zeta)}{d\zeta}\right\|_H, \max_{-w\leq\zeta\leq 0}\left\|A^{\frac{3}{2}}g(t)\right\|_H\right\}.$$

Stability of the difference scheme

For the approximate solution of problem (1), we study the stable difference scheme (DS). Problem (1) discretization is conducted in two stages.

Firstly, the spatial discretization is executed. The equation below defines the grid space:

$$[-t,t]_h=\left\{y=y_n \;\middle|\; y_n=nh,\ -\Gamma\leq n\leq\Gamma,\ \Gamma h=t\right\}.$$

We present the Hilbert space $L_{2h}=L_2([-t,t]_h)$ of the grid functions $\varphi^h(y)=\{\varphi^n\}_{-\Gamma}^{\Gamma}$ defined on $[-t,t]_h$, endowed with the norm

$$\left\|\varphi^h\right\|_{L_{2h}}=\left(\sum_{y\in[-t,t]_h}\left|\varphi^h(y)\right|^2 h\right)^{1/2}.$$

We associate the difference operator A_h^y with the differential operator A that is defined by equation (3), using the following expression

$$A_h^y\varphi^h(y)=\left\{-\left(\delta(y)\varphi_{\overline{y}}^n\right)_y-\beta\left(\delta(-y)\varphi_{\overline{y}}^{-n}\right)_y\right\}_{-\Gamma+1}^{\Gamma-1}, \tag{4}$$

that acts in the space of grid functions $\varphi^h(y)=\{\varphi^n\}_{-\Gamma}^{\Gamma}$ and meeting the requirements

$$\alpha_1 h\varphi^{-\Gamma}-\gamma_1(\varphi^{-\Gamma}-\varphi^{-\Gamma+1})=0,\ \alpha_2 h\varphi^{\Gamma}+\gamma_2(\varphi^{\Gamma}-\varphi^{\Gamma-1})=0.$$

Here

$$\varphi_{\overline{y}}^n=\frac{\varphi^n-\varphi^{n-1}}{h},\ -\Gamma+1\leq n\leq\Gamma,\quad \varphi_y^n=\frac{\varphi^{n+1}-\varphi^n}{h},\ -\Gamma\leq n\leq\Gamma-1.$$

It is properly-established that A_h^y, as defined by equation (4) is a SAPDO in L_{2h}. By making use of A_h^y, the initial discretization step leads to the problem that follows:

$$\begin{cases}\dfrac{\partial^3 u^h(\zeta,y)}{\partial\zeta^3}+A_h^y u^h(\zeta,y)=-bA_h^y u^h(\zeta-w,y)\\ +\Phi^h(\zeta,y),\ y\in[-t,t]_h,\ 0<\zeta<\infty,\\ u^h(\zeta,y)=g^h(\zeta,y),-w\leq\zeta\leq 0,\ y\in[-t,t]_h,\ -w<\zeta<0.\end{cases} \tag{5}$$

Secondly, problem (5) is replaced with the following first order of accuracy DS

$$\begin{cases} \frac{u_{k+2}^h(y)-3u_{k+1}^h(y)+3u_k^h(y)-u_{k-1}^h(y)}{\eta^3}+A_h^y\frac{u_{k+2}^h(y)-u_{k+1}^h(y)}{\eta} \\ =bA_h^y u_{k-\mathrm{M}}^h(y)+\Phi_k^h(y),\Phi_k^h(y)=\Phi^h(\zeta_k,y),k\geq 1, \quad y\in[-t,t]_h, \\ u_k^h(y)=g^h(\zeta_k,y),-\mathrm{M}\leq k\leq 0, \\ (\Upsilon_h+\eta^2A_h^y)\frac{u_1^h(y)-u_0^h(y)}{\eta}=g_\zeta^h(0,y), \\ (\Upsilon_h+\eta^2A_h^y)\frac{u_2^h(y)-2u_1^h(y)+u_0^h(y)}{\eta^2}=g_{\zeta\zeta}^h(0,y),y\in[-t,t]_h, \\ (\Upsilon_h+\eta^2A_h^y)\frac{u_{m\mathrm{M}+1}^h(y)-u_{m\mathrm{M}}^h(y)}{\eta}=\frac{u_{m\mathrm{M}}^h(y)-u_{m\mathrm{M}-1}^h(y)}{\eta},y\in[-t,t]_h, \\ (\Upsilon_h+\eta^2A_h^y)\frac{u_{m\mathrm{M}+2}^h(y)-2u_{m\mathrm{M}+1}^h(y)+u_{m\mathrm{M}}^h(y)}{\eta^2} \\ =\frac{u_{m\mathrm{M}}^h(y)-2u_{m\mathrm{M}-1}^h(y)+u_{m\mathrm{M}-2}^h(y)}{\eta^2},y\in[-t,t]_h,m=1,2,..., \end{cases} \tag{6}$$

here $\eta=1/\mathrm{M}$ and $\zeta_k=k\eta$, $-\mathrm{M}\leq k<\infty$.

Theorem 3. Let h and η be values that are small enough. The following estimates hold for the solution of DS (6):

$$\max_{0\leq k\leq(m+1)\mathrm{M}-2}\left\|\frac{u_{k+2}^h-2u_{k+1}^h+u_k^h}{\eta^2}\right\|_{W_{2h}^1},\max_{1\leq k\leq(m+1)\mathrm{M}}\left\|\frac{u_k^h-u_{k-1}^h}{\eta}\right\|_{W_{2h}^2},$$

$$\max_{0\leq k\leq(m+1)\mathrm{M}}\|u_k^h\|_{W_{2h}^3}\leq\chi_1\Bigg[(2+\eta|b|(\mathrm{M}-2))^m b_0^h$$

$$+\sum_{i=1}^m(2+\eta|b|(\mathrm{M}-2))^{m-i}\eta\sum_{s=(i-1)\mathrm{M}+1}^{i\mathrm{M}}\|\Phi(\zeta_s)\|_{W_{2h}^1}\Bigg],m=0,1,...,$$

$$b_0^h=\max\left\{\max_{-\mathrm{M}\leq k\leq 0}\|g_{\zeta\zeta}^h(\zeta_k)\|_{W_{2h}^1},\max_{-\mathrm{M}\leq k\leq 0}\|g_\zeta^h(\zeta_k)\|_{W_{2h}^2},\max_{-\mathrm{M}\leq k\leq 0}\|g^h(\zeta_k)\|_{W_{2h}^3}\right\}.$$

Here, W_{2h}^1, W_{2h}^2 and W_{2h}^3 represent spaces of all mesh functions $\psi^h(\zeta)$ defined on the interval $[-\rho,\rho]_h$ having the specific norm

$$\left\|\psi^h\right\|_{W_{2h}^k}=\left(\sum_{y\in[-\rho,\rho]}\sum_{i=0}^k\left(\psi^h_{\underbrace{y\cdots y}_{i\text{ time}}}(y)\right)^2h^k\right)^{\frac{1}{2}}.$$

Note that χ_1 does not depend on η, h, $g^h(t_k)$, and $\Phi_k^h(y)$.

Proof. DS (6) can be written in abstract form

$$\begin{cases} \frac{u_{k+2}^h-3u_{k+1}^h+3u_k^h-u_{k-1}^h}{\eta^3}+A_h\frac{u_{k+2}^h-u_{k+1}^h}{\eta}=bA_hu_{k-\mathrm{M}}^h+\Phi_k^h, k\geq 1, \\ u_k^h=g_k^h, -\mathrm{M}\leq k\leq 0, \\ (\Upsilon_h+\eta^2A_h)\frac{u_1^h-u_0^h}{\eta}=g_\zeta^h(0), (\Upsilon_h+\eta^2A_h)\frac{u_2^h-2u_1^h+u_0^h}{\eta^2}=g_{\zeta\zeta}^h(0), \\ (\Upsilon_h+\eta^2A_h)\frac{u_{m\mathrm{M}+2}^h-2u_{m\mathrm{M}+1}^h+u_{m\mathrm{M}}^h}{\eta^2}=\frac{u_{m\mathrm{M}}^h-2u_{m\mathrm{M}-1}^h+u_{m\mathrm{M}-2}^h}{\eta^2}, \\ (\Upsilon_h+\eta^2A_h)\frac{u_{m\mathrm{M}+1}^h-u_{m\mathrm{M}}^h}{\eta}=\frac{u_{m\mathrm{M}}^h-u_{m\mathrm{M}-1}^h}{\eta}, m=1,2,... \end{cases} \quad (7)$$

in L_{2h} which is a Hilbert space with SAPDO $A_h=A_h^y$ that is defined using the formula (4). Where, $g_k^h=g_k^h(y)$, $\Phi_k^h=\Phi_k^h(y)$ and $u_k^h=u_k^h(y)$ are known and unknown abstract mesh functions that are defined on $[-\rho,\rho]_h$ with the values in $H=L_{2h}$. Consequently, Theorem 2 proof relies on the theorem 4 below as well as the self-adjointness and positive definiteness of the space operator A_h (4) [20].

Theorem 4. [21] The following estimate holds for the solution of DS (7):

$$\frac{1}{2}\max_{0\leq k\leq (m+1)\mathrm{M}-2}\left\|A_h^{\frac{1}{2}}\frac{u_{k+2}^h-2u_{k+1}^h+u_k^h}{\eta^2}\right\|_H, \max_{1\leq k\leq (m+1)\mathrm{M}}\left\|A_h\frac{u_k^h-u_{k-1}^h}{\eta}\right\|_H,$$

$$\max_{0\leq k\leq (m+1)\mathrm{M}}\|A_h^{\frac{3}{2}}u_k^h\|_H\leq \chi_1\left[(2+\eta|b|(\mathrm{M}-2))^m b_0^h\right.$$

$$\left.+\sum_{i=1}^{m}(2+\eta|b|(\mathrm{M}-2))^{m-i}\eta\sum_{s=(i-1)\mathrm{M}+1}^{i\mathrm{M}}\|A_H^{\frac{1}{2}}\Phi(\zeta_s)\|_H\right], m=0,1,...,$$

where $b_0=\max\left\{\max_{-\mathrm{M}\leq k\leq 0}\|A_h^{\frac{1}{2}}g''(\zeta_k)\|_H,\ \max_{-\mathrm{M}\leq k\leq 0}\|A_hg_\zeta^h(\zeta_k)\|_H,\ \max_{-\mathrm{M}\leq k\leq 0}\|A_h^{\frac{3}{2}}g^h(\zeta_k)\|_H\right\}$.

2 Numerical algorithm for the third order delay partial differential equation

We give the algorithm for numerically solving the initial BVP for third order delay PDE having involution and Robin boundary condition

$$\begin{cases} \frac{\partial^3u(\zeta,y)}{\partial\zeta^3}-\frac{\partial^3u(\zeta,y)}{\partial\zeta\partial y^2}+8\frac{\partial u(\zeta,y)}{\partial\zeta}-\frac{1}{8}\frac{\partial^3u(\zeta,-y)}{\partial\zeta\partial y^2}+\frac{\partial u(\zeta,-y)}{\partial\zeta} \\ =-0.1(-\frac{\partial^2u(\zeta-1,y)}{\partial y^2}+8u(\zeta-1,y)) \\ -35e^{-2\zeta}\cos 2y+1.2e^{-2(\zeta-1)}\cos 2y, \\ 0<\zeta<\infty,\ -\pi<y<\pi, \\ u(\zeta,y)=e^{-2\zeta}\cos 2y,\ -1\leq\zeta\leq 0,\ -\pi\leq y\leq\pi, \\ u(\zeta,-\pi)-e^{-2\zeta}=28u_y(\zeta,-\pi), -u(\zeta,\pi)+e^{-2\zeta}=28u_y(\zeta,\pi),\ 0\leq\zeta<\infty. \end{cases} \quad (8)$$

The exact solution of problem (8) is $u(\zeta,y)=e^{-2\zeta}\cos 2y, -\pi\leq y\leq\pi, -1\leq\zeta<\infty$. We use the set of grid points $[-1,\infty)_\eta\times[-\pi,\pi]_h=\{(\zeta_k,y_n):\zeta_k=k\eta, -\mathrm{M}\leq k, \mathrm{M}\eta=1, y_n=nh,\ -\Gamma\leq n\leq\Gamma,$

$\Gamma h=\pi\}$, for the approximate solutions of the problem (8), we get the first order of accuracy DS in t

$$\begin{cases} \frac{u_n^{k+2}-3u_n^{k+1}+3u_n^{k}-u_n^{k-1}}{\eta^3}-\frac{u_{n+1}^{k+2}-u_{n+1}^{k+1}-2(u_n^{k+2}-u_n^{k+1})+u_{n-1}^{k+2}-u_{n-1}^{k+1}}{\eta h^2} \\ +8\frac{u_n^{k+2}-u_n^{k+1}}{\eta}-\frac{1}{8}\frac{u_{-n+1}^{k+2}-u_{-n+1}^{k+1}-2(u_{-n}^{k+2}-u_{-n}^{k+1})+u_{-n-1}^{k+2}-u_{-n-1}^{k+1}}{\eta h^2} \\ +\frac{u_{-n}^{k+2}-u_{-n}^{k+1}}{\eta}=-(0.1)\left(-\frac{u_{n+1}^{k-\mathrm{M}}-2u_n^{k-\mathrm{M}}+u_{n-1}^{k-\mathrm{M}}}{h^2}+8u_n^{k-\mathrm{M}}\right) \\ -35e^{-2\zeta_k}\cos 2y_n+1.2e^{-2(\zeta_{k-\mathrm{M}})}\cos 2y_n, \\ t_k=k\eta,\ m\mathrm{M}+1\le k\le (m+1)\mathrm{M}-2, \\ m=0,1,...,\ -\Gamma+1\le n\le \Gamma-1, \\ \mathrm{M}\eta=1,\ y_n=nh,\ -\Gamma+1\le n\le\Gamma-1,\ \Gamma h=\pi, \\ u_n^0=\cos(2nh), \\ \frac{u_n^1-u_n^0}{\eta}+\eta(-\frac{u_{n+1}^1-2u_n^1+u_{n-1}^1}{h^2}+8u_n^1) \\ +\eta(\frac{u_{n+1}^0-2u_n^0+u_{n-1}^0}{h^2}-8u_n^0)=-2\cos(2nh), \\ \frac{u_n^2-2u_n^1+u_n^0}{\eta^2}+(-\frac{u_{n+1}^2-2u_n^2+u_{n-1}^2}{h^2}+8u_n^2) \\ +2(\frac{u_{n+1}^1-2u_n^1+u_{n-1}^1}{h^2}-8u_n^1) \\ +(-\frac{u_{n+1}^0-2u_n^0+u_{n-1}^0}{h^2}+8u_n^0)=4\cos(2nh),\ -\Gamma+1\le n\le\Gamma-1, \\ \frac{u_n^{m\mathrm{M}+1}-u_n^{m\mathrm{M}}}{\eta}+\eta(-\frac{u_{n+1}^{m\mathrm{M}+1}-2u_n^{m\mathrm{M}+1}+u_{n-1}^{m\mathrm{M}+1}}{h^2}+8u_n^{m\mathrm{M}+1}) \\ +\eta(\frac{u_{n+1}^{m\mathrm{M}}-2u_n^{m\mathrm{M}}+u_{n-1}^{m\mathrm{M}}}{h^2}-8u_n^{m\mathrm{M}})=\frac{u_n^{m\mathrm{M}}-u_n^{m\mathrm{M}-1}}{\eta}, \\ \frac{u_n^{m\mathrm{M}+2}-2u_n^{m\mathrm{M}+1}+u_n^{m\mathrm{M}}}{\eta^2}+(-\frac{u_{n+1}^{m\mathrm{M}+2}-2u_n^{m\mathrm{M}+2}+u_{n-1}^{m\mathrm{M}+2}}{h^2}+8u_n^{m\mathrm{M}+2}) \\ +2(\frac{u_{n+1}^{m\mathrm{M}+1}-2u_n^{m\mathrm{M}+1}+u_{n-1}^{m\mathrm{M}+1}}{h^2}-8u_n^{m\mathrm{M}+1})+(-\frac{u_{n+1}^{m\mathrm{M}}-2u_n^{m\mathrm{M}}+u_{n-1}^{m\mathrm{M}}}{h^2}+8u_n^{m\mathrm{M}}) \\ =\frac{u_n^{m\mathrm{M}}-2u_n^{m\mathrm{M}-1}+u_n^{m\mathrm{M}-2}}{\eta^2},\ -\Gamma+1\le n\le\Gamma-1,\ m=1,2,\dots, \\ u_{-\Gamma}^k-e^{-2\zeta_k}=\frac{28}{h}\left(u_{-\Gamma+1}^k-u_{-\Gamma}^k\right), \\ -u_{\Gamma}^k+e^{-2\zeta_k}=\frac{28}{h}\left(u_{\Gamma}^k-u_{\Gamma-1}^k\right),\ 0\le k<\infty, \\ m\mathrm{M}\le k\le (m+1)\mathrm{M},\ m=1,2,\dots \end{cases} \tag{9}$$

We rewrite (9) in the matrix form as in the following:

$$\begin{cases} \Delta\chi^{k+2}+\Theta\chi^{k+1}+\Lambda\chi^{k}+\Omega\chi^{k-1}=\varphi(\chi^{k-\mathrm{M}}), \\ k=1,2,3,\dots \\ \chi^{0}=\begin{bmatrix} \cos\left(2(-\Gamma)h\right) \\ \cos\left(2(-\Gamma+1)h\right) \\ \vdots \\ \cos\left(2(\Gamma-1)h\right) \\ \cos\left(2(\Gamma)h\right) \end{bmatrix}, \\ \chi^{1}=LH\chi^{0}, \\ \chi^{2}=YP\chi^{1}+YQ\chi^{0}, \\ \chi^{m\mathrm{M}+1}=LJ\chi^{m\mathrm{M}}+LW\chi^{m\mathrm{M}-1}, \\ \chi^{m\mathrm{M}+2}=YP\chi^{m\mathrm{M}+1}+YX\chi^{m\mathrm{M}}+YS\chi^{m\mathrm{M}-1} \\ +YZ\chi^{m\mathrm{M}-2}, \\ m=1,2,\dots, \end{cases}$$

where $\Delta,\Theta,\Lambda,\Omega,F,H,J,P,Q,S,V,W,X$ and Z are $(2\Gamma+1)\times(2\Gamma+1)$ matrices, $\varphi(\chi^{k-\mathrm{M}})$, χ^0, χ^1 and χ^r, $r=k,k\pm1,k+2$ are $(2\Gamma+1)\times1$ column vectors defined by

$$\Delta=\begin{bmatrix}
1+\frac{28}{h} & -\frac{28}{h} & 0 & \cdot & 0 & 0 & 0 & \cdot & 0 & 0 & 0 \\
a & b & a & \cdot & 0 & 0 & 0 & \cdot & a^{*} & b^{*} & a^{*} \\
0 & a & b & \cdot & 0 & 0 & 0 & \cdot & b^{*} & a^{*} & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & 0 & 0 & \cdot & a & 0 & a_{*} & \cdot & 0 & 0 & 0 \\
0 & 0 & 0 & \cdot & b & w_1 & b^{*} & \cdot & 0 & 0 & 0 \\
0 & 0 & 0 & \cdot & w_1 & w_2 & w_1 & \cdot & 0 & 0 & 0 \\
0 & 0 & 0 & \cdot & b^{*} & w_1 & b & \cdot & 0 & 0 & 0 \\
0 & 0 & 0 & \cdot & a^{*} & 0 & a & \cdot & 0 & 0 & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & a^{*} & b^{*} & \cdot & 0 & 0 & 0 & \cdot & b & a & 0 \\
a^{*} & b^{*} & a^{*} & \cdot & 0 & 0 & 0 & \cdot & b & a & \\
0 & 0 & 0 & \cdot & 0 & 0 & 0 & \cdot & 0 & -\frac{28}{h} & 1+\frac{28}{h}
\end{bmatrix},$$

$$\Theta=\begin{bmatrix}
0 & 0 & 0 & \cdot & 0 & 0 & 0 & \cdot & 0 & 0 & 0 \\
l & c & l & \cdot & 0 & 0 & 0 & \cdot & l^{*} & c^{*} & l^{*} \\
0 & l & c & \cdot & 0 & 0 & 0 & \cdot & c^{*} & l^{*} & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & 0 & 0 & \cdot & l & 0 & l^{*} & \cdot & 0 & 0 & 0 \\
0 & 0 & 0 & \cdot & c & l+l^{*} & c^{*} & \cdot & 0 & 0 & 0 \\
0 & 0 & 0 & \cdot & l+l^{*} & c+c^{*} & l+l^{*} & \cdot & 0 & 0 & 0 \\
0 & 0 & 0 & \cdot & c^{*} & l^{*}+l & c & \cdot & 0 & 0 & 0 \\
0 & 0 & 0 & \cdot & l^{*} & 0 & l & \cdot & 0 & 0 & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & l^{*} & c^{*} & \cdot & 0 & 0 & 0 & \cdot & c & l & 0 \\
l^{*} & c^{*} & l^{*} & \cdot & 0 & 0 & 0 & \cdot & l & c & l \\
0 & 0 & 0 & \cdot & 0 & 0 & 0 & \cdot & 0 & 0 & 0
\end{bmatrix},$$

$$\Lambda=\begin{bmatrix}0&0&0&0&\cdot&0&0&0\\0&d&0&0&\cdot&0&0&0\\0&0&d&0&\cdot&0&0&0\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\0&0&0&0&\cdot&d&0&0\\0&0&0&0&\cdot&0&d&0\\0&0&0&0&\cdot&0&0&0\end{bmatrix},\ \Omega=\begin{bmatrix}0&0&0&0&\cdot&0&0&0\\0&e&0&0&\cdot&0&0&0\\0&0&e&0&\cdot&0&0&0\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\0&0&0&0&\cdot&e&0&0\\0&0&0&0&\cdot&0&e&0\\0&0&0&0&\cdot&0&0&0\end{bmatrix},$$

$$W=\begin{bmatrix}0&0&0&0&\cdot&0&0&0\\0&w&0&0&\cdot&0&0&0\\0&0&w&0&\cdot&0&0&0\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\0&0&0&0&\cdot&w&0&0\\0&0&0&0&\cdot&0&w&0\\0&0&0&0&\cdot&0&0&0\end{bmatrix},\ S=\begin{bmatrix}0&0&0&0&\cdot&0&0&0\\0&s&0&0&\cdot&0&0&0\\0&0&s&0&\cdot&0&0&0\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\0&0&0&0&\cdot&s&0&0\\0&0&0&0&\cdot&0&s&0\\0&0&0&0&\cdot&0&0&0\end{bmatrix},$$

$$Z=\begin{bmatrix}0&0&0&0&\cdot&0&0&0\\0&z&0&0&\cdot&0&0&0\\0&0&z&0&\cdot&0&0&0\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\0&0&0&0&\cdot&z&0&0\\0&0&0&0&\cdot&0&z&0\\0&0&0&0&\cdot&0&0&0\end{bmatrix},\ F=\begin{bmatrix}1+\frac{28}{h}&-\frac{28}{h}&0&\cdot&0&0&0\\f^*&f&f^*&\cdot&0&0&0\\0&f^*&f&\cdot&0&0&0\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\0&0&0&\cdot&f&f^*&0\\0&0&0&\cdot&f^*&f&f^*\\0&0&0&\cdot&0&-\frac{28}{h}&1+\frac{28}{h}\end{bmatrix},$$

$$P=\begin{bmatrix}0&0&0&\cdot&0&0&0\\p^*&p&p^*&\cdot&0&0&0\\0&p^*&p&\cdot&0&0&0\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\0&0&0&\cdot&p&p^*&0\\0&0&0&\cdot&p^*&p&p^*\\0&0&0&\cdot&0&0&0\end{bmatrix},$$

$$V=\begin{bmatrix}1+\frac{28}{h}&-\frac{28}{h}&0&\cdot&0&0&0\\v*&v&v*&\cdot&0&0&0\\0&v*&v&\cdot&0&0&0\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\0&0&0&\cdot&v&v*&0\\0&0&0&\cdot&v*&v&v*\\0&0&0&\cdot&0&-\frac{28}{h}&1+\frac{28}{h}\end{bmatrix},\ J=\begin{bmatrix}0&0&0&\cdot&0&0&0\\j^*&j&j^*&\cdot&0&0&0\\0&j^*&j&\cdot&0&0&0\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\0&0&0&\cdot&j&j^*&0\\0&0&0&\cdot&j^*&j&j^*\\0&0&0&\cdot&0&0&0\end{bmatrix},$$

$$H=\begin{bmatrix}0&0&0&\cdot&0&0&0\\h^*&e^*&h^*&\cdot&0&0&0\\0&h^*&e^*&\cdot&0&0&0\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\0&0&0&\cdot&e^*&h^*&0\\0&0&0&\cdot&h^*&e^*&h^*\\0&0&0&\cdot&0&0&0\end{bmatrix},\ X=\begin{bmatrix}0&0&0&\cdot&0&0&0\\x^*&s^*&x^*&\cdot&0&0&0\\0&x^*&s^*&\cdot&0&0&0\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\0&0&0&\cdot&s^*&x^*&0\\0&0&0&\cdot&x^*&s^*&x^*\\0&0&0&\cdot&0&0&0\end{bmatrix},$$

$$Q = \begin{bmatrix} 0 & 0 & 0 & \cdot & 0 & 0 & 0 \\ q^* & q & q^* & \cdot & 0 & 0 & 0 \\ 0 & q^* & q & \cdot & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & q & q^* & 0 \\ 0 & 0 & 0 & \cdot & q^* & q & q^* \\ 0 & 0 & 0 & \cdot & 0 & 0 & 0 \end{bmatrix}, \ \varphi^{(k-\mathrm{M})} = \begin{bmatrix} \varphi^k_{-M} \\ \varphi^k_{-M+1} \\ \vdots \\ \varphi^k_{M-1} \\ \varphi^k_M \end{bmatrix}, \ \chi^r = \begin{bmatrix} \chi^r_0 \\ \chi^r_{-M+1} \\ \vdots \\ \chi^r_{M-1} \\ \chi^r_M \end{bmatrix},$$

$r = k, k \pm 1, k+2,$
where

$$\varphi^k_n = -(0.1)\left(-\frac{u^{k-\mathrm{M}}_{n+1} - 2u^{k-\mathrm{M}}_n + u^{k-\mathrm{M}}_{n-1}}{h^2} + 8u^{k-\mathrm{M}}_n\right) - 35e^{-2\zeta_k}\cos 2y_n + 1.2e^{-2(\zeta_{k-\mathrm{M}})}\cos 2y_n,$$

$$\zeta_k = k\eta,\ m\mathrm{M} + 1 \le k \le (m+1)\mathrm{M} - 2,\ m = 0, 1, ...,\ -\Gamma + 1 \le n \le \Gamma - 1.$$

Where, $a = -\frac{1}{\eta h^2},\ a^* = -\frac{1}{8\eta h^2},\ b = \frac{1}{\eta^3} + \frac{2}{\eta h^2} + \frac{8}{\eta},\ b^* = \frac{2}{8\eta h^2} + \frac{1}{\eta},\ c = -\frac{3}{\eta^3} - \frac{2}{\eta h^2} - \frac{8}{\eta},\ c^* = -b^*,\ l = -a,\ l^* = -a^*,\ d = \frac{3}{\eta^3},\ e = -\frac{1}{\eta^3}, w = -\frac{1}{\eta},\ s = -\frac{2}{\eta^2}\ z = \frac{1}{\eta^2},\ f = \frac{2\eta}{h^2} + \frac{1}{\eta} + 8\eta,\ f^* = -\frac{\eta}{h^2},\ p = \frac{2}{\eta^2} + \frac{4}{h^2} + 16,\ p^* = -\frac{2}{h^2},\ v = \frac{1}{2}p,\ v^* = \frac{1}{2}p^*,\ j = f + \frac{1}{\eta},\ j^* = f^*,\ h^* = f^*,\ e^* = f - 2,\ s^* = p^* - 8,\ x^* = -v*,\ q = -\frac{1}{,}\ w_1 = a + a^*,\ w_2 = b + b^*, \eta^2 - \frac{2}{h^2} - 4,\ q^* = x^*,\ L = F^{-1},\ Y = V^{-1}.$

3 Numerical analysis

Provided in Table below are the solutions obtained numerically for various values of M and Γ, with u^k_n representing the solution of this DS at $u(\zeta_k, y_n)$ numerically. The table consist of values for $\mathrm{M} = \Gamma = 30, 60, 120$ in $\zeta \in [0, 1],\ \zeta \in [1, 2],\ \zeta \in [2, 3]$ respectively and the errors are calculated by

$$mE^{\mathrm{M}}_M = \max_{m\mathrm{M}+1 \le k \le (m+1)\mathrm{M},\ -\Gamma \le n \le \Gamma} |u(\zeta_k, y_n) - u^k_n|.$$

T a b l e

Errors of DS (9)

(M, Γ)	$\mathrm{M} = \Gamma = 30$	$\mathrm{M} = \Gamma = 60$	$\mathrm{M} = \Gamma = 120$
$\zeta \in [0, 1]$	0.1933	0.1012	0.0516
$\zeta \in [1, 2]$	0.2350	0.1169	0.0583
$\zeta \in [2, 3]$	0.1692	0.0780	0.0340

If M and Γ are doubled as shown in the above table, the values of the errors decrease by a factor of approximately $\frac{1}{2}$ for DS (9).

4 Conclusion

In this paper, the first order of accuracy DS for the numerical solution of the third order delay PDE with IRC is considered. Numerical results are given for illustration.

Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

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Numerical solution of source identification multi-point problem of parabolic partial differential equation with Neumann type boundary condition

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We study a source identification boundary value problem for a parabolic partial differential equation with multi-point Neumann type boundary condition. Stability estimates for the solution of the overdetermined mixed BVP for multi-dimensional parabolic equation were established. The first and second order of accuracy difference schemes for the approximate solution of this problem were proposed. Stability estimates for both difference schemes were obtained. The result of numerical illustration in test example was given.

Keywords: inverse problem, source identification, parabolic equation, difference scheme, stability, nonlocal condition, boundary value problem, well-posedness, stability estimates, mixed problem.

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Introduction

Various techniques can be used to solve source identification problems (SIPs) for parabolic equations. These may include optimization algorithms, regularization methods, or numerical techniques such as finite element and finite difference methods (see [1–28] a references therein).

In papers [3, 18], SIP for abstract differential equation with self-adjoint positive definite operator A

$$\frac{dv(t)}{dt} + Av(t) = p + f(t),\ 0 < t < 1, \tag{1}$$

$$v(0) = \varphi,\ v(1) = \psi \tag{2}$$

in a Hilbert space H was investigated. In paper [3], for solution of SIP (1), (2), stability estimates in the Hölder norms were obtained.

Some applications to boundary value problems (BVPs) for partial differential equation (PDE) and approximate solutions were studied in [8, 12].

Let $s_1, \mu_1, s_2, \mu_2, ..., s_r, \mu_r$ be given numbers so that

$$\sum_{k=1}^{r} |\mu_k| < 1,\ 0 \le s_1 < s_2 < \ldots < s_r < 1 \tag{3}$$

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and elements φ, $\psi \in H$ and function $f : [0,1] \to H$ are given. In paper [15] SIP to find a pair (p,v) for equation

$$\frac{dv(t)}{dt} + Av(t) = p + f(t), 0 < t < 1,$$

with multi-point nonlocal conditions

$$v(0) = \sum_{k=1}^{r} \mu_k v(s_k) + \psi, \tag{4}$$

$$v(1) = \varphi \tag{5}$$

was studied and stability estimates for the solution were given in the following theorem.

Theorem 1. [15] Assume that conditions (3) for interior points and coefficients are valid, $\varphi \in H$, $\psi \in D(A)$, $f \in C^{\alpha}(H)$ $(\alpha \in (0,1))$ are given. Then, for the solution $(v(t),p)$ of SIP (1), (4), (5) the stability estimates

$$\|v\|_{C(H)} \le M \left[\|\varphi\|_H + \|\psi\|_H + \|f\|_{C(H)}\right]$$

and

$$\|p\|_H \le M \left[\|A\varphi\|_H + \|A\psi\|_H + \frac{1}{\alpha}\|f\|_{C^{\alpha}(H)}\right]$$

are fullfilled, where $M \in R^{+}$ does not depend on f, ψ, φ and α. Here $C(H)$, $C^{\alpha}(H)$ and $C_1^{\alpha}(H)$ are the Banach spaces of H-valued functions $u(t)$ with the corresponding norms

$$\begin{array}{l} \|u\|_{C(H)} = \max\limits_{0\le t\le 1}\|u(t)\|_H, \|u\|_{C^{\alpha}(H)} = \|u\|_{C(H)} + \sup\limits_{0\le t<t+\tau\le 1} \frac{\|u(t+\tau)\|_H - \|u(t)\|_H}{\tau^{\alpha}}, \\ \|u\|_{C_1^{\alpha}(H)} = \|u\|_{C(H)} + \sup\limits_{0\le t<t+\tau\le 1} \frac{(1-t)^{\alpha}\|u(t+\tau)\|_H - \|u(t)\|_H}{\tau^{\alpha}}, \end{array} \tag{6}$$

respectively.

1 SI parabolic problem with multi-point boundary conditions

Now, we study a source identification (SI) BVP for the multi-dimensional PDE.

Let $\Omega = (0,1)^n \subset \mathbb{R}^n$ with boundary $S = \partial\Omega$, $\overline{\Omega} = \Omega \cup S$.

Denote by $L_2(\Omega)$ and $W_2^2(\Omega)$ the Hilbert spaces of integrable functions $u(y)$, defined on Ω, equipped with the corresponding norms

$$\|u\|_{L_2(\Omega)} = \left\{\int_{y\in\Omega} |u(y)|^2 dy_1 \dots dy_n\right\}^{\frac{1}{2}},$$

$$\|u\|_{W_2^2(\Omega)} = \left\{\int_{y\in\Omega}\left(|u(y)|^2 + \sum_{i=1}^{n}\sum_{j=1}^{n}\left|u_{y_iy_j}(y)\right|^2\right) dy_1 \dots dy_n\right\}^{\frac{1}{2}}.$$

Let $\varphi \in L_2(\Omega)$, $\psi \in W_2^2(\Omega)$, $f \in C^{\alpha}(L_2(\Omega))$ be given functions, and $a_i : \Omega \to R^{+}$ be known smooth function for any index $i = 1, \dots, n$.

In $[0,1]\times\Omega$, we study multi-dimensional SIP for parabolic PDE with multi-point boundary and nonlocal conditions

$$\begin{cases} v_t(t,x) - \sum\limits_{i=1}^{n}(a_i(x)v_{x_i}(t,x))_{x_i} + \sigma v(t,x) = f(t,x) + p(x), \\ x = (x_1,\dots,x_n) \in \Omega,\ 0 < t < 1, \\ \frac{\partial}{\partial \vec{n}} v(t,x) = 0,\ x \in S,\ 0 < t < 1, \\ v(0,x) = \sum\limits_{k=1}^{r}\mu_k v(s_k,x) + \psi(x),\ v(1,x) = \varphi(x),\ x \in \overline{\Omega}, \end{cases} \tag{7}$$

where $\overrightarrow{n}$ is the normal vector to Ω at corresponding boundary point.

The differential expression

$$A^x u(x) = -\sum_{i=1}^{n} (a_i(x) u_{x_i}(x))_{x_i} + \sigma u(x)$$

defines SAPD operator A^x, which acts on the Hilbert space $L_2(\Omega)$ with the domain

$$D(A^x) = \left\{ u \mid u \in W_2^2(\Omega),\ \frac{\partial u}{\partial \overrightarrow{n}}(x) = 0 \text{ on } S \right\}.$$

So, the SI BVP (7) for the multi-dimensional parabolic PDE can be replaced with the abstract problem (1), (4), (5) for $H = L_2(\Omega)$. By using stability estimates of Theorem 1, we obtain the following stability estimates for solution of BVP (7).

Theorem 2. Suppose that conditions (3) are satisfied, $\varphi, \psi \in W_2^2(\Omega)$ and $f \in C^\alpha(L_2(\Omega))$. Then, for the solution of multi-dimensional SIP for parabolic PDE (7), the following estimates are valid

$$\|p\|_{L_2(\Omega)} \leq M \left[\|\varphi\|_{W_2^2(\Omega)} + \|\psi\|_{W_2^2(\Omega)} + \frac{1}{\alpha} \|f\|_{C^\alpha(L_2(\Omega))} \right],$$

$$\|v\|_{C(L_2(\Omega))} \leq M \left[\|\varphi\|_{L_2(\Omega)} + \|\psi\|_{L_2(\Omega)} + \|f\|_{C(L_2(\Omega))} \right],$$

where positive number M is independent of f, ψ, φ and α.

2 First and second order of ADSs

We will use the set of uniform grid points

$$[0,1]_\tau = \{t_k = k\tau,\ k = 0, 1, \ldots, N,\ N\tau = 1\}.$$

To discretize problem (7) we use algorithm with two steps. Firstly, we define grid spaces

$$\begin{aligned} \widetilde{\Omega}_h = \{ & x = x_m = (h_1 m_1, \ldots, h_n m_n);\ m = (m_1, \ldots, m_n), \\ & m_j = 0, \ldots, N_j,\ h_j N_j = 1,\ j = 1, \ldots, n\}, \end{aligned}$$

$$\Omega_h = \widetilde{\Omega}_h \cap \Omega,\ S_h = \widetilde{\Omega}_h \cap S.$$

Introduce difference operator A_h^x by formula

$$A_h^x v^h(x) = -\sum_{i=1}^{n} \left(a_i(x) v_{\overline{x}_i}^h(x) \right)_{x_i, j_i} + \sigma v^h(x),$$

which acts in space of grid functions $v^h(x)$ and satisfies the condition $Dv^h(x) = 0$ for all $x \in S_h$.

Applying A_h^x, we arrive at the multi-point nonlocal BVP for some infinite system of ordinary differential equations. Secondly, by using Equation (26) [15; p. 1922], we get the first order of accuracy difference scheme (ADS)

$$\begin{cases} \tau^{-1} \left(v_k^h(x) - v_{k-1}^h(x) \right) + A_h^x v_k^h(x) = f^h(t_k, x) + p^h(x),\ 1 \leq k \leq N,\ x \in \widetilde{\Omega}_h, \\ v_N^h(x) = \varphi^h(x),\ v_0^h(x) = \sum\limits_{i=1}^{r} \mu_i v_{l_i}^h(x) + \psi^h(x),\ x \in \widetilde{\Omega}_h. \end{cases} \tag{8}$$

By using Equations (37)–(39) [15; p. 1925], we get the second order of ADS

$$\begin{cases} \tau^{-1}\left(v_k^h(x)-v_{k-1}^h(x)\right)+A_h^x\left(I+\dfrac{\tau A_h^x}{2}\right)v_k^h(x) \\ =\left(I+\dfrac{\tau A_h^x}{2}\right)\left(f^h(t_{k-\frac{\tau}{2}},x)+p^h(x)\right),\ 1\le k\le N,\ x\in\widetilde{\Omega}_h, \\ v_N^h(x)=\varphi^h(x),\ x\in\widetilde{\Omega}_h, \\ v_0^h(x)=\sum\limits_{i=1}^{r}\left\{\mu_i\left(1-\rho_i\right)v_{l_i}^h(x)+\mu_i\ \rho_i v_{l_i+1}^h(x)\right\}+\psi^h(x),\ x\in\widetilde{\Omega}_h. \end{cases} \tag{9}$$

Denote by $L_{2h}=L_2(\widetilde{\Omega}_h)$ and $W_{2h}^2=W_2^2(\widetilde{\Omega}_h)$, the spaces of the grid functions $u^h(x)=\{u(h_1m_1,\dots,h_nm_n)\}$ defined on $\widetilde{\Omega}_h$, equipped with the corresponding norms

$$\left\|u^h\right\|_{L_{2h}}=\left(\sum_{x\in\widetilde{\Omega}_h}|u^h(x)|^2h_1\cdots h_n\right)^{1/2},$$

$$\left\|u^h\right\|_{W_{2h}^2}=\left\|u^h\right\|_{L_{2h}}+\left(\sum_{x\in\widetilde{\Omega}_h}\sum_{r=1}^{n}\left|(u^h(x))_{x_r\overline{x_r},m_r}\right|^2h_1\cdots h_n\right)^{1/2},$$

and by $\mathcal{C}_\tau(L_{2h})=\mathcal{C}([0,1]_\tau,L_{2h})$, the Banach space of L_{2h}-valued grid functions $u^\tau=\{u_k\}_1^N$ with the suitable norm $\|u^\tau\|_{\mathcal{C}_\tau(L_{2h})}=\max\limits_{1\le k\le N}\|u_k\|_{L_{2h}}$.

Let $\mathcal{C}^\alpha(L_{2h})=\mathcal{C}^\alpha([0,1]_\tau,L_{2h})$ and $\mathcal{C}_\tau^\alpha(L_{2h})=\mathcal{C}_\tau^\alpha([0,1]_\tau,L_{2h})$ be correspondingly Hölder and weighted Hölder spaces with the corresponding norms defined by (6) for $H=L_{2h}$.

Theorem 3. Suppose that τ and $|h|=\sqrt{h_1^2+\cdots+h_n^2}$ are sufficiently small positive numbers, $\varphi^h\in L_{2h}$, $\psi^h\in W_{2h}^2$ and $\left\{f_k^h\right\}_1^N\in\mathcal{C}_\tau^\alpha(L_{2h})$. Then, for the solution of difference schemes (DSs) (8) and (9), the following stability estimates hold

$$\left\|p^h\right\|_{\mathcal{C}_\tau(L_{2h})}\le M\left[\left\|\varphi^h\right\|_{L_{2h}}+\left\|\psi^h\right\|_{W_{2h}}+\frac{1}{\alpha}\left\|\left\{f_k^h\right\}_1^N\right\|_{\mathcal{C}_\tau^\alpha(L_{2h})}\right],$$

$$\left\|\left\{v_k^h\right\}_1^N\right\|_{\mathcal{C}_\tau(L_{2h})}\le M\left[\left\|\varphi^h\right\|_{L_{2h}}+\left\|\psi^h\right\|_{L_{2h}}+\left\|\left\{f_k^h\right\}_1^N\right\|_{\mathcal{C}_\tau(L_{2h})}\right],$$

where M is independent of $\left\{f_k^h\right\}_1^N$, $\psi^h(x)$, $\varphi^h(x)$ and τ.

The proof of Theorem 3 based on Theorems 3.1 and 3.2 of paper [15] on stability estimate for solutions of corresponding DSs for approximate solution of abstract SIP (1), (4), (5) and the theorem on the coercivity inequality for the solution of the elliptic difference problem in L_{2h}.

3 Numerical analysis

For test example, we consider the SIP

$$\begin{cases} v_t(t,x)-(3+2\cos x)v_{xx}\ (t,x)+2\sin x\cdot v_x(t,x)+\ v(t,x)=f(t,x)+p(x), \\ 0<x<\pi,\ 0<t<1, \\ v(1,x)=\varphi\ (x), \\ v(0,x)=v(\frac{1}{3},x)+\psi(x),\ 0\le x\le\pi, \\ v_x(t,0)=0,\ v_x(t,\pi)=0,\ 0\le t\le 1 \end{cases} \tag{10}$$

for one-dimensional parabolic PDE. Here

$$f(t,x)=\left(e^{-t}-e^{-1}\right)(3\cos x+2\cos 2x)-e^{-1}\cos x,\ 0<x<\pi,\ 0<t<1,$$
$$\varphi(x)=\cos x,\ \psi(x)=\left(1-e^{-\frac{1}{3}}\right)\cos x,\ 0\le x\le\pi.$$

It is easy to check that the pair $\left(e^{-1}\left(4\cos x+2\cos 2x\right), e^{-t}\cos x\right)$ is the exact solution of problem (10).

An algorithm of finding the solution of problem (10) contains three stages. In the first stage, we find the solution of auxiliary BVP

$$\begin{cases} u_t(t,x)-(3+2\cos x)u_{xx}(t,x)+2\sin x\cdot u_x(t,x)+u(t,x) \\ =\ (3+2\cos x)\cos x-2\sin x\cdot\sin x+\cos x+f(t,x),\ 0<x<\pi,\ 0<t<1, \\ u(1,x)-u(\frac{1}{3},x)=\psi(x),\ 0\le x\le\pi, \\ u_x(t,0)=0,\ u_x(t,\pi)=0,\ 0\le t\le 1. \end{cases} \tag{11}$$

Then, in the second stage, we find $p(x)$ by

$$p(x)=-(3+2\cos x)u_{xx}(1,x)+2\sin x\cdot u_x(1,x)+u(1,x).$$

In the third stage, we put $p(x)$ in the right side of equation (10) and solve that problem for $v(t,x)$.

Introduce the set of grid points

$$[0,1]_\tau\times[0,\pi]_h=\{(t_k,x_n)\mid t_k=k\tau,\ k=1,\dots,N-1,\ N\tau=1,$$
$$x_n=nh,\ n=1,\dots,M-1,\ Mh=\pi\}.$$

We use notation $l=\left[\frac{\gamma}{\tau}\right]$ for greatest integer function of $\frac{\gamma}{\tau}$ and $\rho=\frac{\gamma}{\tau}-l$.

So, we get the first order of ADS for SIP (10)

$$\begin{cases} \frac{v_n^k-v_n^{k-1}}{\tau}-\frac{(3+2\cos x_n)\left(v_{n+1}^k-2v_n^k+v_{n-1}^k\right)}{h^2}+\frac{\sin(x_n)\left(v_{n+1}^k-v_{n-1}^k\right)}{h}+v_n^k \\ =f(t_k,x_n)+p(x_n),\ k=1,\dots,N,\ n=1,\dots,M-1, \\ v_n^N=\varphi_n,\ v_n^0-\ \rho v_n^l=\psi_n,\ n=0,\dots,M, \\ v_0^k=v_1^k,\ v_M^k=v_{M-1}^k,\ k=0,\dots,N. \end{cases}$$

Later, $p(x_n)$ can be obtained by

$$p(x_n)=-\frac{(3+2\cos(x_n))\left(u_{n+1}^N-2u_n^N+u_{n-1}^N\right)}{h^2}+\frac{\sin(x_n)\left(u_{n+1}^N-u_{n-1}^N\right)}{h}+u_n^N, \tag{12}$$

where $\left\{u_n^k\right\}$ is solution of the difference problem

$$\begin{cases} \frac{u_n^k-u_n^{k-1}}{\tau}-\frac{(3+2\cos(x_n))\left(u_{n+1}^k-2u_n^k+u_{n-1}^k\right)}{h^2} \\ +\frac{\sin(x_n)\left(u_{n+1}^k-u_{n-1}^k\right)}{h}+u_n^k=f(t_k,x_n)-\frac{(3+2\cos(x_n))(\varphi_{n+1}-2\varphi_n+\varphi_{n-1})}{h^2} \\ +\frac{\sin(x_n)(\varphi_{n+1}-\varphi_{n-1})}{h}+\varphi_n,\ k=1,\dots,N,\ n=1,\dots,M-1, \\ u_n^0-u_n^l=\psi_n,\ n=0,\dots,M, \\ u_0^k-u_M^k=0,\ u_M^k-u_{M-1}^k=0,\ k=0,\dots,N, \end{cases} \tag{13}$$

which is the first order of ADS for approximate solution of the nonlocal BVP (11).

For computational reasons it is convenient to write (13) in the following matrix form

$$\begin{gathered} A_n u_{n+1}+B_n u_n+C_n u_{n-1}=I\theta_n,\ n=1,\dots,M-1, \\ u_0=u_1,\ u_M=u_{M-1}. \end{gathered} \tag{14}$$

Here, θ_n is column vector, A_n, B_n, C_n, I are square matrices with $(N+1)$ rows and columns:

$$A_n = \begin{bmatrix} 0 & \dots & 0 & 0 \\ & & & 0 \\ a_n R & & & \vdots \\ & & & 0 \end{bmatrix}, \quad C_n = \begin{bmatrix} 0 & \dots & 0 & 0 \\ & & & 0 \\ c_n R & & & \vdots \\ & & & 0 \end{bmatrix},$$

$$B_n = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & -1 & \cdots & 0 & 0 & 0 & 0 \\ b_n & d & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & b_n & d & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \dots & \dots & \dots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & b_n & d & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & b_n & d \end{bmatrix},$$

$$\begin{array}{l} a_n = -(3+2\cos(x_n))h^{-2} + \sin(x_n)h^{-1}, d = \frac{1}{\tau}, \\ b_n = 1 + d + 2(3+2\cos(x_n))h^{-2}, \\ c_n = -(3+2\cos(x_n))h^{-2} - \sin(x_n)h^{-1}, \end{array}$$

$$\theta_n = \begin{matrix} \theta_n^0 \\ \vdots \\ \theta_n^N \end{matrix}, \quad u_{n\pm1} = \begin{bmatrix} u_{n\pm1}^0 \\ \vdots \\ u_{n\pm1}^N \end{bmatrix}_{(N+1)\times 1}, \quad u_n = \begin{bmatrix} u_n^0 \\ \vdots \\ u_n^N \end{bmatrix}_{(N+1)\times 1}.$$

R is the $N \times N$ identity matrix, as well as

$$\begin{array}{l} \theta_n^0 = \psi_n, \ n = 1, \dots, M-1, \\ \theta_n^k = f(t_k, x_n) - \frac{(3+2\cos(x_n))(\varphi_{n+1} - 2\varphi_n + \varphi)}{h^2} + \frac{\sin(x_n)(\varphi_{n+1} - \varphi_{n-1})}{h} + \varphi_n, \\ k = 1, \dots, N, \ n = 1, \dots, M-1. \end{array}$$

We search solution of (14) by reccurence formula

$$u_n = \alpha_{n+1} u_{n+1} + \beta_{n+1}, \ n = M-1, \dots, 1,$$

where α_n and β_n $(n = 1, \dots, M-1)$ are column vectors with $(N+1)$ elements. For the solution of difference equation (14) we use the following formulas for α_n, β_n

$$\begin{array}{rcl} \alpha_n & = & -(B_n + C_n \alpha_{n-1})^{-1} A_n, \\ \beta_n & = & (B_n + C_n \alpha_{n-1})^{-1} (R\theta_n - C_n \beta_{n-1}), \ n = 1, \dots, M-1, \end{array}$$

where α_1 is the $(N+1) \times (N+1)$ identity matrix and β_1 is the column vector with $(N+1)$ zeros. u_M is computed by formula

$$u_M = (A_M + B_M + C_M \alpha_M)^{-1} (I\theta_M - C_M \beta_M).$$

Second, applying appropriate approximation formulas for derivatives in the nonlocal BVP (10), we get the second order of ADS in t and x

$$\begin{cases} \frac{v_n^k - v_n^{k-1}}{\tau} + \frac{q_2\left(v_{n+1}^k - v_{n-1}^k\right)}{2h} + \frac{q_3\left(v_{n+1}^k - 2v_n^k + v_{n-1}^k\right)}{h^2} \\ + \frac{\tau q_0}{2} \frac{\left(v_{n+2}^k - 3v_{n+1}^k + 3v_n^k - v_{n-1}^k\right)}{h^3} + \frac{\tau q_1}{2} \frac{\left(v_{n+2}^k - 4v_{n+1}^k + 6v_n^k - 4v_{n-1}^k + v_{n-2}^k\right)}{h^4} \\ = \theta_n^k + p(x_n) - \frac{\tau}{2} \cdot \frac{(3+2\sin x_n)\ (p(x_{n+1}) - 2p(x_n) + p(x_{n-1}))}{h^2} \\ -\frac{\tau}{2} \cdot \frac{\cos(x_n)(p(x_{n+1}) - p(x_{n-1}))}{h} + \frac{\tau p(x_n)}{2}, \\ k = 1, \dots N,\ n = 2, \dots, M-2, \\ -3v_0^k + 4v_1^k - v_2^k = 0, \ -3v_M^k + 4v_{M-1}^k - v_{M-2}^k = 0, \\ 10v_0^k - 15v_1^k + 6v_2^k - v_3^k = 0, \\ 10v_M^k - 15v_{M-1}^k + 6v_{M-2}^k - v_{M-3}^k = 0,\ k = 0, \dots, N,\ n = 0, \dots, M, \\ v_n^N = \varphi_n,\ v_n^0 - (1-\rho)v_n^l - \rho v_n^{l+1} = \psi(x_n),\ n = 0, \dots, M \end{cases}$$

for the approximate solution of the nonlocal BVP (10).

Later, we calculate $p(x_n)$ by using (12), where $\{u_n^k\}$ is solution of the difference problem

$$\begin{cases} \frac{u_n^k - u_n^{k-1}}{\tau} + \frac{q_2\left(u_{n+1}^k - u_{n-1}^k\right)}{2h} + \frac{q_3\left(u_{n+1}^k - 2u_n^k + u_{n-1}^k\right)}{h^2} + \frac{\tau}{2} \frac{q_0\left(u_{n+2}^k - 2u_{n+1}^k + 2u_{n-1}^k - u_{n-2}^k\right)}{2h^3} \\ + \frac{\tau}{2} \frac{q_1\left(u_{n+2}^k - 4u_{n+1}^k + 6u_n^k - 4u_{n-1}^k + u_{n-2}^k\right)}{h^4} = \theta_n^k, k = 1, \dots, N,\ n = 2, \dots, M-2, \\ -3u_0^k + 4u_1^k - u_2^k = 0, \ -3u_M^k + 4u_{M-1}^k - u_{M-2}^k = 0, \\ 10u_0^k - 15u_1^k + 6u_2^k - u_3^k = 0, \\ 10u_M^k - 15u_{M-1}^k + 6u_{M-2}^k - u_{M-3}^k = 0,\ k = 0, \dots, N, \\ u_n^0 - (1-\rho)u_n^l - \rho u_n^{l+1} = \psi(x_n),\ n = 0, \dots, M, \end{cases} \tag{15}$$

which is the second order of ADS for nonlocal BVP (11).

For computational reasons it is convenient to rewrite the system (15) in the following matrix form

$$\begin{aligned} & A_n u_{n+2} + B_n u_{n+1} + C_n u_n + D_n u_{n-1} + E_n u_{n-2} = I\theta_n,\ n = 2, \dots, M-2, \\ & -3u_0 + 4u_1 - u_2 = \vec{0}, \ -3u_M + 4u_{M-1} - u_{M-2} = \vec{0}, \\ & 10u_0 - 15u_1 + 6u_2 - u_3 = \vec{0},\ 10u_M - 15u_{M-1} + 6u_{M-2} - u_{M-3} = \vec{0}, \end{aligned} \tag{16}$$

where θ_n is column vector, A_n, B_n, C_n, D_n, E_n, I are $(N+1)\times(N+1)$ square matrices, R is $N \times N$ identity matrix,

$$A_n = \begin{bmatrix} 0 & \dots & 0 & 0 \\ & & & 0 \\ e_n R & & & \vdots \\ & & & 0 \end{bmatrix}, \quad B_n = \begin{bmatrix} 0 & \dots & 0 & 0 \\ & & & 0 \\ y_n R & & & \vdots \\ & & & 0 \end{bmatrix},$$

$$D_n = \begin{bmatrix} 0 & \dots & 0 & 0 \\ & & & 0 \\ z_n R & & & \vdots \\ & & & 0 \end{bmatrix}, \quad E_n = \begin{bmatrix} 0 & \dots & 0 & 0 \\ & & & 0 \\ w_n R & & & \vdots \\ & & & 0 \end{bmatrix},$$

$$C_n = \begin{bmatrix} 1 & 0 & 0 & \cdots & -(1-\rho) & \rho & \cdots & 0 & 0 & 0 \\ r_n & d & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & r_n & d & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \cdots & \cdots & \cdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & r_n & d & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & r_n & d \end{bmatrix}, \quad \theta_n = \begin{bmatrix} \theta_n^0 \\ \vdots \\ \theta_n^N \end{bmatrix},$$

$$
\begin{aligned}
e_n &= \tfrac{\tau q_0}{4h^3} + \tfrac{\tau q_1}{2h^4},\ y_n = \tfrac{q_2}{2h} + \tfrac{1}{h^2}q_3 - \tfrac{\tau q_0}{2h^3} - \tfrac{2\tau q_1}{h^4},\\
r_n &= 1 + \tfrac{1}{\tau} + \tfrac{\tau}{2} - \tfrac{2}{h^2}q_3 + \tfrac{3\tau q_1}{h^4},\\
z_n &= -\tfrac{q_2}{2h} + \tfrac{1}{h^2}q_3 + \tfrac{\tau q_0}{h^3} - \tfrac{2\tau q_1}{h^4},\\
w_n &= -\tfrac{\tau q_0}{4h^3} + \tfrac{\tau q_1}{2h^4},\ n = 2, \dots, M-2.
\end{aligned}
$$

We search solution of linear system equation (16) in the next form

$$
\begin{aligned}
&u_n = \alpha_{n+1}u_{n+1} + \beta_{n+1}u_{n+2} + \gamma_{n+1},\ n = M-2, \dots, 0,\\
&u_M = D_M^{-1}\left((3I - 2\alpha_{M-2})\gamma_{M-1} - 3\gamma_{M-2}\right),\\
&u_{M-1} = D_M^{-1}\left[(4I - \alpha_{M-2})\gamma_{M-1} - \gamma_{M-2}\right],\\
&D_M = (\beta_{M-2} + 5I) - (4I - \alpha_{M-2})\alpha_{M-1},
\end{aligned}
$$

where

$$\gamma_0 = \gamma_1 = \overrightarrow{0},\ \alpha_0 = \tfrac{4}{3}I,\ \beta_0 = -\tfrac{1}{3}I,\ \alpha_1 = \tfrac{8}{5}I,\ \beta_1 = -\tfrac{3}{5}I$$

$$\gamma_{M-2} = \gamma_{M-3} = \overrightarrow{0},\ \alpha_{M-2} = 4I,\ \beta_{M-2} = -3I,\ \alpha_{M-3} = \tfrac{8}{5}I,\ \beta_{M-3} = -\tfrac{3}{5}I,$$

and

$$
\begin{aligned}
&F_n = \left(C_n + D_n\alpha_{n-1} + E_n\beta_{n-2} + E_n\alpha_{n-2}\alpha_{n-1}\right),\ n = 2, \dots, M-4.\\
&\alpha_n = -F_n^{-1}\left(B_n + D_n\beta_{n-1} + E_n\alpha_{n-2}\beta_{n-1}\right),\ \beta_n = -F_n^{-1}A_n,\\
&\gamma_n = -F_n^{-1}\left(I\varphi_n - D_n\gamma_{n-1} - E_n\alpha_{n-2}\gamma_{n-1} - E_n\gamma_{n-2}\right),
\end{aligned}
$$

$$
\begin{aligned}
&Q_{11} = -3B_{M-2} - 8C_{M-2} - 8D_{M-2}\alpha_{M-3} - 3D_{M-2}\beta_{M-3}\\
&-8E_{M-2}\alpha_{M-4}\alpha_{M-3} - 3E_{M-2}\alpha_{M-4}\beta_{M-3} - 8E_{M-2}\beta_{M-4},\\
&Q_{12} = A_{M-2} + 4B_{M-2} + 9C_{M-2} + 9D_{M-2}\alpha_{M-3} + 4D_{M-2}\beta_{M-3}\\
&+9E_{M-2}\alpha_{M-4}\alpha_{M-3} + 4E_{M-2}\alpha_{M-4}\beta_{M-3} + 9E_{M-2}\beta_{M-4},\\
&Q_{21} = A_{M-1} - 3C_{M-1} - 8D_{M-1} - E_{M-1}(8\alpha_{M-3} + 3\beta_{M-3}),\\
&Q_{22} = B_{M-1} + 4C_{M-1} + 9D_{M-1} + E_{M-1}(9\alpha_{M-3} + 4\beta_{M-3}),\\
&G_1 = I\varphi_{M-2} - D_{M-2}\gamma_{M-3} - E_{M-2}\alpha_{M-4}\gamma_{M-3} - E_{M-2}\gamma_{M-3},\\
&G_2 = I\varphi_{M-1} - E_{M-1}\gamma_{M-3},\\
&u_M = (Q_{11} - Q_{12}Q_{22}^{-1}Q_{21})^{-1}(G_1 - Q_{12}Q_{22}^{-1}G_2),\\
&u_{M-1} = Q_{22}^{-1}(G_2 - Q_{21}u_M).
\end{aligned}
$$

Numerical illustration is carried out by using MATLAB program. Solutions of DSs are computed for different values of (N, M). v_n^k and u_n^k correspond to the corresponding numerical values of $v(t,x)$ and $u(t,x)$ at $(t,x) = (t_k, x_n)$ and p_n represents the numerical value of $p(x)$ at point $x = x_n$. The errors are computed by

$$
\begin{aligned}
&Ev_M^N = \max_{0\le k\le N}\left(\sum_{n=1}^{M-1}\left|v(t_k,x_n) - v_n^k\right|^2 h\right)^{\frac{1}{2}},\\
&Eu_M^N = \max_{0\le k\le N}\left(\sum_{n=1}^{M-1}\left|u(t_k,x_n) - u_n^k\right|^2 h\right)^{\frac{1}{2}},\\
&Ep_M = \left(\sum_{n=1}^{M-1}\left|p(x_n) - p_n\right|^2 h\right)^{\frac{1}{2}}.
\end{aligned}
$$

Tables 1 and 2 illustrate the errors between the exact and approximate solutions of DSs for various N and M, respectively. It can be seen from output results that the second order of ADS is more accurate than the first order of ADS. The error analysis shown in Tables 1 and 2 indicate that both DSs have correct convergence rates.

Table 1

Mesh grid absolute value of difference between exact solution and solution of first order of ADS

N=M	20	40	80	160
Ev_M^N	0.034277	0.016674	0.008483	0.004278
Ep_M	0.086716	0.043925	0.022123	0.011104
Ev_M^N	0.152320	0.075113	0.037321	0.018601

Table 2

Mesh grid absolute value of difference between exact solution and solution of second order of ADS

N=M	20	40	80	160
Ev_M^N	0.020123	0.004141	0.000920	0.000217
Ep_M	0.08946	0.024373	0.006796	0.001926
Ev_M^N	0.089678	0.018803	0.004188	0.000969

Conclusion

In this work, SIP for a multi-dimensional parabolic partial differential equation with multi-point nonlocal boundary condition was studied. Stability estimates for solution of inverse problem were obtained. Well-posedness of three SIPs for the reverse parabolic partial differential equations was established.

Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

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Controllability and Optimal Fast Operation of Nonlinear Systems

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A new method for solving the problem of controllability and optimal transient behavior of nonlinear systems subject to boundary conditions and constraints on control values was proposed. Unlike existing methods, this new approach is based on constructing a general solution of the integral equation for a linear controlled system, followed by transforming the original problem into a special initial optimal control problem. We propose a new method for studying the global asymptotic stability of dynamical systems with a cylindrical phase space with a countable equilibrium position based on a non-singular transformation of the equation of motion and estimation of improper integrals along the solution of the system. Conditions for global asymptotic stability were obtained without involving any periodic Lyapunov function, as well as the frequency theorem. The effectiveness of the proposed method is shown with an example.

Keywords: optimal performance, integrity constraints, functional gradient, integral equation.

2020 Mathematics Subject Classification: 49K15, 49K20, 49K21.

Introduction

The first work on controllability of linear systems without constraints on control values is the paper by R.E. Kalman [1]. In [1], minimal norm control is constructed for systems with constant coefficients, and a rank criterion for controllability is established. Controllability of linear systems based on l-problem methods is explored in [2]. Various issues such as minimal control vector dimension, controllability of nonlinear systems with small parameters, and consequences of controllability for linear systems are discussed in [3]. Positional control of linear systems based on Lyapunov functions is examined in [4]. Geometric interpretations of controllability of linear systems are studied in [5], and the relationship between controllability and stabilization of dynamic systems is investigated in [6].

The problem of optimal transient performance was first studied by L.S. Pontryagin and his students [7]. Optimal fast operation under phase coordinate constraints is detailed in [8], and solutions under uncertainty conditions are considered in [9]. Applications of the maximum principle to various specific problems are presented in [10].

It is noteworthy that the problem of optimal fast operation is closely related to controllability. The aforementioned works explore specific cases of the general problems of controllability and fast operation without phase or integral constraints and without boundary condition restrictions. Current and unresolved issues in controllability and optimal fast operation include obtaining necessary and sufficient conditions for the solvability of general controllability and fast operation problems and developing constructive methods for solving general problems of controllability and fast operation of ordinary differential equations.

This paper proposes a new method for investigating controllability and optimal transient behavior of ordinary differential equations based on the study of solvability and the construction of a general solution of a Fredholm integral equation of the first kind with a fixed parameter.

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The solvability and construction of solutions of Fredholm integral equations of the first kind are among the complex and unresolved problems in mathematics [11]. Known results on the solvability of integral equations apply when the operator kernel is symmetric [12].

Results on solvability and construction of solutions of Fredholm integral equations of the first kind and their applications to the qualitative theory of differential equations are presented in [12, 13]. Specific results on applying the study of Fredholm integral equations of the first kind to solving problems of controllability and optimal control are found in [13]. A general theory of boundary value problems for dynamic systems is provided in [12], and research on the dynamics of processes described by integro-differential equations is detailed in [9].

The theory of controllability for nonlinear systems described by ordinary differential equations remains a relatively underexplored area in the mathematical theory of control. It is shown that the problem of controllability of ordinary differential equations, by constructing a general solution of a Fredholm integral equation of the first kind with a fixed parameter, can be reduced to an initial optimal control problem. Solutions to the problem of optimal fast operation can be derived from solving the general controllability problem.

1 Problem Statement

Consider a controlled process described by ordinary differential equations:

$$\dot{x} = A(t)x + B(t)\ f(x, u, t), \quad t \in \mathrm{I} = [t_0,\ t_1], \tag{1}$$

with boundary conditions

$$x(t_0) = x_0 \in R^n, \quad x(t_1) = x_1 \in R^n, \tag{2}$$

subject to control constraints

$$u(t) \in \Lambda(t) = \{u(t) \in L_2(I, R^{m_1}) | u(t) \in V(t) \subset R^{m_1} \ \text{ almost everywhere } \ t \in I\}. \tag{3}$$

Here, $A(t),\ B(t)$ are matrices with piecewise continuous elements of sizes $n \times n$ and $n \times m$, respectively. The vector function $f(x,\ u,\ t)$ is continuous in all variables $(x,\ u,\ t) \in R^n \times R^{m_1} \times I$, satisfying conditions

$$|f(x,\ u,\ t) - f(y,\ u,\ t)| \leq l(t)|x - y|, \ \ \forall (x, u, t),\ (y,\ u,\ t) \in R^n \times R^{m_1} \times I, \tag{4}$$

$$|f(x,\ u,\ t)| \leq c_0(|x| + |u|^2) + c_1(t), \ \ t \in I, \tag{5}$$

$$l(t) > 0, \ \ l(t) \in L_1(I,\ R^1), \ \ c_0 = const > 0, \ \ c_1(t) \geq 0, \ \ c_1(t) \in L_2\,(I,\ R^1). \tag{6}$$

From (4)–(6) it follows that differential equation (1) with initial condition $x(t_0) = x_0$, for any fixed control $u(t) \in L_2(I, R^{m_1})$, has a unique solution. Assume $\Lambda(t),\ t \in I$ is a given bounded convex closed set in $L_2(I,\ R^{m_1})$. In particular, if $A(t) \equiv 0,\ B(t) = I_n$, where I_n, is the $n \times n$, identity matrix, then equation (1) takes the form $\dot{x} = f(x,\ u,\ t)$.

Definition 1. The system (1)–(3) is called controllable, if there exists a control $u(t) \in \Lambda(t)$, that transforms the solution of differential equation (1) from initial state $x_0 = x(t_0)$ at time t_0 to state $x_1 = x(t_1)$ at time t_1.

Along with system (1)–(3), consider the linear controllable system

$$\dot{y} = A(t)y + B(t)\ w(t), \quad t \in I = [t_0, t_1], \tag{7}$$

$$y(t_0) = x_0 \in R^n, \ y(t_1) = x_1 \in R^n, \tag{8}$$

$$w(t) \in L_2(I,\ R^m). \tag{9}$$

The following problems are solved:

Problem 1. Find all control sets $U(t) \subset L_2(I,\ R^m)$, where each element $U(t)$ function $w(t) \in U(t)$ transforms the solution of differential equation (7) under conditions (8), (9) from initial point $x_0 = y(t_0)$ to point $x_1 = y(t_1)$.

Problem 2. Find control $u(t) \in \Lambda(t)$, that transforms the trajectory of system (1)–(3) from initial state $x_0 = x(t_0)$ at time t_0, to state $x_1 = x(t_1)$ at time t_1.

Problem 3. (Optimal Quick Action). Find control $u(t) \in \Lambda(t) \subset L_2(I,\ R^m)$ that moves the trajectory of system (1)–(3) from poin $x_0 = x(t_0)$ to point $x_1 = x(t_1)$ in the shortest time, where t_0 is fixed and t_1 is not fixed.

The problem of optimal quick action is formulated as

$$J(x,\ u,\ t_1) = \int_{t_0}^{t_1} 1 \cdot dt = t_1 - t_0\ \to \inf$$

subject to conditions (1)–(3).

2 Linear Controllable System

Consider solving Problem 1.

The solution of differential equation (7) takes the form

$$y(t) = \Phi(t, t_0)\ x_0 + \int_{t_0}^{t} \Phi(t, \tau) B(\tau) w(\tau) d\tau, \ \ t \in I, \tag{10}$$

where $\Phi(t,\ \tau) = \theta(t)\theta^{-1}(\tau)$, $\theta(t)$ is the fundamental matrix of solutions of the linear homogeneous equation $\dot{\xi} = A(t)\xi$. Note that the matrix $\theta(t),\ t \in I$ of order $n \times n$ is a solution of the matrix equation $\dot{\theta}(t) = A(t)\ \theta(t),\ \ \theta(t_0) = I_n$, where I_n is the identity matrix of order $n \times n$. From (10) at $t \in t_1$, considering $y(t_1) = x_0$, we obtain

$$y(t_1) = x_1 = \Phi(t_1, t_0)\, x_0 + \int_{t_0}^{t_1} \Phi\,(t_1, t)\, B(t)\, w(t) dt.$$

Then

$$\int_{t_0}^{t_1} \Phi\,(t_1, t)\, B(t)\, w(t) dt = x_1 - \Phi(t_1, t_0)\, x_0.$$

Here, considering $\Phi(t_1, t) = \Phi(t_1, t_0)\ \Phi(t_0, t),\ \ \Phi^{-1}(t_1, t_0) = \Phi(t_0, t_1)$, we have

$$\int_{t_0}^{t_1} \Phi(t_0, t)\, B(t)\, w(t)\, dt = \Phi\,(t_0, t_1) x_1 - x_0. \tag{11}$$

Let

$$K(t) = \Phi(t_0, t) B(t),\ \ a = \Phi(t_0, t_1) x_1 - x_0,\ \ t \in I,\ \ a \in R^n. \tag{12}$$

From (11) it follows that the control $w(t) \in L_2(I,\ R^m)$ drives the trajectory of system (7)–(9) from any point x_0 to any point x_1, when $u(t)$ satisfies the integral equation (11). The following theorem establishes the necessary and sufficient condition for the solvability of integral equation (11) for any vector $a \in R^n$ from (12).

Theorem 1. The integral equation (11) has solutions for any vector $a \in R^n$ if and only if the matrix

$$W(t_0, t_1) = \int_{t_0}^{t_1} \Phi(t_0, t)\ B(t)\ B^*(t)\ \Phi^*(t_0, t)\, dt = \int_{t_0}^{t_1} K(t)\ K^*(t) dt, \tag{13}$$

of order $n \times n$ is positive definite, where (*) denotes transposition.

The proof of Theorem 1 can be found in reference [1]. The following two theorems present new results in the theory of controllability of linear systems.

Theorem 2. Suppose the matrix $W(t_0, t_1)$ defined by formula (13) is positive definite. Then the general solution of the integral equation (11) for any $a \in R^n$ is given by

$$w(t) = v(t) + \lambda_1(t,\ x_0,\ x_1) + N_1(t)\ z(t_1,\ v) \in L_2(I, R^m), \tag{14}$$

where $v(t) \in L_2(I,\ R^m)$ is any function. The function $z(t) = z(t,\ v),\ \ t \in I$ is the solution of the differential equation

$$\dot{z} = A(t)z + B(t)\ v(t),\ \ z(t_0) = 0,\ \ v(t) \in L_2(I, R^m), \tag{15}$$

where

$$\lambda_1(t,\ x_0,\ x_1) = B^*(t)\ \Phi^*(t_0, t)\ W^{-1}(t_0, t_1)\ a,\ \ N_1(t) = -B^*(t)\ \Phi^*(t_0, t)\ W^{-1}(t_0, t_1)\ \Phi(t_0, t_1),\ \ t \in I. \tag{16}$$

Proof. Introduce the following sets

$$W = \{w\,(t) \in L_2(I, R^m)|\int_{t_0}^{t_1} K(t)\ w(t)\,dt = a\}, \tag{17}$$

$$U = \{w(t) \in L_2(I, R^m)|w(t) = v(t) + \lambda_1(t,\ x_0,\ x_1) + N_1(t)\ z(t_1, v),\ \ v(t) \in L_2(I, R^m) - \text{any function}\}. \tag{18}$$

The set W contains all solutions of the integral equation (11), when $W(t_0, t_1) > 0$. The theorem asserts that a function $w(t) \in L_2(I,\ R^m)$ belongs to W if and only if it belongs to U. To prove $W = U$, it suffices to show $U \subset W$ and $W \subset U$.

Show $U \subset W$. Indeed, if $w(t) \in U$, then from (18) the equality

$$\int_{t_0}^{t_1} K(t)\,w(t)dt = \int_{t_0}^{t_1} K(t)\,[v(t) + \lambda_1(t,\ x_0,\ x_1) + N_1(t)\ z(t_1,\ v)]dt = \int_{t_0}^{t_1} K(t)\ v(t)dt +$$

$$+\int_{t_0}^{t_1} K(t)\ \lambda_1(t,\ x_0,\ x_1)dt + \int_{t_0}^{t_1} K(t)\ N_1(t)dt\ z(t_1,\,v) =$$

$$= \int_{t_0}^{t_1} K(t)\ v(t)d + \int_{t_0}^{t_1} K(t)\,B^*(t)\Phi^*(t_0, t)dt\ W^{-1}(t_0, t_1)\,a +$$

$$+\int_{t_0}^{t_1} K(t)\,[-B^*(t)\ \Phi^*(t_0, t)]dt\ W^{-1}(t_0, t_1)\ \Phi(t_0, t_1)\ z(t_1,\ v).$$

Hence, considering that the solution of differential equation (15) has the form

$$z(t) = \Phi(t, t_0)\ z(t_0) + \int_{t_0}^{t} \Phi(t, \tau)\ B(\tau)\ v(\tau)d\tau = \int_{t_0}^{t} \Phi(t, \tau)\ B(\tau)\ v(\tau)d\tau,$$

$$z(t_1) = \int_{t_0}^{t_1} \Phi(t_1, t)\ B(\tau)\ v(\tau)dt = \Phi(t_1, t_0) \int_{t_0}^{t} \Phi(t_0, t)\ B(t)\ v(t)dt,$$

we get $(K(t) = \Phi(t_0, t)\ B(t))$

$$\int_{t_0}^{t_1} K(t)\ w(t)dt = \int_{t_0}^{t_1} \Phi(t_0, t)\,B(t)\ v(t)\,dt + \int_{t_0}^{t} \Phi(t_0, t)\ B(t)\ B^*(t)\ \Phi^*(t_0, t)dt\ W^{-1}(t_0, t_1)\ a -$$

$$-\int_{t_0}^{t} \Phi(t_0,t)\ B(t)\ B^*(t)\Phi^*(t_0,t)dt\ W^{-1}(t_0,t_1)\ \Phi(t_0,t_1)\ \Phi(t_1,t_0)\ \int_{t_0}^{t_1} \Phi(t_0,t)\ B(t)\ v(t)\,dt =$$

$$= \int_{t_0}^{t_1} \Phi(t_0,t)\ B(t)\ v(t)\,dt + a - \int_{t_0}^{t_1} \Phi(t_0,t)\ B(t)\ v(t)\,dt = a.$$

Therefore, $w(t) \in W,\ \ U \subset W$.

Show that $W \subset U$. Suppose $w_*(\tau) \in W$. Then from (17) it follows that

$$\int_{t_0}^{t_1} K(t)\ w_*(t)dt = a.$$

Note that in relation (14), the function $v(t) \in L_2(I,\ R^m)$ is arbitrary. In particular, we can choose $v(t) = w_*(\tau),\ \ t \in I$. Now, the function $w(t) \in U$ can be expressed as.

$$w(t) = v(t) + \lambda_1(t,\ x_0,\ x_1) + N_1(t)\ z(t_1,\, v) = w_*(t) + B^*(t)\,\Phi^*(t_0,t)\,W^{-1}(t_0,t_1)a-$$

$$-B^*(t)\ \Phi^*(t_0,t)\ W^{-1}(t_0,t_1)\ \Phi(t_0,t_1)\ \Phi(t_1,t_0)\ \int_{t_0}^{t} \Phi(t_0,t)\,B(t)\ w_*(t)dt = w_*(t)+$$

$$+B^*(t)\,\Phi^*(t_0,t)\ W^{-1}(t_0,t_1)\,a - B^*(t)\ \Phi^*(t_0,t)\ W^{-1}(t_0,t_1)a\ \ = w_*(t) \in U.$$

Therefore, $w_*(\tau) = w(\tau) \in U$. Hence, $W \subset U$. From $U \subset W$ and $W \subset U$, it follows that $U = W$. The theorem is proved.

From (14)–(18), it follows that all control sets, each element of which transforms the trajectory of the system (7)–(9) from point x_0 to point x_1, are determined by formula (18).

Key properties of solutions to integral equation (11):

1. Function $w(t) \in U$ can be represented as $w(t) = w_1(t) + w_2(t)$, where $w_1(t) = K^*(t)W^{-1}(t_0,t_1)a$ is a particular solution of integral equation (11), and

$$w_2(t) = v(t) - K^*(t)W^{-1}(t_0,t_1)\int_{t_0}^{t_1} K(\eta)\,v(\eta)\,d\eta,\ \ t \in I$$

is a solution of the homogeneous integral equation.

$$\int_{t_0}^{t_1} K(t)\ w_2(t)dt = 0.$$

Indeed,

$$\int_{t_0}^{t_1} K(t)\ w_1(t)dt = \int_{t_0}^{t_1} K(t)\ K^*(t)\ W^{-1}(t_0,t_1)\,a = a,$$

$$\int_{t_0}^{t_1} K(t)\ w_2(t)dt = \int_{t_0}^{t_1} K(t)\ v(t)dt - \int_{t_0}^{t_1} K(t)\ K^*(t)W^{-1}(t_0,t_1)\ \int_{t_0}^{t_1} K(\eta)\ v(\eta)d\eta = 0.$$

2. Functions $w_1(t) \in L_2(I,\ R^m)$, $w_2(t) \in L_2(I,\ R^m)$ are orthogonal in L_2, $w_1 \perp w_2$. Indeed,

$$\langle w_1,\ w_2\rangle_{L_2} = \int_{t_0}^{t_1} w_1^*(t)\ w_2(t)dt = a^*\,W^{-1}(t_0,t_1)\int_{t_0}^{t_1} K(t)\ v(t)dt-$$

$$-a^*\,W^{-1}(t_0,t_1)\int_{t_0}^{t_1} K(t)K^*(t)W^{-1}(t_0,t_1)\ \int_{t_0}^{t_1} K(\eta)\ v(\eta)d\eta = 0.$$

3. Function $w_1(t) = K^*(t)W^{-1}(t_0,t_1)a,\ \ t \in I$, is a solution of integral equation (11) with minimal norm in $L_2(I,\ R^m)$. Indeed, $\|w\|^2 \geq \|w_1\|^2 + \|w_2\|^2$, due to $w_1 \perp w_2$. Hence, $\|w\|^2 \geq \|w_1\|^2$. If the

function $v(t) \equiv 0,\ t \in I$, then the function $w_2(t) \equiv 0,\ t \in I$. Therefore $\|w\| = \|w_1\|,\ w(t) = w_1(t)$, $t \in I$.

4. The set of solutions of integral equation (11) is convex. Since $w(t) \in U$, U is a convex set.

Theorem 3. Let the matrix $W(t_0, t_1) > 0$. Then the solution of the differential equation (7) corresponding to the control $w(t) \in U$ is determined by the formula

$$y(t) = z(t_1,\ v) + \lambda_2(t,\ x_0,\ x_1) + N_2(t) z(t_1,\ v),\ \ t \in I,\ \ \forall v,\ \ v(t) \in L_2(I, R^m), \tag{19}$$

where

$$\begin{aligned} &\lambda_2(t,\ x_0,\ x_1) = \Phi(t,t_0)\, W(t,t_1)\, W^{-1}(t_0,t_1) x_0 + \Phi(t,t_0) W(t_0,t)\, W^{-1}(t_0,t_1)\Phi(t_0,t_1) x_1, \\ &N_2(t) = -\Phi(t,t_0)\, W(t_0,t)\, W^{-1}(t_0,t_1)\Phi(t_0,t_1),\ \ t \in I, \\ &W(t_0,t) = \textstyle\int_{t_0}^{t} K(\tau)\, K^*(\tau) d\tau,\ \ W(t,t_1) = \int_{t}^{t_1} K(\tau)\, K^*(\tau) d\tau,\ \ t \in I. \end{aligned} \tag{20}$$

Proof. Suppose the control is determined by formula (14). Then the function.

$$y(t) = \Phi(t,t_0)\ x_0 + \int_{t_0}^{t} \Phi(t,\tau)\ B(\tau)[v(\tau) + \lambda_1(\tau,\ x_0,\ x_1) + N_1(\tau)\ z(t_1,\ v)] d\tau =$$

$$= \int_{t_0}^{t} \Phi(t,\tau)\ B(\tau)\ v(\tau)\ d\tau + \Phi(t,t_0)\ x_0 + \int_{t_0}^{t} \Phi(t,\tau)\ B(\tau)\ B^*(\tau)\ \Phi^*(t_0,\tau)\, d\tau\ \ W^{-1}(t_0,t_1),$$

$$[\Phi(t_1,t_0) x_1 - x_0] - \int_{t_0}^{t} \Phi(t,\tau)\ B(\tau)\ B^*(\tau)\ \Phi^*(t_0,\tau)\, d\tau\ \ W^{-1}(t_0,t_1)\ \Phi(t_1,t_0)\ z(t_1,\ v).$$

Thus, considering that

$$W(t_0,t) = \int_{t_0}^{t} K(\tau)\, K^*(\tau) d\tau = \int_{t_0}^{t} \Phi(t_0,\tau)\, B(\tau)\ B^*(\tau)\ \Phi^*(t_0,\tau) d\tau,\quad W(t,t_1) = W(t_0,t_1) - W(t_0,t),$$

we obtain

$$y(t) = z(t,\ v) + [\Phi(t,t_0) - \Phi(t,t_0)\ W(t_0,t) W^{-1}(t_0,t_1)]\ x_0 + \Phi(t,t_0)\ W(t_0,t)\ W^{-1}(t_0,t_1)\ \Phi(t_1,t_0)\ x_1 -$$

$$-\Phi(t,t_0)\ W(t_0,t)\ W^{-1}(t_0,t_1)\ \Phi(t_1,t_0)\ z(t_1,\ v) = z(t,\ v) + \Phi(t,t_0)\ W(t,t_1)\ W^{-1}(t_0,t_1)\ x_0 +$$

$$+\Phi(t,t_0)\ W(t_0,t)\ W^{-1}(t_0,t_1)\ \Phi(t_0,t_1)\, x_1 - \Phi(t,t_0)\, W(t_0,t) W^{-1}(t_0,t_1)\ \Phi(t_0,t_1) z(t_1,\ v) =$$

$$= z(t,\ v) + \lambda_2(t,\ x_0,\ x_1) + N_2(t) z(t_1,\ v),$$

where $\lambda_2(t,\ x_0,\ x_1),\ N_2(t),\ t \in I,$ are from (20). The theorem is proved.

3 Controllability of Nonlinear Systems

Consider the solution to problem 2.

Comparing systems (1)–(3) and (7)–(9), it is easy to see that they coincide when replacing the function $w(t)$ with $f(x,\ u,\ t)$. This leads to considering the following optimization problem: minimize the functional

$$J(v,u) = \int_{t_0}^{t_1} |v(t) + \lambda_1(t,\ x_0,\ x_1) + N_1(t) z(t_1, v) - f(y(t), u(t), t)|^2 dt \to \inf, \tag{21}$$

subject to the constraints

$$\dot{z} = A(t) z + B(t) v(t),\quad z(t_0) = 0,\ \ t \in I = [t_0, t_1], \tag{22}$$

$$v(t) \in L_2(I, R^m),\ u(t) \in \Lambda(t) \subset L_2(I, R^{m_1}), \tag{23}$$

where the function $y(t),\ t \in I$, is determined by formula (19).

Theorem 4. Suppose the matrix $W(t_0, t_1) > 0$. Then the system (1)–(3) is controllable if and only if the value $J(v_*, u_*) = 0$, where the pair $(v_*(t), u_*(t)) \in L_2(I, R^m) \times \Lambda(t)$ is the optimal control in problem (21)–(23).

Proof. Necessity. Suppose the system (1)–(3) is controllable. We will show that $J(v_*, u_*) = 0$. From the controllability of the system (1)–(3), it follows that there exists a solution to the differential equation (1) the function $x(t) = x(t;\ t_0, x_0, u_*),\ t \in I$, such that $x(t_0) = x_0,\ x(t_1) = x_1$ for $u_* = u_*(t),\ t \in I$. Then $f(x(t;\, t_0,\, x_0,\, u_*),\, u_*(t), t) = w_*(t) \in L_2(I, R^m)$, and the system (1)–(3) can be written as $(x(t) = x(t;\ t_0, x_0, u_*))$.

$$\dot{x}(t; t_0, x_0, u_*) = A(t)x(t; t_0, x_0, u_*) + B(t)\ w_*(t),\ \ t \in I = [t_0, t_1],$$

$$x(t_0; t_0, x_0, u_*) = x_0,\ \ x(t_1; t_0, x_0, u_*) = x_1,\ \ u_*(t) \in L_2(I, R^m).$$

Let $y(t) = x(t;\, t_0, x_0, u_*),\ t \in I$. The function $y(t),\ t \in I$ satisfies $\dot{y} = A(t)y + B(t)w_*(t),\ y(t_0) = x_0$, $y(t_1) = x_1$. Therefore, the function $w_*(t) \in L_2(I, R^m)$ translates the trajectory $y(t),\ t \in I$ from the point x_0 the point x_1. According to Theorem 1, $w_*(t) \in U$, where $w_*(t) = v_*(t) + \lambda_1(t,\ x_0,\ x_1) + N_1(t)z(t_1, v_*),\ \ t \in I$. Thus,

$$J(v_*, u_*) = \int_{t_0}^{t_1} |v_*(t) + \lambda_1(t,\ x_0,\ x_1) + N_1(t)z(t_1, v_*) - f(y(t),\ u_*(t), t)|^2 dt = 0.$$

Necessity is proved.

Sufficiency. Let the functional value $J(v_*, u_*) = 0$, for the pair $(v_*(t),\ u_*(t)) \in L_2(I, R^m) \times \Lambda(t)$. We will demonstrate that the system (1)–(3) is controllable. Note that $J(v, u) \geq 0$. Hence, $J(v_*, u_*) = 0$ if and only if

$$v_*(t) + \lambda_1(t,\ x_0,\ x_1) + N_1(t)\ z(t_1, v_*) = f(y(t,\, v_*),\ u_*(t), t),\ \ t \in I,$$

where we denote

$$w_*(t) = v_*(t) + \lambda_1(t,\ x_0,\ x_1) + N_1(t)\ z(t_1, v_*) = f(y(t,\, v_*),\ u_*(t), t),\ \ t \in I,$$

with $y(t_0,\, v_*) = x_0,\ \ y(t_1,\, v_*) = x_1$. Now the system (7)–(9) can be written as

$$\dot{y}(t, v_*) = A(t)y(t, v_*) + B(t)\ w_*(t),\ \ y(t_0) = x_0,\ \ y(t_1) = x_1,\ \ w_*(t) \in L_2(I, R^m).$$

From this, it follows that $y(t,\, v_*) = x(t;\, t_0, x_0, u_*),\ x(t_0) = x_0,\ x(t_1) = x_1$. Therefore, system (1)–(3) is controllable. Sufficiency is proven. The theorem is proved.

Below are solutions to the optimization problem (21)–(23). It should be noted that: 1) in the optimization problem (21)–(23), unlike the original boundary value problem (1)–(3), boundary conditions are absent; 2) the optimization problem (21)–(23) is an initial problem of optimal control and can be solved using known methods of successive approximations.

Let us introduce the following notations:

$$F_0(q_0, t) = \left|v + T_1(t)\, x_0 + T_2(t)\, x_1 + N_1(t)\ z(t_1, v) - f(y, u.t)\right|^2, \tag{24}$$

where

$$\lambda_1(t,\ x_0,\ x_1) = T_1(t)\, x_0 + T_2(t)\, x_1,$$

$$T_1(t) = -B^*(t)\Phi^*(t_0, t)\, W^{-1}(t_0, t_1),\ T_2(t) = B^*(t)\Phi^*(t_0, t)\, W^{-1}(t_0, t_1)\ \Phi^*(t_0, t_1),$$

$$\lambda_2(t,\ x_0,\ x_1) = C_1(t)\,x_0 + C_2(t)\,x_1,$$
$$C_1(t) = \Phi(t,t_0)\,W(t,t_1),\ W^{-1}(t_0,t_1),\ C_2(t) = \Phi(t,t_0)\,W(t_0,t),\ W^{-1}(t_0,t_1)\ \Phi(t_0,t_1),$$
$$y(t) = z(t,v)\ + C_1(t)\,x_0 + C_2(t)\,x_1 + N_2(t)\,z(t_1,v),\ t \in I,$$
$$q = (v, u, z,\ z(t_1)) \in R^m \times R^{m_1} \times R^n \times R^n.$$

Lemma 1. Suppose matrix $W\,(t_0,t_1) > 0$, the function $f(y,\ u,\ t)$ is defined and continuous with respect to $(y,u,t) \in R^n \times R^{m_1} \times I$ together with partial derivatives with respect to $(y,u) \in R^n \times R^{m_1}$. Then the partial derivatives are

$$\frac{\partial\ F_0(q,t)}{\partial v} = 2[v + T_1(t)x_0 + T_2(t)x_1 + N_1(t)z(t_1) - f(z + C_1(t) + C_2(t) + N_2(t)z(t_1),\ u,\ t)], \quad (25)$$

$$\frac{\partial\ F_0(q,t)}{\partial u} = -2f_u(y,u,t)[v + T_1(t)x_0 + T_2(t)x_1 + N_1(t)z(t_1) - f(y,\ u,\ t)], \quad (26)$$

$$\frac{\partial\ F_0(q,t)}{\partial z} = -2f_x(y,u,t)[v + T_1(t)x_0 + T_2(t)x_1 + N_1(t)z(t_1) - f(y,\ u,\ t)], \quad (27)$$

$$\frac{\partial\ F_0(q,t)}{\partial z(t_1)} = 2[N_1^*(t) + N_2^*(t)f_x(y,\ u,\ t)][v + T_1(t)x_0 + T_2(t)x_1 + N_1(t)z(t_1) - f(y,\ u,\ t)]. \quad (28)$$

Relations (25)–(28) are derived directly from (24) by differentiation.

Lemma 2. Suppose the conditions of Lemma 1 hold and the inequality

$$\langle F_{0q}(q_1,t) - F_{0q}(q_2,t),\ q_1 - q_2\rangle \geq 0,\ \ \forall q_1,\ q_2 \in R^{m+m_1+2n}, \quad (29)$$

is satisfied, where

$$F_{0q}(q,t) = \frac{\partial\,F_0(q,t)}{\partial q} = (\frac{\partial\,F_0}{\partial v},\ \frac{\partial\,F_0}{\partial u},\ \frac{\partial\,F_0}{\partial z},\frac{\partial\,F_0}{\partial z(t_1)}),\ \ t \in I.$$

Then the functional (21) under conditions (22), (23) is convex.

Proof. Inequality (29) is a necessary and sufficient condition for the convexity of the function $F_0(q,t)$ with respect to q. Therefore,

$$F_0(\alpha q_1 + (1-\alpha)q_2)\ \ \leq \alpha F_0(q_1,t) + (1-\alpha)F_0(q_2,t),\ \ t \in I,$$
$$\forall q_1,\ q_2 \in R^N,\ \ N = m_1 + m + 2n,\ \ \forall \alpha, \alpha \in [0,1].$$

Since for any $v_1(t),\ v_2(t) \in L_2(I, R^m)$, the value $z(t, \alpha\, v_1 + (1-\alpha)\, v_2) = \alpha\, z(t,\ v_1) + (1-\alpha)\, z\,(t,\ v_2)$, $\forall \alpha, \alpha \in [0,1],\ \ t \in I$, then

$$J(\alpha\, v_1 + (1-\alpha)\, v_2,\ \alpha\, u_1 + (1-\alpha)\, u_2) = \int_{t_0}^{t_1} F_0(\alpha\, v_1 + (1-\alpha)\, v_2,\ \alpha\, u_1 + (1-\alpha)\, u_2),$$

$$z(t, \alpha\, v_1 + (1-\alpha)\, v_2),\ z(t_1,\ \alpha v_1 + (1-\alpha)\, v_2))dt \leq \alpha \int_{t_0}^{t_1} F_0(q_1,t)dt + (1-\alpha)\int_{t_0}^{t_1} F_0(q_2,t)dt =$$
$$= \alpha\, J\,(v_1,u_1) + (1-\alpha)J\,(v_2,u_2),\ \ \forall\, v_1,\ v_2 \in L_2(I,R^m),\ \ \forall\, u_1,\ u_2 \in L_2(I,R^{m_1}).$$

Thus, the lemma statement follows. Lemma is proved.

Definition 2. The partial derivatives (25)–(28) are said to satisfy the Lipschitz condition if

$$\begin{vmatrix} \frac{\partial\,F_0(q+\Delta q,t)}{\partial v} - \frac{\partial\,F_0(q,t)}{\partial v} \end{vmatrix} \leq L_1\,|\Delta q|\,, \qquad \begin{vmatrix} \frac{\partial\,F_0(q+\Delta q,t)}{\partial u} - \frac{\partial\,F_0(q,t)}{\partial u} \end{vmatrix} \leq L_2\,|\Delta q|\,,$$
$$\begin{vmatrix} \frac{\partial\,F_0(q+\Delta q,t)}{\partial z} - \frac{\partial\,F_0(q,t)}{\partial z} \end{vmatrix} \leq L_3\,|\Delta q|\,, \qquad \begin{vmatrix} \frac{\partial\,F_0(q+\Delta q,t)}{\partial z(t_1)} - \frac{\partial\,F_0(q,t)}{\partial z(t_1)} \end{vmatrix} \leq L_4\,|\Delta q|\,, \quad (30)$$

where $L_i = const > 0$, $i = \overline{1,4}, \Delta q = (\Delta v,\ \Delta u,\ \Delta z,\ \Delta z(t_1))$.

Theorem 5. Suppose the conditions of Lemma 1 and inequalities (30). Then the functional (21) under conditions (22), (23) is continuously differentiable in the Frechet sense, and the gradient

$$J'(v,u) = (J'_v(v,u),\ (J'_u(v,u)) \in L_2(I,R^m) \times L_2(I,R^{m_1})$$

at any point $(v,u) \in L_2(I,R^m) \times L_2(I,R^{m_1})$ is defined by

$$J'_v(v,u) = \frac{\partial F_0(q(t),t)}{\partial v} - B^*(t)\psi(t), \quad J'_u(v,u) = \frac{\partial F_0(q(t),t)}{\partial u}, \tag{31}$$

where $q(t) = (v(t),\ u(t),\ z(t,v),\ z(t_1,v))$, the function $z(t) = z(t,v)$, $t \in I$ is a solution of differential equation (22), and $\psi(t)$, $t \in I$ is a solution of equation

$$\dot{\varphi} = \frac{\partial F_0(q(t),t)}{\partial z} - A^*(t)\psi, \quad \psi(t_1) = -\int_{t_0}^{t_1} \frac{\partial F_0(q(t),t)}{\partial z(t_1)}dt. \tag{32}$$

Moreover, the gradients $J'(v,u)$ satisfy the Lipschitz condition

$$\begin{aligned} &\|J'(v_1,u_1) - J'(v_2,u_2)\| \le l_1(\|v_1 - v_2\|^2 + \|u_1 - u_2\|^2)^{1/2}, \\ &\forall (v_1,v_2) \in L_2(I,R^m), \quad \forall (u_1,u_2) \in L_2(I,R^{m_1}). \end{aligned} \tag{33}$$

Proof. Note that for any $v(t)$, $v(t) + h(t) \in L_2(I,R^m)$, $\Delta z(t) = z(t,v+h) - z(t,v)$ satisfies the differential equation

$$\Delta\dot{z}(t) = A(t)\Delta z(t) + B(t)h(t),\ \Delta z(t_0) = 0,\ t \in I,$$

where

$$\Delta z(t) = \int_{t_0}^{t} \Phi(t,\tau)\ B(\tau)\ h(\tau)d\tau, \quad |\Delta z(t)| \le \int_{t_0}^{t_1} \|\Phi(t,\tau)\|\ \|l(\tau)\|\ |h(\tau)|d\tau \le c_1\|h\|_{L_2}.$$

The increment of the functional

$$\Delta J = J(v+h,\ u+\Delta u) - J(v,u) = \int_{t_0}^{t_1} [h^*(\tau)F_{ov}(q(t),t) + \Delta u^*(t)F_{ou}(q(t),t)+$$

$$+z^*(t)F_{oz}(q(t),t) + \Delta z^*(t_1)F_{oz(t_1)}(q(t),t)]dt + R,$$

where $|R| \le c_2(\|h\|^2 + \|\Delta u\|^2)$, due to estimate (30),

$$F_{0v}(q,t) = \frac{\partial F_0(q,t)}{\partial v},\ F_{0u}(q,t) = \frac{\partial F_0(q,t)}{\partial u},$$

$$F_{0z}(q,t) = \frac{\partial F_0(q,t)}{\partial z},\ F_{0z(t_1)}(q,t) = \frac{\partial F_0(q,t)}{\partial z(t_1)}.$$

The term

$$\Delta z^*(t_1)\int_{t_0}^{t_1} F_{oz(t_1)}(q(t),t) = -\int_{t_0}^{t_1} \Delta z^*(t)\ \psi(t)dt - \int_{t_0}^{t_1} \Delta z^*(t)\ \dot{\psi}(t)dt =$$

$$= -\int_{t_0}^{t_1} h^*(t)\ B^*(t)\ \psi(t) - \int_{t_0}^{t_1} \Delta z^*(t)F_{oz}(q(t),t)dt.$$

Thus, the increment of the functional

$$\Delta J = \int_{t_0}^{t_1} \{h^*(t)\ [F_{ov}(q(t),t) - B^*(t)\ \psi(t)] + \Delta u^*(t)\ F_{ou}(q(t),t)\}dt + R.$$

From here, the first statement (31) of the theorem follows. Let's show that estimate (33), where $\psi(t),\ t \in I$ is a solution of differential equation (32).

Let $\xi(t) = (v(t),\ u(t)),\ t \in I$. Then,

$$J'(\xi_1) - J'(\xi_2) = (F_{ov}(q(t) + \Delta q(t), t) - F_{ov}(q(t), t) - B^*(t)\ \Delta\psi(t),$$

$$F_{ou}(\,q(t) + \Delta q(t), t) - F_{ou}(q(t), t)),\quad \xi_1 = (v_1,\ u_1),\ \ \xi_2 = (v_2,\ u_2).$$

Therefore,

$$|J'(\xi_1) - J'(\xi_2)| = |F_{ov}(q(t) + \Delta q(t), t) - F_{ov}(q(t), t)| + B^*_{\max}\ |\Delta\psi(t)| +$$

$$+|F_{ou}(\,q(t) + \Delta q(t), t) - F_{ou}(q(t), t)| \le (L_1 + L_2)\ |\Delta q(t), t)| + B^*_{\max}\ |\Delta\psi(t)|,$$

where $B^*_{\max} = \sup\limits_{t_0 \le t \le t_1} \|B^*(t)\|$. Norm

$$\|J'(\xi_1) - J'(\xi_2)\|^2 = \int_{t_0}^{t_1} |J'(\xi_1) - J'(\xi_2)|^2 dt \le 2\ (L_1 + L_2)\ \int_{t_0}^{t_1} |\Delta q(t)|^2 dt + 2(B^*_{\max})^2 \int_{t_0}^{t_1} |\Delta\psi(t)|^2 dt \le$$

$$\le 2\, c_3^2\, (L_1 + L_2)\ \|\Delta\xi\|^2 + 2\,(B^*_{\max})^2 \int_{t_0}^{t_1} |\Delta\psi(t)|^2 dt,$$

where $\|\Delta q\| \le c_3\, \|\Delta\xi\|^2,\ \|\Delta\xi\|^2 = (\|h\|^2 + \|\Delta u\|^2),\ \ \Delta\xi = (h,\ \Delta u)$. It can be shown that $|\Delta\psi(t)| \le (L_4\, c_3\ \sqrt{t_1 - t_0} + L_3\, c_3\ \sqrt{t_1 - t_0})\ e^{A^*_{\max}(t_1 - t_0)} \|\Delta\xi\|,\ \ t \in I$, where $A^*_{\max} = \sup\limits_{t_0 \le t \le t_1} \|A^*(t)\|$. Then

$\|J'(\xi_1) - J'(\xi_2)\|^2 \le l_1^2\ \|\Delta\xi\|^2$, where

$$l_1 = [2\, c_3^2\, (L_1 + L_2)^2 + 2\,(B^*_{\max})^2\, (t_1 - t_0)^2 (L_3 + L_4)^2\ c_3^2\, e^{A^*_{\max}(t_1 - t_0)}]^{1/2}.$$

Hence, estimate (33) is proven. Theorem is proved.

Theorem 6. Suppose the conditions of Theorem 5 are satisfied, and the sequences $\{v_n\} \subset L_2(I, R^m)$, $\{u_n\} \subset \Lambda(t) \subset L_2(I, R^{m_1})$ are defined by relations

$$\begin{array}{l} v_{n+1} = v_n - \alpha_n\, J'_v(v_n,\, u_n),\ \ u_{n+1} = P_\Lambda[u_n - \alpha_n\, J'_u(v_n,\, u_n)],\quad n = 0,\, 1,\, 2,\, \dots\, . \\ 0 < \varepsilon_0 \le \alpha_n \le \frac{2}{l_1 + 2\varepsilon_1},\ \ \varepsilon_1 > 0,\ \ n = 0,\, 1,\, 2,\, \dots\, , \end{array} \tag{34}$$

where $P_\Lambda[\cdot]$ is the projection of a point onto the set Λ. Then:

1) The numerical sequence $\{J(v_n,\, u_n)\}$ strictly decreases;

2) $\|v_n - v_{n+1}\| \to 0,\quad \|u_n - u_{n+1}\| \to 0$ as $n \to 0$.

If, in addition, inequality (29), is satisfied, the set $M(v_0, u_0) = \{(v, u) \in L_2(I, R^m) \times \Lambda(t) | J(v, u) \le J(v_0, u_0)\}$ is bounded, then

3) The sequences $\{v_n\}$, $\{u_n\}$ are minimizing sequences,

$$\lim_{n\to\infty} (v_n, u_n) = J_* = \inf J(v, u),\ \ (v, u) \in X \in L_2(I, R^m) \times \Lambda(t);$$

4) The sequences $\{v_n\}$, $\{u_n\}$, weakly converge to the set U_*, where

$$U_* = \{(v_*, u_*) \in X | J(v_*, u_*) = J_* = \inf J(v, u) = \min J\,(v, u),\ \ (v, u) \in X\};$$

5) The rate of convergence estimate is valid:

$$0 \le J(v_n, u_n) - J_* \le \frac{m_0}{n},\ \ n = 1,\, 2,\, \dots\, ,\ \ m_0 = const > 0;$$

6) The controllability problem (1), (2), (4) has a solution if and only if $J(v_*, u_*) = J_* = 0$, in which case $x_*(t) = z(t, v_*) + \lambda_2(t, x_0, x_1) + N_2(t)\ z(t_1, v_*),\ \ t \in I$;

7) If $J(v_*, u_*) > 0$, это $x_*(t),\ \ t \in I$ is the best approximate solution to the controllability problem (1), (2), (4).

Proof. From the property of projection onto sets (34), we have

$$\langle v_{n+1} - v_n + \alpha_n\, J'_v\,(v_n, u_n),\ v - v_{n+1}\rangle_{L_2} = 0,\ \ \forall v,\ \ v \in L_2(I, R^m) \tag{35}$$

$$\langle u_{n+1} - u_n + \alpha_n\, J'_u\,(v_n, u_n),\ u - u_{n+1}\rangle_{L_2} \geq 0,\ \ \forall u,\ u \in \Lambda. \tag{36}$$

Let $\theta = (v,\, u),\ \ \theta_n = (v_n, u_n),\ \ J'(v_n, u_n) = (J'_v(v_n, u_n),\ \ J'_u(v_n, u_n))$. Then (35), (36) can be written as

$$\langle J'(\theta_n),\ \theta - \theta_{n+1}\rangle_{L_2} \geq \frac{1}{\alpha_n}\langle \theta_n - \theta_{n-1},\ \theta - \theta_{n-1}\rangle,\ \ \forall \theta,\ \theta \in X. \tag{37}$$

From the inclusion $J(v,u) \in C^{1,1}(X)$ the inequality

$$J(\theta^1) - J(\theta^2) \geq \langle J'(\theta^1),\ \theta^1 - \theta^2\rangle_H - \frac{l_1}{2}\|\theta^1 - \theta^2\|^2,\ \ \forall \theta^1,\ \theta^2 \in X.$$

Therefore, specifically for $\theta^1 = \theta_n,\ \ \theta^2 = \theta_{n+1}$, we obtain

$$J(\theta_n) - J(\theta_{n-1}) \geq \langle J'(\theta_n),\ \theta_n - \theta_{n+1}\rangle - \frac{l_1}{2}\ \|\theta_n - \theta_{n-1}\|^2. \tag{38}$$

From (37), (38), (34), we have

$$J(\theta_n) - J(\theta_{n-1}) \geq (\frac{1}{\alpha_n} - \frac{l_1}{2})\ \|\theta_n - \theta_{n-1}\|^2 \geq \varepsilon_1\ \|\theta_n - \theta_{n-1}\|^2,\ \ n = 0,\, 1,\, 2,\, ... \tag{39}$$

From here, statements 1) and 2) of the theorem follow.

If inequality (29), is satisfied, then the functional (21) under conditions (22), (23) is convex, the set $M(v_0, u_0)$ is bounded, closed, and convex in H. Therefore, the set $M(v_0, u_0)$ is weakly precompact. The functional $J(v, u)$ is weakly lower semicontinuous on the set $M(v_0, u_0)$ and achieves its infimum, $U_* \neq \oslash,\ \ \oslash$ empty set.

Let's show that the sequence $\{\xi_n\} = \{v_n,\, u_n\}$ is minimizing. Indeed, from the convexity of $J(\xi) \in C^{1,1}(M(v_0, u_0))$, it follows that

$$J(\xi_n) - J(\xi_*) \leq \langle J'(\xi_n),\ \xi_n - \xi_*\rangle_H \leq \|J'(\xi_n)\|\ \|\xi_n - \xi_*\| \leq \|J'(\xi_n)\|\ D, \tag{40}$$

where $\xi_* = (v_*,\ \ u_*) \in U_* \subset M(v_0, u_0),\ \ D$ is diameter of $M(v_0, u_0)$.

From (40), it follows that the sequence $\{\xi_n\} \subset M(\xi_0)$ is minimizing, and $\xi_n \overset{weak}{\to} \xi_*$ weakly as $n \to \infty$, where $\xi_n \overset{weak}{\to} \xi_*$ as $n \to \infty$ means a special convergence of the sequence $\{\xi_n\}$ to an element ξ_*. Thus, statements 3) and 4) are proven.

Let $a_n = J(\xi_n) - J(\xi_*)$. Then from (39), (40) we have

$$a_n - a_{n-1} \geq \frac{1}{2l_1}\|J'(\xi_n)\|^2,\ \ a_n \leq D\ \|J'(\xi_n)\|. \tag{41}$$

From (41) the rate of convergence estimate 5) follows. The theorem is proven.

Optimal Performance. Let t_0 be fixed, t_1 be unfixed. It is necessary to find the smallest value $t_1 = t_*$, for which the system (1), (2), (4) is controllable. It is necessary to find a pair $(t_*,\ u_*(t))$, where $u_*(t) \in \Lambda(t) \subset L_2(I, R^{m_1})$.

I. Setting $t_1 > t_*,$. Using the algorithm outlined above, we find the control $u_{*t_1}(t)$, where $t_0,\ t_1$ are known quantities.

Next, we choose $t_{11} = \frac{t_1}{2}$. We find a pair $(v_{**},\ u_{**}) \in X,\ \ t \in [t_0,\ t_{11}]$. If $J(v_{**},\ u_{**}) = 0$, for this pair, then we choose $t_{12} = \frac{t_1}{4},\ \ t_{12} < t_{11}$ and solve optimization problem (41).

In case where $J(v_{**},\ u_{**}) > 0$, optimization problem (41) is solved for $\frac{3t_1}{4}$ and so on. As a result, the value t_* is determined with the given accuracy $\varepsilon = t_{1n} - t_*$.

II. Sequential Approximation Method. Consider the following optimization problem: minimize the functional

$$J(v,u,t_1) = \int_{t_0}^{t_1} |v(t) + \lambda_1(t,\ x_0,\ x_1) + N_1(t)z(t_1,v) - f(y(t),\ u(t),t)|^2 dt = \int_{t_0}^{t_1} F_0(q(t),\ t_1,\ t)dt \to \inf$$

subject to conditions (42), (43), $t_1 > t_0$. Find Frechet derivatives, $J'_v(v,u,t_1),\ \ J'_u(v,u,t_1)$,

$$J'_{t_1}(v,u,t_1) = F_0(q(t_1),\ t_1,\ t_1) + \int_{t_0}^{t_1} \frac{\partial F_0(q(t),t_1,\ t)}{\partial t_1} dt.$$

Next, we construct sequences $\{v_n\},\ \ \{u_n\},\ \ \{t_{1n}\}$, where

$$t_{1n+1} = t_{1n} - \alpha_n J'_{t_1}(v_n,\ \ u_n,\ \ t_{1n}),\ \ n = 0,\ 1,\ 2,\ ...$$

4 Solution of the Model Problem

As an example, consider the Duffing equation with control [12].

$$\ddot{x} + x + 2x^3 = u(t),\ \ t \in I = [0,\ t_1].$$

This equation can be represented as

$$\dot{x}_1 = x_2,\ \dot{x}_2 = -x_1 - 2x_1^3 + u(t),\ \ t \in [0,\ t_1] = I, \tag{42}$$

where

$$x_1(0) = 1,\ \ x_2(0) = 0,\ x_1(t_1) = 0,\ \ x_2(t_1) = 0, \tag{43}$$

$$u(t) \in \Lambda = \{u(t) \in L_2(I, R^1)| - 2 \le u(t) \le +2 \text{ almost everywhere } t \in I\}. \tag{44}$$

The system (42)–(44) is a mathematical model describing the motion of a rigid spring under the influence of external force $u(t) \in \Lambda$. Consider the problem of optimal performance. For (42)–(44), the linear controllable system takes the form

$$\dot{y}_1 = y_2,\ \dot{y}_2 = w(t),\ \ t \in [0,\ t_1] = I,\ u(t) \in \Lambda,$$

$$y_1(0) = 1,\ \ y_2(0),\ y_1(t_1) = 0,\ \ y_2(t_1) = 0.$$

For this example,

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix},\quad y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix},\quad y(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = x_0,\quad y(t_1) = x_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Matrices

$$e^{At} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix},\quad e^{-At} = \begin{pmatrix} 1 & -t \\ 0 & 1 \end{pmatrix},\quad \theta(t) = e^{At},\ \ \Phi(t,\ \tau) = e^{A(t-\tau)}.$$

Calculate the following vectors and matrices:

$$a = \Phi(\tau, t_1)x_1 - x_0 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \quad W(0,t_1) = \int_0^{t_1} e^{-At}BB^*e^{-A^*\tau}d\tau = \begin{pmatrix} \frac{t_1^3}{3} & -\frac{t_1^2}{2} \\ -\frac{t_1^2}{2} & t_1 \end{pmatrix} > 0,$$

$$W^{-1}(0,t_1) = \begin{pmatrix} \frac{12}{t_1^3} & \frac{6}{t_1^2} \\ \frac{6}{t_1^2} & \frac{4}{t_1} \end{pmatrix}, \quad \lambda_1(t, x_0, x_1) = T_1(t)x_0 + T_2(t)x_1 = \frac{12}{t_1^3} - \frac{6}{t_1^2},$$

$$N_1(t) = \left(\frac{12}{t_1^3} - \frac{6}{t_1^2}, \ -\frac{6t}{t_1^2} - \frac{2}{t_1}\right), \quad \lambda_2(t, x_0, x_1) = \begin{pmatrix} \frac{t_1^3+2t^3-3t_1t^2}{t_1^3} \\ \frac{6t^2-6t\,t_1}{t_1^3} \end{pmatrix},$$

$$N_2(t) = \begin{pmatrix} \frac{2t^3-3t^2t_1}{t_1^3} & \frac{-t^3+t_1t^2}{t_1^2} \\ \frac{6t^3-6t\,t_1}{t_1^3} & \frac{-3t^2+2t\,t_1}{t_1^2} \end{pmatrix}.$$

Then

$$\begin{aligned} &w(t) = v(t) + \left(\tfrac{12t}{t_1^3} - \tfrac{6}{t_1^2}\right) + \left(\tfrac{12t}{t_1^3} - \tfrac{6}{t_1^2}\right) \ z_1(t_1, v) \left(-\tfrac{6t}{t_1^2} + \tfrac{2}{t_1^3}\right) \ z_2(t_1, v), \\ &y(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}, \ y_1(t) = z_1(t) + \tfrac{t_1^3+2t^3-3t_1t^2}{t_1^3} + \left(\tfrac{2t^3-3t^2t_1}{t_1^3}\right) \ z_1(t_1, v) + \tfrac{-t^3+t_1t^2}{t_1^2} \ z_2(t_1, v), \\ &y_2(t) = z_2(t) + \tfrac{6t^2-6t\,t_1}{t_1^3} + \left(\tfrac{6t^2-3t^2t_1}{t_1^3}\right) \ z_1(t_1, v) + \left(\tfrac{-3t^2+2t\,t_1}{t_1^2}\right) \ z_2(t_1, v). \end{aligned} \tag{45}$$

The optimal control problem (1) (21)–(23) for this example takes the form

$$J(v,u) = \int_{t_0}^{t_1} |v(t) + \lambda_1(t, \ x_0, \ x_1) + N_1(t)z(t_1, v) - (-y_1 - 2y_1^3 + u(t))|^2 dt \to \inf \tag{46}$$

subject to conditions

$$\dot{z}_1 = z_2, \ \dot{z}_2 = v(t), \ z_1(0) = 0, \ z_2(0) = 0, \ v(t) \in L_2(I, R^1), \ u \in \Lambda, \tag{47}$$

where $f(y, \ u, \ t) = -y_1 - 2y_1^3 + u(t), \ F_0 = |w(t) - (-y_1 - 2y_1^3 + u)|^2$.

Partial derivatives:

$$\frac{\partial F_0}{\partial v} = 2[v(t) - (-y_1 - 2y_1^3 + u(t)], \quad \frac{\partial F_0}{\partial u} = -2[w(t) - (-y_1 - 2y_1^3 + u(t)],$$

$$\frac{\partial F_0}{\partial z_1} = -2(-1 - 6y_1^2)[w(t) - (-y_1 - 2y_1^3 + u], \quad \frac{\partial F_0}{\partial z_2} = 0,$$

$$\frac{\partial F_0}{\partial z_1(t_1)} = 2[N_1^*(t) + N_2^*(t)f_x(y,u,t)[w(t) - (-y_1 - y_1^3 + u],$$

where $f_x(y,u,t) = \begin{pmatrix} -1 - 3y_1^2 \\ 0 \end{pmatrix}$, $w(t), \ y_1(t), \ y_2(t), \ t \in I$ are determined by formula (45).

The Frechet derivative of the functional (46) under condition (47) is $J'(v,u) = (J'_v(v,u), \ J'_u(v,u))$, where $J'_v(v,u) = \frac{\partial F_0}{\partial v} - B^*\psi(t), \ J'_u(v,u) = \frac{\partial F_0}{\partial u}$. The function $\psi(t), \ t \in I = [0, t_1]$ solving a differential equation

$$\dot{\psi} = \frac{\partial F_0}{\partial z} - A^*\psi, \ \psi(t_1) = -\int_0^{t_1} \frac{\partial F_0}{\partial z(t_1)} dt.$$

Sequences $\{v_n\}$, $\{u_n\}$ are determined by the formulas:

$$v_{n+1} = v_n - \alpha_n J'_v(v_n, u_n), \ u_{n+1} = P_\Lambda[u_n - \alpha_n J'_u(v_n, u_n)], \quad n = 0, \, 1, \, 2, \, ...$$

The solution of the optimization problem (44), (45) for $t_1 = 4$ is:

$$v_*(t) = \begin{cases} -1, & 0 \le t < \frac{5}{4}, \\ +1, & \frac{5}{4} \le t < \frac{13}{4} \\ -1, & \frac{13}{4} \le t < 4, \end{cases}, \qquad u_*(t) = \begin{cases} -\frac{t^2}{2} + 2(1 - \frac{t^2}{2})^3, & \tau \le t < \frac{5}{4}, \\ \frac{t^2}{6} + \frac{5t}{2} + \frac{57}{16} + 2(\frac{t^2}{2} - \frac{5t}{2} + \frac{41}{16})^3, & 0 \le t < \frac{13}{4}, \\ (\frac{t^2}{2} + 4t - 9) + 2(-\frac{t^2}{2} + 4t - 8)^3, & \frac{13}{4} \le t < 4. \end{cases}$$

$$-2 \le u_*(t) \le +2, \quad t \in I = [0,\ 4],$$

$$x_{1*}(t) = \begin{cases} 1 - \frac{t^2}{2}, & 0 \le t \le \frac{5}{4}, \\ \frac{t^2}{2} - \frac{5t}{2} + \frac{41}{16}, & \frac{5}{4} \le t \le \frac{13}{4}, \\ -\frac{t^2}{2} + 4t - 8, & \frac{13}{4} \le t \le 4, \end{cases} \qquad x_{2*}(t) = \begin{cases} -t, & 0 \le t \le \frac{5}{4}, \\ t - \frac{5t}{2}, & \frac{5}{4} \le t \le \frac{13}{4}, \\ -t + 4, & \frac{13}{4} \le t \le 4. \end{cases}$$

The solution to the optimal performance problem for $t_{1*} = 2$ is:

$$v_*(t) = \begin{cases} -1, & 0 \le t < 1, \\ 1, & 1 \le t < 2, \end{cases} \qquad u_*(t) = \begin{cases} -\frac{t^6}{4} + \frac{3t^4}{2} - \frac{7t^2}{2} + 2, & 0 \le t < 1, \\ \frac{t^6}{4} - 3t^5 + 15t^4 - 40t^3 + \frac{121t^2}{2} - 50t + 19, & 1 \le t < 2, \end{cases}$$

$$-2 \le u_*(t) \le 2, \quad t \in I = [0,\ 2].$$

$$x_{1*}(t) = \begin{cases} 1 - \frac{t^2}{2}, & 0 \le t \le 1, \\ \frac{t^2}{2} - 2t + 2, & 1 \le t \le 2, \end{cases} \qquad x_{2*}(t) = \begin{cases} -t, & 0 \le t \le 1, \\ t - 2, & 1 \le t \le 2. \end{cases}$$

5 *Conclusion*

A new method for solving the controllability problem of nonlinear systems described by ordinary differential equations has been developed. The scientific novelty of the obtained results lies in the following:

– all sets of controls for linear systems have been found, each element of which transforms the system trajectory from any initial state to any desired final state (Theorem 2);

– a general solution to the linear controllable system corresponding to the control from the selected set of all controls has been constructed (Theorem 3);

– necessary and sufficient conditions for the controllability of nonlinear systems have been derived (Theorem 4);

– the controllability problem has been reduced to solving the initial optimal control problem for nonlinear control systems (Lemmas 1, 2);

– the gradient of the functional has been found, minimizing sequences have been constructed, and their convergence has been studied (Theorems 5, 6);

– an algorithm for solving the problem of optimal speed was formulated;

– theoretical research results have been demonstrated using an example by solving the nonlinear Duffing equation control problem.

This completes the summary and conclusions of the paper regarding the methods and results obtained for solving the optimal speed control problem for nonlinear systems.

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Author Contributions

S.A. Aisagaliev collected and analyzed data, and led manuscript preparation. G.T. Korpebay assisted in data collection and analysis. S.A. Aisagaliev, G.T. Korpebay served as the principal investigator of the research grant and supervised the research process. All authors participated in the revision of the manuscript and approved the final submission. All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

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Hessian measures in the class of m-convex $(m-cv)$ functions

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The theory of m-convex $(m-cv)$ functions is a new direction in the real geometry. In this work, by using the connection $m-cv$ functions with strongly m-subharmonic (sh_m) functions and using well-known and rich properties of sh_m functions, we show a number of important properties of the class of $m-cv$ functions, in particular, we study Hessians $H^k(u)$, $k=1,2,...,n-m+1$, in the class of bounded $m-cv$ functions.

Keywords: Convex function, m-convex function, Strongly m-subharmonic function, Borel measures, Hessians.

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Introduction

It is well known that m-convex functions are a real analogue in $\mathbb{R}^n$ strongly m-subharmonic (sh_m) functions in the complex space $\mathbb{C}^n$. Let us recall the definition of the class sh_m of functions, which at this time has become the subject of research by many authors (Z. Błocki [1], S. Dinew and S. Kolodziej [2–4], S. Li [5], H.C. Lu [6, 7], H.C. Lu and V.D. Nguyen [8], A. Sadullaev and his students [9–11], etc.).

A twice differentiable function $u(z)\in C^2(D),\ D\subset\mathbb{C}^n$, is said to be strongly m-subharmonic, if at each point of the domain D it holds inequalities

$$(dd^cu)^k\wedge\beta^{n-k}\geq 0,\quad k=1,2,...,n-m+1,$$

where $\beta=dd^c\|z\|^2$ is the standard volume form in $\mathbb{C}^n$.

It's clear that $psh=sh_1\subset sh_2\subset...\subset sh_n=sh$. Operators $(dd^cu)^k\wedge\beta^{n-k}$ are closely related to the Hessians. For a twice differentiable function $u\in C^2(D)$, the second-order differential $dd^cu=\frac{i}{2}\sum\limits_{j,t}\frac{\partial^2u}{\partial z_j\partial\bar z_t}dz_j\wedge d\bar z_t$ (at a fixed point $o\in D$) is a Hermitian quadratic form. After a suitable unitary coordinate transform, it is reduced to the diagonal form $dd^cu=\frac{i}{2}\left[\lambda_1dz_1\wedge d\bar z_1+...+\lambda_ndz_n\wedge d\bar z_n\right]$, where $\lambda_1,...,\lambda_n$ are the eigenvalues of the Hermitian matrix $\left(\frac{\partial^2u}{\partial z_j\partial\bar z_t}\right)$, which are real: $\lambda=(\lambda_1,...,\lambda_n)\in\mathbb{R}^n$. Note that the unitary transformation does not change the differential form. $\beta=dd^c\|z\|^2$. Therefore, it is not difficult to see that

$$(dd^cu)^k\wedge\beta^{n-k}=k!(n-k)!H_o^k(u)\beta^n,$$

where $H_o^k(u)=\sum\limits_{1\leq j_1<...<j_k\leq n}\lambda_{j_1}...\lambda_{j_k}$ is the Hessian of dimension k of the vector $\lambda=\lambda(u)\in\mathbb{R}^n$.

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Consequently, the twice differentiable function $u(z) \in C^2(D), \ \ D \subset \mathbb{C}^n$, is strongly m-subharmonic, if at each point $o \in D$ the next inequalities hold

$$H^k(u) = H^k_o(u) \geq 0, \quad k = 1, 2, ..., n - m + 1. \tag{1}$$

The following theorem is important

Theorem 1. (see [1]). For any twice differentiable $sh_m \cap C^2(D)$ functions $v_1, ..., v_k \in sh_m(D) \cap C^2(D)$, $1 \leq k \leq n - m + 1$, the relation

$$dd^c v_1 \wedge ... \wedge dd^c v_k \wedge \beta^{m-1} \geq 0$$

is valid. In particular, for $u \in sh_m(D) \cap C^2(D)$ and for any $v_1, ..., v_{n-m} \in sh_m(D) \cap C^2(D)$ it holds

$$dd^c u \wedge dd^c v_1 \wedge ... \wedge dd^c v_{n-m} \wedge \beta^{m-1} \geq 0. \tag{2}$$

The last property has dual character: if a twice differentiable function u, it satisfies (2) for all $v_1, ..., v_{n-m} \in sh_m(D) \cap C^2(D)$, then the function u is certainly sh_m function. Moreover, the class of second-order polynomials of the form is sufficient here (see [1, 2])

$$v_j = \sum_{k=1}^{n} c_{j,k} |z_k|^2 \in sh_m(\mathbb{C}^n), c_{j,k} \in \mathbb{R} \text{ is } const. \tag{3}$$

Theorem 1 allows us to define sh_m functions in the class L^1_{loc}.

Definition 1. A function $u \in L^1_{loc}(D)$ is called sh_m in the domain $D \subset \mathbb{C}^n$, if it is upper semi-continuous and for any twice differentiable sh_m functions $v_1, ..., v_{n-m}$ of the form (3), the current $dd^c u \wedge dd^c v_1 \wedge ... \wedge dd^c v_{n-m} \wedge \beta^{m-1}$ defined as

$$\left[dd^c u \wedge dd^c v_1 \wedge ... \wedge dd^c v_{n-m} \wedge \beta^{m-1}\right](\omega) =$$
$$= \int u dd^c v_1 \wedge ... \wedge dd^c v_{n-m} \wedge \beta^{m-1} \wedge dd^c \omega, \ \ \omega \in F^{0,0}$$

is positive, $\int u \, dd^c v_1 \wedge ... \wedge dd^c v_{n-m} \wedge \beta^{m-1} \wedge dd^c \omega \geq 0, \ \ \forall \omega \in F^{0,0}, \ \omega \geq 0.$

1 m-convex functions and associated measures

In this section, similarly to (1), we define Hessians $H^k(u), \ \ k = 1, 2, ..., n - m + 1$, in the class of bounded m-convex functions as Borel measures. This method of defining $H^k(u)$ as a measure belongs to A. Sadullaev.

Let $D \subset \mathbb{R}^n$ and $u(x) \in C^2(D)$. Then matrix $\left(\frac{\partial^2 u}{\partial x_j \partial x_t}\right)$ is orthogonal, $\frac{\partial^2 u}{\partial x_j \partial x_t} = \frac{\partial^2 u}{\partial x_t \partial x_j}$. Therefore, after a suitable orthonormal transformation, it is transformed into a diagonal form,

$$\left(\frac{\partial^2 u}{\partial x_j \partial x_t}\right) \to \begin{pmatrix} \lambda_1 & 0 & ... & 0 \\ 0 & \lambda_2 & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & \lambda_n \end{pmatrix},$$

where $\lambda_j = \lambda_j(x) \in \mathbb{R}$ are the eigenvalues of the matrix $\left(\frac{\partial^2 u}{\partial x_j \partial x_t}\right)$. Let

$$H^k(u) = H^k(\lambda) = \sum_{1 \leq j_1 < ... < j_k \leq n} \lambda_{j_1} ... \lambda_{j_k}$$

be Hessian of the dimension k of the vector $\lambda = (\lambda_1, \lambda_2, ..., \lambda_n)$.

Definition 2. A twice differentiable function $u \in C^2(D)$ is called m-convex in $D \subset \mathbb{R}^n$, $u \in m - cv(D)$, if its eigenvalue vector $\lambda = \lambda(x) = (\lambda_1(x), \lambda_2(x), ..., \lambda_n(x))$ satisfies the conditions

$$m - cv \cap C^2(D) = \left\{ H^k(u) = H^k(\lambda(x)) \geq 0, \ \forall x \in D, \ \ k = 1, ..., n - m + 1 \right\}.$$

Potential theory of $m - cv$ functions is poorly-studied and is a new direction in the theory of real geometry. However, when $m = 1$, this class $1 - cv \cap C^2(D) = \{H^1(\lambda) \geq 0\} = \{\lambda_1 \geq 0, \lambda_2 \geq 0, ..., \lambda_n \geq 0\}$ coincides with the convex functions in $\mathbb{R}^n$, and when $m = n$, the class $n - cv \cap C^2(D) = \{\lambda_1 + \lambda_2 + ... + \lambda_n \geq 0\}$ coincides with the class of subharmonic functions in $\mathbb{R}^n$, $cv = 1 - cv \subset 2 - cv \subset ... \subset n - cv = sh$. The class of convex functions is well studied A. Aleksandrov [12], I. Bakelman [13], A. Pogorelov [14], A. Artykbaev [15] and others. When $m > 1$ this class has been studied in a series of works N. Trudinger, H. Wang, N. Ivochkina and other mathematicians (see [16–22].

Principal difficulties in the theory of $m - cv$ are the introduction of the class $m - cv \cap L^1_{loc}$, i.e. definition $m - cv(D)$ of functions in the class of upper semicontinuous, locally integrable or bounded functions and the definition of Hessians $H^k(u), \ u \in m - cv \cap L^1_{loc}$. So for $m = n$ (the case of subharmonic functions) in the class of upper semicontinuous, locally integrable functions $u(x) \in n - cv(D)$ are defined as a distribution and the Laplace operator $\Delta u = dd^c u \wedge \beta^{n-1}$ is a Borel measure.

To define operators $(dd^c u)^k \wedge \beta^{n-k} \geq 0, \ \ k = 1, 2, ..., n - m + 1$ for the function $u(z) \in sh_m(D)$ in a domain $D \subset \mathbb{C}^n$ the function $u(z)$ must be locally bounded, i.e. $u(z) \in L^\infty_{loc}(D)$. In this case, the operators $(dd^c u)^k \wedge \beta^{n-k} \geq 0, \ \ k = 1, 2, ..., n - m + 1$ are also positive Borel measures (see [10]).

In this work, by using the connection of $m - cv$ functions with strongly m-subharmonic functions and using well-known and rich properties sh_m of functions, we show a number of important properties of the class of $m - cv$ functions, in particular, of the Hessians $H^k(u), \ \ k = 1, 2, ..., n - m + 1$, in the class of bounded $m - cv$ functions.

We embed $\mathbb{R}^n_x$ into $\mathbb{C}^n$, by $\mathbb{R}^n_x \subset \mathbb{C}^n_z = \mathbb{R}^n_x + i\mathbb{R}^n_y \ (z = x + iy)$, as a real $n-$dimensional subspace of the complex space $\mathbb{C}^n$.

Proposition 1. (see [23]). A twice differentiable function $u(x) \in C^2(D), \ \ D \subset \mathbb{R}^n_x$, is $m - cv$ in D, if and only if a function $u^c(z) = u^c(x + iy) = u(x)$ that does not depend on variables $y \in \mathbb{R}^n_y$, is sh_m in the domain $D \times \mathbb{R}^n_y$.

Proof. We establish a connection between the Hessians $H^k(u)$ and $H^k(u^c)$. We have,

$$\frac{\partial u^c}{\partial z_j} = \frac{1}{2}\left[\frac{\partial u^c}{\partial x_j} - \frac{\partial u^c}{\partial y_j}\right] = \frac{1}{2}\frac{\partial u^c}{\partial x_j};$$

$$\frac{\partial^2 u^c}{\partial z_j \partial \bar{z}_t} = \frac{1}{2}\frac{\partial}{\partial \bar{z}_t}\left[\frac{\partial u^c}{\partial x_j}\right] = \frac{1}{4}\left[\frac{\partial^2 u^c}{\partial x_j \partial x_t} + \frac{\partial^2 u^c}{\partial x_j \partial y_t}\right] = \frac{1}{4}\frac{\partial^2 u^c}{\partial x_j \partial x_t}.$$

Thus, $\left(\frac{\partial^2 u^c}{\partial z_j \partial \bar{z}_t}\right) = \frac{1}{4}\left(\frac{\partial^2 u}{\partial x_j \partial x_t}\right)$ and therefore, $H^k(u) = 4^k H^k(u^c)$, that is the proof of the proposition.

Let now $u(x)$ be an upper semicontinuous function in the domain $D \subset \mathbb{R}^n_x$. Then $u^c(z)$ also will be upper semicontinuous function in the domain $D \times \mathbb{R}^n_y \subset \mathbb{C}^n_z$.

Definition 3. An upper semicontinuous function $u\,(x)$ in a domain $D \subset \mathbb{R}^n_x$ is called m-convex in D, if the corresponding function $u^c(z)$ is strongly m-subharmonic, $u^c(z) \in sh_m\left(D \times \mathbb{R}^n_y\right)$.

Let $u(x)$ be a locally bounded m-convex function in the domain $D \subset \mathbb{R}^n_x$. Then $u^c(z)$ will be also locally bounded, strongly m-subharmonic function in the domain $D \times \mathbb{R}^n_y \subset \mathbb{C}^n_z$. Therefore, the operators

$$(dd^c u^c)^k \wedge \beta^{n-k}, \quad k = 1, 2, ..., n - m + 1$$

are defined as Borel measures in the domain $D \times \mathbb{R}^n_y \subset \mathbb{C}^n_z, \ \ \mu_k = (dd^c u^c)^k \wedge \beta^{n-k}$.

Since for a twice differentiable function $(dd^cu^c)^k \wedge \beta^{n-k} = k!(n-k)!H^k(u^c)\beta^n$, then for a bounded, strongly m-subharmonic function in the domain $D \times \mathbb{R}^n_y \subset \mathbb{C}^n_z$, it is natural to determine its Hessians, equating them to the measure

$$H^k(u^c) = \frac{\mu_k}{k!(n-k)!} = \frac{1}{k!(n-k)!}(dd^cu^c)^k \wedge \beta^{n-k}.$$

We can now define Hessians H^k, $k = 1, 2, ..., n-m+1$ in the class of locally bounded, m-convex domain $D \subset \mathbb{R}^n_x$ functions.

Definition 4. Let a function $u(x)$ be locally bounded and m-convex in the domain $D \subset \mathbb{R}^n_x$. Let us define Borel measures in the domain $D \times \mathbb{R}^n_y \subset \mathbb{C}^n_z$,

$$\mu_k = (dd^cu^c)^k \wedge \beta^{n-k}, k = 1, 2, ..., n-m+1.$$

Since $u^c \in sh_m(D \times \mathbb{R}^n_y)$ does not depend on $y \in \mathbb{R}^n_y$, then for any Borel sets $E_x \subset D$, $E_y \subset \mathbb{R}^n_y$, the measures $\frac{4^k}{mesE_y}\mu_k(E_x \times E_y)$ do not depend on the set $E_y \subset \mathbb{R}^n_y$, i.e. $\frac{4^k}{mesE_y}\mu_k(E_x \times E_y) = \nu_k(E_x)$. The Borel measures

$$\nu_k: \quad \nu_k(E_x) = \frac{4^k}{mesE_y}\mu_k(E_x \times E_y), \quad k = 1, 2, ..., n-m+1,$$

we call by Hessians H^k, $k = 1, 2, ..., n-m+1$, for a locally bounded, m-convex function $u(x) \in m-cv(D)$ in the domain $D \subset \mathbb{R}^n_x$.

For twice differentiable function $u(x) \in m-cv(D) \cap C^2(D)$ the Hessians are ordinary functions, however, for a non-twice differentiable, bounded semicontinuous function $u(x) \in m-cv(D) \cap L^\infty(D)$, the Hessians H^k, $k = 1, 2, ..., n-m+1$ are positive Borel measures.

Using Theorem 1 and Preposition 1 (see also Definition 3) $m-cv$ functions are defined as

Definition 5. A function $u(x) \in L^1_{loc}(D)$ is called m-convex function in the domain $D \subset \mathbb{R}^n_x$, $u(x) \in m-cv(D)$, if it is upper semicontinuous and for any twice differentiable $m-cv(D)$ functions $v_1, ..., v_{n-m}$, the current $dd^cu^c \wedge dd^cv^c_1 \wedge ... \wedge dd^cv^c_{n-m} \wedge \beta^{m-1}$ defined as

$$\left[dd^cu^c \wedge dd^cv^c_1 \wedge ... \wedge dd^cv^c_{n-m} \wedge \beta^{m-1}\right](\omega) =$$

$$= \int u^c dd^cv^c_1 \wedge ... \wedge dd^cv^c_{n-m} \wedge \beta^{m-1} \wedge dd^c\omega, \omega \in F^{0,0}\left(D \times \mathbb{R}^n_y\right)$$

is positive.

2 General definitions of m-convex functions

In various works (see, for example, [18, 19]) m-convex functions in the class of bounded upper semicontinuous $m-cv(D)$ functions define using the "viscosity" definition: an upper semicontinuous function $u(x)$ is called $m-cv(D)$, $u(x) \in m-cv(D)$, if any quadratic polynomial $q(x)$ for which the difference $u(x)-q(x)$ achieves a local maximum only at a finite number of points $x^1, ..., x^q \in D$, is $m-cv(D)$, $q(x) \in m-cv(D)$.

The following important proposition belongs to Trudinger-Wang [19]

Lemma 1. A semicontinuous function $u(x)$ is in $m-cv(D)$, if for each domain $G \subset\subset D$ and each function $v(x) \in C^2(D): H_m(v) \leq 0$ from $u|_{\partial G} \leq v|_{\partial G} \Rightarrow u_G \leq v|_G$.

Lemma 2. A semicontinuous function $u(x)$ is in $m-cv(D)$, if and only if for any domain $G \subset\subset D$ there exists $u_j(x) \in C^2(G) \cap m-cv(G): \ u_j(x) \downarrow u(x)$.

Lemma 3. If $m < \frac{n}{2} + 1$, then $m - cv(D) \subset C^{0,\gamma} = Lip_\gamma$, where $\gamma = 2 - \frac{n}{n-m+1}$, $0 < \gamma \leq 1$.

Corollary 1. If $m < \frac{n}{2} + 1$, then $u(x) \in m - cv(D)$ continuous.

For our purpose, it is convenient to use the Trudenger-Wang's definition based on Lemma 2:

Definition 6. An upper semicontinuous function $u(x)$ is called m-convex $m - cv(D)$, if for any domain $G \subset\subset D$ there exists a sequence of functions $u_j(x) \in C^2(G) \cap m - cv(G) : u_j(x) \downarrow u(x)$.

In fact, the two main ones, Definition 3 and Definition 6, are equivalent.

Theorem 2. A function $u(x)$ is $m - cv(D)$ in the sense of Definition 3, if and only if it is $m - cv(D)$ in the sense of Definition 6.

Proof. Let the function $u(x)$ have a monotonically decreasing sequence of functions $u_j(x) \in m - cv(G)$: $u_j(x) \downarrow u(x)$. Let us put $\mathbb{R}^n_x$ in $\mathbb{C}^n_z$, $\mathbb{R}^n_x \subset \mathbb{C}^n_z = \mathbb{R}^n_x + i\mathbb{R}^n_y$ $(z = x + iy)$, and construct a monotonically decreasing sequence $u^c_j(z) = u_j(x) \in sh_m\left(G \times \mathbb{R}^n_y\right)$. Then $\lim\limits_{j\to\infty} u^c_j(z) = u^c(z) \in sh_m\left(G \times \mathbb{R}^n_y\right)$ and $u(x) = u^c(x)$ is $m - cv(G)$.

On the other side, let the function $u(x)$ be such that $u^c(z) = u(x) \in sh_m\left(D \times \mathbb{R}^n_y\right)$. Let us construct a standard approximation $u^c_j(z) = u^c \circ K_{\frac{1}{j}}(w - z)$, $j = 1, 2, ...$ (see [10]). For any compact domain $G \subset\subset D$, starting from a certain number $j \geq j_0$, they are defined, infinitely smooth functions $u^c_j(z) \in sh_m(G)$: $u^c_j(z) \downarrow u^c(z)$. Moreover, it is easy to see that $u^c_j(z)$ do not depend on $y \in \mathbb{R}^n_y$. Therefore, $u^c_j(x) = u_j(x) \downarrow u(x), u_j(x) \in m - cv(G) \cap C^\infty(G)$.

3 *Example (fundamental solution)*

$$\chi_m(x, 0) = \begin{cases} |x|^{2-\frac{n}{n-m+1}} & if \quad m < \frac{n}{2} + 1, \\ \ln|x| & if \quad m = \frac{n}{2} + 1, \\ -|x|^{2-\frac{n}{n-m+1}} & if \quad m > \frac{n}{2} + 1. \end{cases}$$

Thus, when $m < \frac{n}{2} + 1$, the fundamental solution is bounded and Lipschitz, when $m \geq \frac{n}{2} + 1$, it is equal $-\infty$ at the point $x = 0$. Note that at $m = n$, i.e. for the subharmonic case it coincides with fundamental solution of the Laplass operator Δ.

4 *Weakly convergence of m-convex functions*

We will continue our study of Borel measures

$$\left\{H^k(u) \geq 0,\ \forall x \in D,\ \ k = 1, 2, ..., n - m + 1\right\}$$

in the class $u(x) \in m - cv(D) \cap L^\infty_{loc}(D)$.

Theorem 3. If $u(x) \in m - cv(D) \cap L^\infty_{loc}(D)$ and $u_j(x) \in m - cv(D)$ are sequences of monotonically decreasing functions, converging to $u(x), u_j(x) \downarrow u(x)$, then there is weakly convergence of measures $H^k(u_j) \mapsto H^k(u)$, $k = 1, 2, ..., n - m + 1$.

Proof. Let us continue the functions $u(x)$, $u_j(x)$ from $D \subset \mathbb{R}^n_x$ to $D \times \mathbb{R}^n_y$, as sh_m- functions $u^c(z)$, $u^c_j(z) \in sh_m\left(D \times \mathbb{R}^n_y\right)$. Then $u^c(z) \in sh_m\left(D \times \mathbb{R}^n_y\right) \cap L^\infty_{loc}\left(D \times \mathbb{R}^n_y\right)$ and $u^c_j(z) \downarrow u^c(z)$. According to Theorem Sadullaev-Abdullaev (see. [10]), Borel measures

$$H^k(u^c_j) = \frac{\mu^k}{k!(n-k)!} = \frac{1}{k!(n-k)!}\left(dd^c u^c_j\right)^k \wedge \beta^{n-k}$$

weakly converges: $H^k(u_j^c) \mapsto H^k(u^c)$, $k = 1, 2, ..., n - m + 1$. This implies weakly convergence $H^k(u_j) \mapsto H^k(u)$, $k = 1, 2, ..., n - m + 1$.

As is known, if $\{u_\alpha(z)\} \subset sh_m\left(D \times \mathbb{R}_y^n\right)$, $D \times \mathbb{R}_y^n \subset \mathbb{C}^n$, a family of uniformly bounded, strongly m-subharmonic functions, then for any compact set $K \subset\subset D$ there exists a constant $C(K)$, such that the integral averages

$$\int\limits_K (dd^c u_\alpha)^k \wedge \beta^{n-k} \leq C(K), \quad k = 1, 2, ..., n - m + 1$$

(see. [10]). From this it follows that the Hessians

$$H^k(u_\alpha) = \frac{1}{k!(n-k)!}(dd^c u_\alpha)^k \wedge \beta^{n-k},$$

which are Borel measures, are uniformly bounded on average on compact subsets of the domain D. This fact, discovered by Chern-Levine-Nirenberg [24] for a class of psh functions, then it played a main role in the construction of the theory of potential in the class psh and sh_m functions.

Here we will prove a similar fact for Hessians $H^k(u)$, $k = 1, 2, ..., n-m+1$, in the class of $m-cv(D)$, $D \subset \mathbb{R}^n$, functions. At the same time, we note that, if in a class $sh_m\left(D \times \mathbb{R}_y^n\right)$, $D \times \mathbb{R}_y^n \subset \mathbb{C}^n$, the proof is based on differential forms and Stokes' Theorem, then for the estimate $H^k(u)$, $k = 1, 2, ..., n-m+1$, in the class of $m - cv(D)$, $D \subset \mathbb{R}^n$, we do not have this technique.

Theorem 4. If $\{u_\alpha(x)\} \subset m - cv(D)$, $D \subset \mathbb{R}_x^n$, is a family of locally uniformly bounded m-convex functions, then the family of measures $\left\{H^k(u_\alpha)\right\}$, $k = 1, 2, ..., n - m + 1$, in Hessians are uniformly bounded on average on compact subsets of the domain D. In other words, for any compact set $K \subset\subset D$ there is a constant $C(K)$ that is upper bound for integral averages

$$\int\limits_K H^k(u_\alpha) \leq C(K), \quad k = 1, 2, ..., n - m + 1.$$

Proof. Let us use Proposition 1 and Definition 3. We put $\mathbb{R}_x^n$ in $\mathbb{C}^n$, $\mathbb{R}_x^n \subset \mathbb{C}_z^n = \mathbb{R}_x^n + i\,\mathbb{R}_y^n$ $(z = x + iy)$, as a real $n-$dimensional subspace of a complex space $\mathbb{C}^n$ and construct a family of locally uniformly bounded functions. $\{u_\alpha^c(z)\} \subset sh_m\left(D \times \mathbb{R}_y^n\right)$. For this family Borel measures $\left\{H^k(u_\alpha^c)\right\}$, $k = 1, 2, ..., n - m + 1$ is uniformly bounded on average on compact subsets of the domain. $D \times \mathbb{R}_y^n$. From the definition of measures $\left\{H^k(u_\alpha)\right\}$ in Hessians it follows that the family of measures $\left\{H^k(u_\alpha)\right\}$, $k = 1, 2, ..., n - m + 1$ is uniformly bounded on average on compact subsets of the domain D.

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Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

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https://doi.org/10.31489/2024M3/101-111 *Research article*

Synthesis of uniformly distributed optimal control with nonlinear optimization of oscillatory processes

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In the article the problem of synthesizing uniformly distributed optimal control for nonlinear optimization of oscillatory processes described by integro-differential partial differential equations with the Volterra integral operator was explored. The study was conducted according to the Bellman-Egorov scheme and an algorithm for constructing a uniformly distributed optimal control in the form of a functional from the state of the controlled process was developed. Sufficient conditions for the solvability of the synthesis problem in nonlinear optimization were established.

Keywords: generalized solution, Volterra operator, nonlinear optimization, Bellman functional, Frechet differential, Bellman type equations, synthesis of optimal control.

2020 Mathematics Subject Classification: 49K20.

Introduction

With the advent of studies [1–7], methods of the theory of optimal control with distributed parameter systems began to penetrate into various fields of science and attract the attention of researchers.

However, despite the large flow of research, methods for solving optimal control problems with processes described by integrodifferential partial differential equations [8] have not been sufficiently developed. In particular, the development of methods for solving the synthesis problem is one of the most pressing problems. Research is continuing in this direction, and several papers have been published [9–13]. This article examines the solvability of the problem of synthesis of uniformly distributed optimal control, with nonlinear optimization of oscillatory processes described by integro-differential partial differential equations with the Volterra integral operator. Building on the methodology outlined in [10], we developed synthesis problem-solving method based on the Bellman-Egorov scheme. A Bellman type equation is obtained, which is a non-linear integro-differential equation of a nonstandard form. The structure of its solution is found, which makes it possible to transform Bellman-type equations into a system of two equations, one of which is solved independently of the second. This circumstance significantly simplifies the procedure for constructing a synthesizing control.

The issues of constructing a generalized solution to the boundary value problem of a controlled process with an integral Volterra operator are described in detail and sufficient conditions for unambiguity of the solvability of the synthesis problem are established.

1 A generalized solution to the boundary value problem of a controlled process

Let's consider an oscillatory process described by the function $V(t,x)$, which in the domain $Q_T = Q \times (0,T)$ satisfies the integro-differential equation

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$$V_{tt} - AV = \lambda \int_0^t K(t,\tau)V(\tau,x)d\tau + g(t,x)f[u(t)], (t,x) \in Q_T, \tag{1}$$

along with initial conditions at the boundary of the domain Q_T

$$V(0,x) = \psi(x), \quad V_t(0,x) = \psi_2(x), \quad x \in Q \subset R^n, \tag{2}$$

and a boundary condition

$$\Gamma V(t,x) \equiv \sum_{i,k=1}^{n} a_{ik}(x)V_{x_k}(t,x)\cos(\sigma, x_i) + a(x)V(t,x) = 0, (t,x) \in \gamma_T = \gamma \times (0,T), \tag{3}$$

where A is an elliptic operator, Q is a domain in Euclidean space $R^n c$ with a piecewise-smooth boundary γ, σ is the normal vector coming from the point $x\epsilon\gamma$; λ is a parameter; $K(t,\tau)$ is a function defined in the domain $\{0 \le t, \tau \le T\}$ and satisfies the condition

$$\int_0^T \int_0^T K^2(t,\tau)d\tau dt = K_0 < \infty.$$

The functions $\psi_1(x) \in H_1(Q)$, $\psi_2(x) \in H(Q)$, $g(t,x) \in H(Q_T)$, $a_{ik}(x)$, $a(x)$ are considered known; the external source function $f[u(t)] \in H(0,T)$ is nonlinear and monotonic with respect to the functional variable $u(t), t \in [0,T]$; $u(t) \in H(0,T)$ is the control function; H(Y) $-$ Y denotes a Hilbert space of square-integrable functions defined on the set Y; $H_1(Q)$ is the first-order Sobolev space; T is a fixed point in time.

In the context of the problem under consideration, the given functions may be discontinuous, and the existence of a classical solution to the boundary value problem is unlikely. In this regard, following the methodology of reference [9], we will use the following definition of a generalized solution.

Definition 1. A generalized solution of the boundary value problem (1)–(3) is a function $V(t,x) \in H_1(Q_T)$ that satisfies the integral identity

$$\int_Q (V_t(t,x)\Phi(t,x))_{t_1}^{t_2} \quad dx = \int_{t_1}^{t_2} \Big\{ \Big[\int_Q V_t(t,x)\Phi_t(t,x) \; -$$

$$- \sum_{i,k=1}^{\infty} a_{ik}(x)V_{x_k}(t,x)\Phi_{x_i}(t,x) - c(x)V(t,x)\Phi(t,x) +$$

$$+ \left(\lambda \int_0^t K(t,\tau)V(\tau,x)d\tau + g(t,x)f[u(t)] \right) \Phi(\mathrm{t},x) \Big) \Big] dx -$$

$$- \int_\gamma a(x)V(T,x)\Phi(t,x) \Big\} dt \tag{4}$$

for all $t\,(0 \le t \le t_2 \le T)$ and for any function $\Phi(t,x) \in H_1(Q_T)$, as well as the initial condition (2) in the weak sense, i.e., as $t_1 \to 0$

$$\int_Q [(V(t,x) - \psi_1(x))]\,\Phi_0(x)dx = 0, \int_Q [V_t(t,x) - \psi_2(x)]\,\Phi_1(x)dx = 0$$

for any functions $\Phi_0(x) \in H(Q)$ and $\Phi_1(x) \in H(Q)$.

We seek the generalized solution of the boundary value problem (1)–(3) in the form

$$V(t,x) = \sum_{n=1}^{\infty} V_n(t) z_n(x), \quad V_n(t) = \int_Q V(t,x) z_n(x) dx, \tag{5}$$

where $z_n(x)$ is a generalized eigenfunction of the boundary value problem of the form [9]

$$\begin{aligned} &D_n\left[V(t,x), z_j(x)\right] \equiv \\ &\equiv \int_Q \left[\sum_{i,k=1}^{n} a_{ik}(x) V_{x_k}(t,x) z_{jx_i}(x) + c(x) V(t,x) z_{j(x)}\right] dx + \\ &+ \int_\gamma a(x) V(t,x) z_j(x) dx = \lambda_j^2 \int_Q V(t,x) z_j(x) dx, \\ &\Gamma z_j(x) = 0, \ j = 1,2,3,\ldots \end{aligned}$$

and the corresponding eigenvalues satisfy the properties

$$\lambda_j \le \lambda_{j+1} \le \cdots \text{ and } \lim_{j\to\infty} \lambda_j = \infty$$

and the functions $z_n(x), n = 1,2,3,\ldots$, form a complete orthonormal system of generalized eigenfunctions of the boundary value problem (6) in a closed domain $\bar{Q} = Q \cup \gamma$.

According to the methodology of [9], it can be shown that the Fourier coefficients $V_n(t)$ are determined as the solution of the Cauchy problem

$$\begin{gathered} V_n''(t) + \lambda_n^2 V_n(t) = \lambda \int_0^t K(t,\tau) V_n(\tau) d\tau + g_n(t) f[u(t)], \forall t \in [t_1, t_2], \\ V_n(t_1) = \int_Q V(t_1, x) z_n(x) dx, \quad V_n'(t_1) = \int_Q V_t(t_1, x) z_n(x) dx, \\ g_n(t) = \int_Q g(t,x) z_n(x) dx, \quad n = 1,2,3,\ldots, \end{gathered}$$

which is obtained from the integral identity (4) with $\Phi(t,x) \equiv z_n(x)$.

We find the solution of this problem using the formula

$$\begin{aligned} V_n(t) = \int_Q V(t_1,x) z_n(x) dx \cos\lambda_n t + \frac{1}{\lambda_n} \int_Q V_t(t_1,x) z_n(x) dx \sin\lambda_n t + \\ + \frac{1}{\lambda_n} \int_{t_1}^t \sin\lambda_n(t-\tau)\left[\lambda \int_0^\tau K(\tau,y) V_n(y) dy + g_n(\tau) f[u(\tau)]\right] d\tau \end{aligned}$$

which as $t_1 \to 0$ becomes

$$V_n(t) = \psi_{1n}\cos\lambda_n t + \frac{1}{\lambda_n}\psi_{2n}\sin\lambda_n t + \frac{1}{\lambda_n}\int_0^t \sin\lambda_n(t-\tau)\left[\lambda\int_0^t K(\tau,y)V_n(y)dy + q_n(\tau) f[u(\tau)]\right] d\tau,$$

where

$$\psi_{1n} = \lim_{t_1\to 0}\int_Q V(t_1,x) z_n(x) dx, \quad \psi_{2n} = \lim_{t_1\to 0}\int_Q V_t(t_1,x) z_n(x) dx.$$

Using the Liouville approach, this solution can be represented as a linear integral equation of the Volterra 2nd kind of the following form:

$$V_n(t) = \lambda \int_0^t K_n(t,y) V_n(y) dy + q_n(t), \quad n = 1, 2, \ldots, \tag{6}$$

where

$$K_n(t,y) = \int_y^t \frac{1}{\lambda_n} \sin \lambda_n (t-\tau) K(\tau, y) d\tau, \tag{7}$$

$$q_n(t) = \psi_{1n} \cos \lambda_n t + \frac{1}{\lambda_n} \psi_{2n} \sin \lambda_n t + \int_0^t \frac{1}{\lambda_n} \sin \lambda_n (t-\tau) g_n(\tau) f[u(\tau)] d\tau.$$

The solution of equation (6) is found using the formula [14]

$$V_n(t) = \lambda \int_0^t R_n(t,y,\lambda) q_n(y) dy + q_n(t), \tag{8}$$

where the resolvent $R_n(t,y,\lambda)$ is the sum of the Neumann series, i.e.,

$$R_n(t,y,\lambda) = \sum_{i=1}^{\infty} \lambda^{i-1} K_{n,i}(t,y),$$

$$K_{n,i+1}(t,y) = \int_y^t K_n(t,\tau) K_{n,i}(\tau,y) d\tau, i = 1, 2, 3, \ldots.$$

By direct calculations, we establish the estimates

$$|K_{n,i}(t,y)| \le \frac{K_0^i T^{i-1}}{\lambda_n^i} \cdot \frac{(t-y)^i}{i!}, \quad i = 1, 2, 3 \ldots$$

which imply the ratio

$$|R_n(t,y,\lambda)| \le \sum_{i=1}^{\infty} |\lambda|^{i-1} |K_{n,i}(t,y)| \le$$
$$\le \frac{1}{|\lambda| T} \left(\sum_{i=0}^{\infty} \frac{1}{i!} \left[\frac{|\lambda| K_0 T}{\lambda_n} (t-y) \right]^i - 1 \right) \frac{1}{|\lambda| T} \left(e^{\frac{|\lambda| K_0 T}{\lambda_n}(t-y)} - 1 \right),$$

from which it follows that the resolvent $R_n(t,y,\lambda)$ for each $n = 1, 2, 3 \ldots$, for any value of the parameter $\lambda \neq 0$ is a continuous function of the arguments. Note that for the resolvent, there is an estimate

$$\int_0^{]T} |R_n(t,y,\lambda)|^2 dy \le \int_0^T \left[\frac{1}{|\lambda| T^2} \left(e^{\frac{|\lambda| K_0 T}{\lambda_n}(t-y)} - 1 \right) \right]^2 dy \le$$
$$\le \frac{2}{|\lambda|^2 T^2} \int_0^T \left(\int_0^T \left(e^{\frac{2|\lambda| K_0 T}{\lambda_n}(t-y)} + 1 \right) \right) dy \le \tag{9}$$

$$\le \frac{2}{|\lambda|^2 T} \left(\frac{e^{\alpha_n} - 1}{\lambda_n} + 1 \right) \le \frac{2}{|\lambda|^2 T} \left(\frac{e^{\lambda_1} - 1}{\lambda_1} + 1 \right)$$

since

$$\alpha_n = \frac{2|\lambda| K_0 T^2}{\lambda_n} \text{ and } \lim_{\alpha_n \to 0} \frac{e^{\alpha_n} - 1}{\alpha_n} = 1, \text{ as } \lambda_n \to \infty.$$

Next, we substitute the Fourier coefficients $V_n(t)$ found by formula (8) into (5), and we find the formal solution of the boundary value problem (1)–(3) using the formula

$$V(t,x)=\sum_{n=1}^{\infty}\left(\lambda\int_0^t R_n(t,y,\lambda)q_n(y)dy+q_n(t)\right)z_n(x), \tag{10}$$

where the function $q_n(t)$ has the form (7).

Lemma 1.1 The function (10) is an element of the space $H_1\left(Q_T\right)$.

Proof. Differentiating (10) by t, we obtain the function

$$V_t(t,x)=\sum_{n=1}^{\infty}\left(\lambda\int_0^t R'_{nt}(t,y,\lambda)q_n(y)dy+\lambda R_n(t,y,\lambda)q_n(t)+q'_n(t)\right)z_n(x). \tag{11}$$

Taking into account (9) and (10)-(11), by direct calculations, the following relations are established:

$$\|V(t,x)\|^2_{H(Q_T)}\le\frac{6T}{{\lambda_1}^2}\left(1+2\left(\frac{e^{\alpha_1-1}}{\alpha_1}+1\right)\right)\times$$

$$\times\left(\|\psi_1(x)\|^2_{H_1(Q)}+\|\psi_2(x)\|^2_{H_{(Q)}}+\|g(t,x)\|^2_{H(Q_T)}\|f[u(t)]\|^2_{H(0,T)}\right)<\infty;$$

$$\|V_t(t,x)\|^2_{H(Q_T)}\le 9T\left(1+\frac{\lambda^2K_0^2T^2}{\lambda_1^4}\cdot\frac{e^{\alpha_1-1}}{\alpha_1}\right)\times$$

$$\times\left(\|\psi_1(x)\|^2_{H_1(Q)}+\|\psi_2(x)\|^2_{H_{(Q)}}+\|g(t,x)\|^2_{H(Q_T)}\|f[u(t)]\|^2_{H(0,T)}\right)<\infty.$$

From these relations, the statement of the lemma follows.

Theorem 1.1 Let the given functions and parameters satisfy the conditions of the boundary value problem (1)–(3). Then the boundary value problem (1)–(3), for any value of the parameter λ, has a unique generalized solution of the form (10).

Proof. According to Lemma 1.1, the function of the form (10) belongs to the space $H_1\left(Q_T\right)$. By construction, it satisfies the integral identity, and its Fourier coefficients are uniquely determined as the solution to the Cauchy problem. It is also worth noting that due to the monotonicity of the function $f[u(t)]$ with respect to the functional variable, there is a one-to-one correspondence between the elements of the control space $\{u(t)\}=H(0,T)$ and the space of states of the controlled process $\{V(t,x)\}$. If we assume the existence of two generalized solutions, we will arrive at a contradiction.

2 Formulation of the optimal control synthesis problem

Let's consider a nonlinear optimization problem, where the goal is to minimize the integral functional

$$I[u(t)]=\int_Q\|w(T,x)-\xi(x)\|^2dx+\beta\int_0^T p[t,u(t)]dt,\quad \beta>0, \tag{12}$$

over the set of generalized solutions of the boundary value problem (1)-(3). Here, the symbol $\|\cdot\|$ denotes the norm of a vector; $w(t,x)=\{V(t,x),V_t(t,x)\}$ is the vector function describing the state of the controlled process, the vector function $\xi(x)=\{\xi_1(x),\ \xi_2(x)\}\in H^2(Q)=H(Q)\times H(Q)$ is considered known; and the function $p[t,u(t)]\in H(0,T)$ is convex with respect to the functional variable $u(t)\in H(0,T)$.

In the optimal control synthesis problem, the sought control $u^0(t)\in H(0,T)$ is to be found as a function (functional) of the state vector of the controlled process, i.e., in the form $u^0(t)=u[t,w(t,x)]$.

First, let's note one property of the functional (12).

Lemma 1.2 Suppose the function $f[t, u(t)]$ is monotonic and the function $p[t, u(t)]$ is convex with respect to the functional variable $u(t), \forall t \in [0, T]$. Then, the functional $I[u(t)]$ attains its minimum value at a unique element $u^0(t) \in H(0, T)$.

Proof. The monotony condition of the function $f[t, u(t)]$ implies, that each control $[u(t)]$ corresponds to a unique state of the controlled process $w(t, x)$. For example, for the control $u_1(t) + u_2(t)$ the corresponding state of the controlled process is $w_1(t, x) + w_2(t, x)$, leading to the relationship:

$$I\left(\frac{u_1(t)+u_2(t)}{2}\right) = \int_Q \left\| \frac{w_1(T,x)+w_2(T,x)}{2} - \xi(x) \right\|^2 dx + \beta \int_0^T p\left[t, \frac{u_1(t)+u_2(t)}{2}\right] dt. \tag{13}$$

By analogy with the known methodology [1], direct calculations easily establish the equality

$$I\left[u_1(t)\right] + I\left[u_2(t)\right] = 2\int_Q \left\| \frac{w_1(T,x)+w_2(T,x)}{2} - \xi(x) \right\|^2 dx +$$
$$+\frac{1}{2}\int_Q \|w_1(T,x) - w_2(T,x)\|^2 dx + \beta \int_0^T \left(p\left[t, u_1(t)\right] + p_2\left[t, u_2(t)\right]\right) dt,$$

from which, considering the convexity property of the function $p[t, u(t)]$, we obtain the inequality

$$I\left[u_1(t)\right] + I\left[u_2(t)\right] \geq 2\int_Q \left\| \frac{w_1(T,x)+w_2(T,x)}{2} - \xi(x) \right\|^2 dx +$$
$$+\frac{1}{2}\int_Q \|w_1(T,x) - w_2(T,x)\|^2 dx + 2\beta \int_0^T p\left(t, \frac{u_1(t)+u_2(t)}{2}\right) dt >$$
$$> 2I\left[\frac{u_1(t)+u_2(t)}{2}\right]. \tag{14}$$

Suppose that the functional $I[u(t)]$ attains its minimum value $I_{\min}$ for the controls $u_1(t)$ and $u_2(t)$. Then, according to (13)-(14), we obtain the inequality

$$I\left[\frac{u_1(t)+u_2(t)}{2}\right] < I\left[u_1(t)\right] + I\left[u_2(t)\right] = 2I_{\min},$$

which contradicts the optimality of the controls $u_1(t)$ and $u_2(t)$.

3 About the solvability of the synthesis problem

According to (12), the Bellman functional takes the form

$$S[t, w(t,x)] = \min_{\substack{u(\tau)\in U \\ t\leq\tau\leq T}} \left\{ \beta \int_t^T p[\tau, u(\tau)]d\tau + \int_Q \|w(T,x) - \xi(x)\|^2 dx \right\}, \tag{15}$$

where U is the set of admissible control values $u(t) \in H(0, T)$. According to the Bellman-Egorov scheme, assuming that $S[t, w(t, x)]$ as a function is differentiable by t and as a functional is differentiable by Fresche, the relation (15) is reduced to the form

$$-\frac{\partial S[t, w(t,x)]}{\partial t}\Delta t = \min_{\substack{u(\tau)\in U \\ t\leq\tau\leq t}} \left\{ \beta \int_t^{t+\Delta t} p[\tau, u(\tau)]d\tau + ds[t, w(t,x); \Delta w(t,x)] + \right.$$
$$\left. + o_1(\Delta t) + \delta[t, w(t,x); \Delta w(t,x)] \right\} = \min_{u(\tau)\in U} \left\{ \beta \int_t^{t+\Delta t} p[\tau, u(\tau)]d\tau + \int_Q m^*(t,x)\Delta w(t,x)dx + o_1(\Delta t) + \right.$$
$$\left. +\delta[t, w(t,x); \Delta w(t,x)] \right\}, \tag{16}$$

where $m(t,x)=\{m_1(t,x),m_2(t,x)\}$ is the gradient of the functional $S[t,w(t,x)]$; $o_1(\Delta t),\delta[t,w(t,x);\Delta w(t,x)$ are infinitesimal quantities; * denotes transposition.

Further, using an identity of the form
$m^*(t,x)\Delta w(t,x)=m_1(t,x)\Delta V(t,x)-\Delta m_2(t,x)V_t(t+\Delta t,x)+\ (V_t(t,x)m_2(t,x))\,{}_t^{t+\Delta t}$; and an integral identity

$$\int_Q (V_t(t,x)m_2(t,x))_t^{t+\Delta t}\,dx=\int_t^{t+\Delta t}\left\{\int_Q [V_t(y,x)m_{2t}(y,x)\ -\right.$$
$$-\sum_{i,k=1}^{k} a_{i,k}(x)V_{x_k}(y,x)m_{2x_i}(y,x)-c(x)V(y,x)m_2(y,x)+$$
$$\left.+\left(\lambda\int_0^y K(y,\tau)V(\tau,x)d\tau+g(y,x)f[y,u(y)]\right)m_2(y,x)\right]dx-$$
$$\left.-\int_\gamma a(x)V(y,x)m_2(y,x)dx\right\}dy,$$

which is derived from (4) with $t_1=t,t_2=t+\Delta t,\Phi(t,x)\equiv m_2(t,x)$, relation (16) can be represented as

$$-\frac{\partial S[t,w(t,x)]}{\partial t}\Delta t=\min_{\substack{u(\tau)\in U\\ t\le\tau\le t+\Delta t}}\left\{\beta\int_t^{t+\Delta t}p[\tau,u(\tau)]d\tau\ +\right.$$
$$+\int_Q (m_1(t,x)\Delta V(t,x)-\Delta m_2(t,x)V_t(t+\Delta t,x))\ +$$
$$+\int_t^{t+\Delta t}\left(\int_Q[\ V_t(y,x)m_{2t}(y,x)-\sum_{i,k=1}^{n} a_{i,k}(x)V_{x_k}(y,x)m_{2x_i}(y,x)-c(x)V(y,x)m_2(y,x)+\right.$$
$$\left.+\left(\lambda\int_0^y K(y,\tau)V(\tau,x)d\tau+g(y,x)f[y,u(y)]\right)m_2(y,x)\right]dx-$$
$$\left.\left.-\int_\gamma a(x)V(y,x)m_2(y,x)dx\right)dy+o_1(\Delta t)+\delta[t,w(t,x);\Delta w(t,x)]\right\}.$$

We divide this equality by Δt and for Δt → 0, after simple calculations, we have equality in the limit

$$-\frac{\partial S[t,w(t,x)]}{\partial t}=\min_{\substack{u(\tau)\in U\\ t\le\tau\le t}}\left\{\beta p[t,u(t)]+\int_Q g(t,x)m_2(t,x)dxf[t,u(t)]\ +\int_Q [m_1(t,x)V_t(t,x)\ -\right.$$
$$-\sum_{i,k=1}^{n} a_{i,k}(x)V_{x_k}(t,x)m_{2x_i}(t,x)-c(x)V(t,x)m_2(t,x)+$$
$$\left.+\left(\lambda\int_0^t K(t,\tau)V(\tau,x)d\tau\right)m_2(t,x)\right]dx-$$
$$\left.-\int_\gamma a(x)V(t,x)m_2(t,x)dx\right\},\tag{17}$$

which we will call the Bellman-type equation. Note that here the equality holds for the variable $t\in(0,\mathrm{T})$ almost everywhere. We will consider this equation together with the condition

$$S[T,w(T,x)]=\int_Q \|w(T,x)-\xi(x)\|^2dx.\tag{18}$$

Thus, the Bellman functional $S[t, w(t, x)]$ should be found as a solution for the Cauchy-Bellman problem (17)-(18), which is called the Cauchy-Bellman problem.

In the first stage of solving equation (17), we will consider the minimization problem over the control $u(t)$, $\forall t \in [0, T]$, which, depending on the properties of the set U, is solved by different methods.

Let U be an open set. Then, the extremal problem is solved by the classical method, and the first-order optimality condition is given by

$$\beta p_u[t, u(t)] + \int_Q g(t, x) m_2(t, x) dx f_u[t, u(t)] = 0, \tag{19}$$

and the second-order optimality condition is determined by a differential inequality of the form

$$\beta p_{uu}[t, u(t)] + \int_Q g(t, x) m_2(t, x) dx f_{uu}[t, u(t)] > 0,$$

which, with (19) taken into account, can be transformed into the form [10–13]

$$f_u[t, u(t)] \left(\frac{p_u[t, u(t)]}{f_u[t, u(t)]} \right)_u > 0. \tag{20}$$

This inequality is one of the constraints, meaning that the problem of optimal control synthesis in nonlinear optimization of controlled processes is solvable only for those pairs of functions $(f[t, u(t)], p[t, u(t)])$ that satisfy condition (20). When condition (20) is met, according to the implicit function theorem, equation (19) is uniquely solvable for the control $u(t)$. In other words, there exists a unique function $\varphi(\cdot)$, such that

$$u^0(t) = \varphi \left[t, \int_Q g(t, x) m_2(t, x) dx, \beta \right]. \tag{21}$$

Substituting the found $u^0(t)$ into (17), we obtain a simplified version of the Bellman type equation.

$$\begin{aligned}
-\frac{\partial S[t, w(t, x)]}{\partial t} &= \beta p \left[t, \int_Q g(t, x) m_2(t, x) dx, \beta \right] + \\
&+ \int_Q g(t, x) m_2(t, x) dx f \left[t, \int_Q g(t, x) m_2(t, x) dx, \beta \right] + \\
&+ \int_Q [m_2(t, x) V_t(t, x) - \\
&- \sum_{i,k=1}^{n} a_{i,k}(x) V_{x_k}(t, x) m_{2x_i}(t, x) - c(x) V(t, x) m_2(t, x) + \\
&+ \lambda \int_0^t K(t, \tau) V(\tau, x) d\tau \Big) m_2(t, x) \Big) \Big] dx - \\
&- \int_\gamma a(x) V(t, x) m_2(t, x) dx.
\end{aligned} \tag{22}$$

This equation is a nonlinear integro-differential equation of a complex nature and is not of a standard form. According to the methodology developed by A. Kerimbekov [10–13], we seek the solution to equation (22) in the form

$$S[t, w(t, x)] = S_0[t, w(t, x)] + \lambda S_1(t), \tag{23}$$

where $S_0[t,w(t,x)]$ and $S_1(t)$ are to be determined. In this case, equation (22) splits into two equations, and the functional $S_0[t,w(t,x)]$ is determined as the solution to a problem of the form

$$\begin{aligned} -\frac{\partial S_0[t,w]}{\partial t} &= \beta p\left[t, \int_Q g(t,x)m_2(t,x)dx, \beta\right] + \\ &+ \int_Q g(t,x)m_2(t,x)dxf\left[t, \int_Q g(t,x)m_2(t,x)dx, \beta\right] + \\ &+ \int_Q \Big[m_1(t,x)V_t(t,x) - \\ &- \sum_{i,k=1}^{n} a_{i,k}(x)V_{x_k}(t,x)m_{2x_i}(t,x) - c(x)V(t,x)m_2(t,x)\Big]dx - \\ &- \int_\gamma a(x)V(t,x)m_2(t,x)dx, \qquad (24) \\ S_0[T,w(T,x)] &= \int_Q \|w(T,x)-\xi(x)\|^2 dx, \qquad (25) \end{aligned}$$

and the function $S_1(t)$ is determined as the solution to the following problem

$$-\frac{\partial S_1(t)}{\partial t} = \int_Q m_2(t,x)\int_0^t K(t,\tau)V(\tau,x)d\tau dx, \qquad (26)$$

$$S_1(T) = 0. \qquad (27)$$

According to (23), the equality

$$\operatorname{grad} S[t,w(t,x)] = \operatorname{grad} S_0[t,w(t,x)]$$

holds, which implies that in formula (21) the function $m_2(t,x)$ can be determined by solving problem (24)-(25). This circumstance significantly simplifies the procedure of constructing optimal control depending on the state of the controlled process, i.e. the solution of the synthesis problem.

Let $S_0[t,w(t,x)]$ be the solution to problem (24)-(25), and $S_1(T)$ be the solution to problem (26)-(27). Then, according to (23) and (15), the minimum value of the functional (12) is found by the formula

$$\begin{aligned} I\left[u^0(t)\right] = S[0,w(0,x)] &= S_0\left[0,V(0,x),V_t(0,x)\right] + \lambda S_1(0) = \\ &= S_0\left[0,\psi_1(x),\psi_2(x)\right] + \lambda S_1(0). \end{aligned}$$

In conclusion, it should be noted that in the general case, methods for solving problem (24)-(25) are not developed. However, in some particular cases, it is possible to find the solution to problem (24)-(25) and, using formula (21), to write down the explicit form of the sought control $u^0(t)$ depending on the state of the controlled process.

Author Contributions

1. A. K.: Development of the solution method for the nonlinear integro-differential equation of the Bellman type and general analysis of erroneous results.
2. Zh. A.: Justification of results on the application of implicit function theory and analysis.
3. A. B.: Justification of results on the application of integral equations theory and analysis.

Conflict of Interest

The authors declare no relevant financial or non-financial competing interests.

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https://doi.org/10.31489/2024M3/112-125 *Research article*

Existence of extremal solutions for a class of fractional integro-differential equations

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In the study the existence of solutions of a class of fractional integro-differential equations with boundary conditions was considered. The main tool, we employ, is the conventional monotone iterative technique, which is highly effective method to examine the quantitative and qualitative characteristics of various nonlinear problems. This technique produces monotone sequences whose iterations are unique solutions of the certain linear problems. These bounds converge uniformly to the maximal solutions of the given problems. Some types of coupled solutions are considered to obtain the claim of the main results under suitable conditions.

Keywords: Caputo derivative, integro-differential equation, Riemann-Liouville integral, extremal solutions, monotone iterative technique, upper and lower solutions.

2020 Mathematics Subject Classification: 26A33, 26D10, 34B05, 34B60.

Introduction

The basis of our understanding of the world is frequently based on classical calculus, which involves the operation of derivatives and integrals on integer orders. However, many real-world phenomena exhibit memory effects and non-local interactions that cannot be fully captured by these integer-order operations. At this point in the discussion, the concept of fractional calculus presents itself as a relevant topic that should be considered [1]. Fractional calculus is a fascinating field of mathematics that extends the concepts of differentiation and integration to non-integer orders [2, 3]. This extension facilitates a more sophisticated representation of memory-dependent processes, in which the current state is affected by the entire history of the process. See [4–10] for recent works.

Fractional integro-differential equations (FIDEs) are of great importance in the area of fractional calculus. Fractional derivatives and integral terms are combined in FIDEs, making them effective tools for modeling many systems. For instance, FIDEs provide a flexible framework for modeling intricate financial systems with memory effects, such as long-range dependencies in market behavior, and non-classical diffusion processes characterized by varying anomalous diffusion rates, diverging from classical diffusion. Furthermore, FIDEs effectively capture the delayed response of viscoelastic materials to external forces, as these materials exhibit a combination of elastic and viscous properties [11–13].

Monotone iterative technique (MIT) proposes a powerful combination of theoretical and practical tools for nonlinear problems. It provides a theoretical framework to determine the existence and uniqueness of solutions for certain equations, while also offering an efficient iterative algorithm to approximate these solutions numerically, making it valuable for various applications. MIT produces a sequence of functions in which each iteration is derived by substituting the preceding one into the specified linear differential equation. The fundamental principle of MIT is the notion of monotonicity, which guarantees that the sequence is either consistently growing or consistently decreasing, hence

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gradually converging towards the solution of the given nonlinear problem. Under specific conditions, MIT guarantees that the generated sequences converges uniformly in a closed set to the unique solution of the differential equation lying between the initial lower and upper solutions (LUSs) [14]. Recently MIT was adapted for some types of fractional differential or integro-differential equations involving initial or boundary conditions. See [15–22] and the references therein.

In this work, we discuss the following FIDE with boundary conditions of the form:

$$^{C}D^{q_1}u\left(t\right)=F\left(t,u\left(t\right),I^{q_2}u\left(t\right)\right),h\left(u\left(0\right),u\left(T\right)\right)=0, \tag{1}$$

where $F\in C\left[J\times\mathbb{R}\times\mathbb{R},\mathbb{R}\right]$, $J=[0,T]$, $h\in C\left[\mathbb{R}^2,\mathbb{R}\right]$, and $0<q_2\le q_1<1$.

It should be observed that supplementary conditions $h\left(u\left(0\right),u\left(T\right)\right)=0$ may indicate initial, boundary or other general conditions, depending on the selection of the function h. Therefore, problem (1) can be seen as a more comprehensive version of the boundary value problems that were previously mentioned.

The basic objective of the study is to utilize the MIT in order to solve the problem (1), consequently getting the extremal (minimal and maximal) solutions as the limit of the functions of sequences which converge uniformly, by considering several types of coupled lower and upper solutions (LUSs) of (1).

The remainder of this article is structured as follows: Section 1 provides a brief overview of fractional calculus and FIDEs with necessary definitions and lemmas, required for the proofs of main results. The subsequent part presents the main results including the existence and uniqueness theorem for the solution via selection of coupled LUSs. Final section offers concluding remarks and potential directions for future research.

1 *Mathematical preliminaries*

Definition 1. [3] Let $[0,T]\subset\mathbb{R}$, $Re(\theta)>0$ and $f\in L_1[0,T]$. Then the Riemann-Liouville(R-L) fractional integrals I_{0+}^{θ} of order θ is given by

$$I_{0+}^{\theta}f\left(x\right)=\frac{1}{\Gamma\left(\theta\right)}\int\limits_{0}^{x}\frac{f\left(t\right)dt}{\left(x-t\right)^{1-\theta}},\ x\in\left(0,T\right].$$

Definition 2. The Caputo derivative of order $0\le\theta<1$ for $t\in[0,T]$, designated by $^{c}D_{0+}$ is given by

$$^{c}D_{0+}f(x):=I_{0+}^{1-\theta}Df\left(x\right)=\frac{1}{\Gamma\left(1-\theta\right)}\int\limits_{0}^{x}\frac{f'\left(t\right)dt}{\left(x-t\right)^{\theta}}.$$

We offer multiple definitions regarding coupled LUSs to problem (1).

Definition 3. Let $\vartheta,\omega\in C^1[J,\mathbb{R}]$. Then ϑ and ω are said to be

(i) natural LUSs of (1) if

$$^{C}D^{q_1}\vartheta\left(t\right)\le F\left(t,\vartheta\left(t\right),I^{q_2}\vartheta\left(t\right)\right),\quad h\left(\vartheta\left(0\right),\vartheta\left(T\right)\right)\le0,$$
$$^{C}D^{q_1}\omega\left(t\right)\ge F\left(t,\omega\left(t\right),I^{q_2}\omega\left(t\right)\right),\ \ h\left(\omega\left(0\right),\omega\left(T\right)\right)\ge0;$$

(ii) coupled LUSs of type 1 of (1) if

$$^{C}D^{q_1}\vartheta\left(t\right)\le F\left(t,\vartheta\left(t\right),I^{q_2}\omega\left(t\right)\right),\quad h\left(\vartheta\left(0\right),\vartheta\left(T\right)\right)\le0,$$
$$^{C}D^{q_1}\omega\left(t\right)\ge F\left(t,\omega\left(t\right),I^{q_2}\vartheta\left(t\right)\right),\ \ h\left(\omega\left(0\right),\omega\left(T\right)\right)\ge0;$$

(iii) coupled LUSs of type 2 of (1) if

$$^{C}D^{q_1}\vartheta(t) \le F(t,\omega(t), I^{q_2}\vartheta(t)), \quad h(\vartheta(0),\vartheta(T)) \le 0,$$
$$^{C}D^{q_1}\omega(t) \ge F(t,\vartheta(t), I^{q_2}\omega(t)), \quad h(\omega(0),\omega(T)) \ge 0;$$

(iv) coupled LUSs of type 3 of (1) if

$$^{C}D^{q_1}\vartheta(t) \le F(t,\omega(t), I^{q_2}\omega(t)), \quad h(\vartheta(0),\vartheta(T)) \le 0,$$
$$^{C}D^{q_1}\omega(t) \ge F(t,\vartheta(t), I^{q_2}\vartheta(t)), \quad h(\omega(0),\omega(T)) \ge 0.$$

Definition 4. The functions ϱ and r, both belonging to the space $C^1[J,R]$, are called to be coupled minimal and maximal solutions (MMSs) of (1), if, for any coupled solutions ϑ and ω, it holds that $\varrho \le \vartheta, \omega \le r$.

Next result is related to the solution of a linear fractional integro-differential equation.

Lemma 1. Let $\varphi \in C^1[J,\mathbb{R}]$, $0 < q_2 \le q_1 < 1$ and L, M be real numbers. Then, there exists a unique solution $\varphi \in C^1[J,\mathbb{R}]$ of the problem

$$^{C}D^{q_1}\varphi(t) = L\varphi(t) + MI^{q_2}\varphi(t), \varphi(0) = \varphi_0, \tag{2}$$

such that

$$\varphi(t) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{(M)^n (L)^m \binom{n+m}{m} t^{q_1(n+m)+nq_2}}{\Gamma(q_1(n+m)+nq_2+1)}\varphi_0.$$

Proof. The proof and more general form of this result can be found in [23, 24].

Lemma 2. [23] Suppose that ϑ and ω are natural LUSs of (1). Moreover following condition holds

$$F(t,u_1(t),v_1(t)) - F(t,u_2(t),v_2(t)) \le L(u_1-u_2) + M(v_1-v_2),$$

$L, M \ge 0$, whenever $u_1 \ge u_2, v_1 \ge v_2$.

Then $\vartheta(0) \le \omega(0)$ implies $\vartheta(t) \le \omega(t)$ on J.

Corollary 1. ([23]) Let p belongs to the space $C^1[J,\mathbb{R}]$ and $L \ge 0$, $M \ge 0$. If the inequality

$$^{C}D^{q_1}p(t) \le Lp(t) + MI^{q_2}p(t), p(0) \le 0,$$

holds, then we get $p(t) \le 0$ on J.

Analogously, $^{C}D^{q_1}p(t) \ge -Lp(t) - MI^{q_2}p(t), p(0) \ge 0$ implies $p(t) \ge 0$ on J.

2 Main results

In this section, we formulate the monotone technique for the problem (1) via coupled LUSs with the aid of the method of LUSs. We construct monotone functions of sequences, whose iterations are generated by unique solutions of corresponding Caputo type fractional linear initial value problems, hence converging uniformly and monotonically to the minimal and maximal solutions of the given BVP problem (1).

In the following theorem, we first employ natural LUSs to reach the main objective.

Theorem 1. Assume that

(A_1) $\vartheta_0, \omega_0 \in C^1[J,\mathbb{R}]$ are natural LUSs of problem (1) with $\vartheta_0(t) \le \omega_0(t)$ on J;

(A_2) $h(u,v) \in C[\mathbb{R}^2,\mathbb{R}]$ is non-increasing in the second variable and there is a positive constant M satisfying

$$h(u_1,v) - h(u_2,v) \le M(u_1-u_2),$$

for $\vartheta_0(0) \le u_2 \le u_1 \le \omega_0(0)$, $\vartheta_0(T) \le v \le \omega_0(T)$;

(A_3) the function $F \in C[J \times \mathbb{R} \times \mathbb{R}, \mathbb{R}]$ satisfies

$$F(t, u_1(t), v_1(t)) - F(t, u_2(t), v_2(t)) \geq -L(u_1 - u_2) - M(v_1 - v_2), \tag{3}$$

where $\vartheta_0 \leq u_2 \leq u_1 \leq \omega_0$ and $\vartheta_0 \leq v_2 \leq v_1 \leq \omega_0$ and $L > 0$, $M > 0$.

Then there exist monotone sequences $\{\vartheta_n(t)\}$, $\{\omega_n(t)\}$ converging uniformly and monotonically to the functions ϱ and r on J, indicating that ϱ and r serve as minimal and maximal solutions of (1), respectively.

Proof. For any function $\mu \in C^1[J, \mathbb{R}]$, we define the linear initial value problem

$$
\begin{aligned}
{}^C D^{q_1} u(t) &= F(t, \mu(t), I^{q_2}\mu(t)) - L(u - \mu) - MI^{q_2}(u - \mu), && (4)\\
u(0) &= \mu(0) - \frac{1}{M} h(\mu(0), \mu(T)). && (5)
\end{aligned}
$$

where $\vartheta_0 \leq \mu \leq \omega_0$. Pay attention to the fact that the right-hand side of the equation (4) is Lipschitzian, thus unique solution exits for every μ.

Consider A as an operator, such that $A\mu = u$, which assists in the construction the sequences $\{\vartheta_n\}$ and $\{\omega_n\}$.

We have to prove that

(i) $\vartheta_0 \leq A\,\vartheta_0$ and $\omega_0 \geq A\omega_0$;

(ii) the operator A is monotone on the sector $[\vartheta_0, \omega_0] = \{u \in C^1[J, \mathbb{R}] : \vartheta_0 \leq u \leq \omega_0\}$.

To prove (i), set $A\vartheta_0 = \vartheta_1$, where ϑ_1 is the unique soluion of (4)-(5) with $\mu = \vartheta_0$. Setting $p(t) = \vartheta_1(t) - \vartheta_0(t)$ for $t \in J$, we see that

$$
\begin{aligned}
{}^C D^{q_1} p(t) &= {}^C D^{q_1}\vartheta_1(t) - {}^C D^{q_1}\vartheta_0(t)\\
&\geq F(t, \vartheta_0(t), I^{q_2}\vartheta_0(t)) - L(\vartheta_1 - \vartheta_0) - MI^{q_2}(\vartheta_1 - \vartheta_0)\\
&\quad - F(t, \vartheta_0(t), I^{q_2}\vartheta_0(t))\\
&= -Lp(t) - MI^{q_2}p(t),
\end{aligned}
$$

and

$$
\begin{aligned}
p(0) &= \vartheta_1(0) - \vartheta_0(0)\\
&= \vartheta_0(0) - \frac{1}{M} h(\vartheta_0(0), \vartheta_0(T)) - \vartheta_0(0)\\
&\geq 0.
\end{aligned}
$$

This gives, from Corollary 1, $p(t) \geq 0$ on J, hence $\vartheta_0 \leq \vartheta_1$. In the similar way, one can form $p(t) = \omega_0(t) - \omega_1(t)$, where $A\omega_0 = \omega_1$. Then, we obtain

$$
\begin{aligned}
{}^C D^{q_1} p(t) &= {}^C D^{q_1}\omega_0(t) - {}^C D^{q_1}\omega_1(t)\\
&\geq F(t, \omega_0(t), I^{q_2}\omega_0(t)) - (F(t, \omega_0(t), I^{q_2}\omega_0(t)) - L(\omega_1 - \omega_0) - MI^{q_2}(\omega_1 - \omega_0))\\
&= -Lp(t) - MI^{q_2}p(t)
\end{aligned}
$$

and

$$
\begin{aligned}
p(0) &= \omega_0(0) - \omega_1(0)\\
&= \omega_0(0) - \left(\omega_0(0) - \frac{1}{M} h(\omega_0(0), \omega_0(T))\right)\\
&= \frac{1}{M} h(\omega_0(0), \omega_0(T))\\
&\geq 0.
\end{aligned}
$$

This ensures that $p(t) \geq 0$, thus meaning $\omega_0 \geq \omega_1$ on J.

To achieve (ii), consider $\mu_1, \mu_2 \in [\vartheta_0, \omega_0]$, such that $\mu_1 \leq \mu_2$. Suppose that $A\mu_1 = u_1$ and $A\mu_2 = u_2$. Set $p(t) = u_2(t) - u_1(t)$, then

$$\begin{aligned} {}^C D^{q_1} p(t) &= {}^C D^{q_1} u_2(t) - {}^C D^{q_1} u_1(t) \\ &= F(t, \mu_2(t), I^{q_2}\mu_2(t)) - L(u_2 - \mu_2) - MI^{q_2}(u_2 - \mu_2) \\ &\quad - F(t, \mu_1(t), I^{q_2}\mu_1(t)) + L(u_1 - \mu_1) + MI^{q_2}(u_1 - \mu_1) \\ &= F(t, \mu_2(t), I^{q_2}\mu_2(t)) - F(t, \mu_1(t), I^{q_2}\mu_1(t)) + L(u_1 - \mu_1 - u_2 + \mu_2) \\ &\quad + MI^{q_2}(u_1 - \mu_1 - u_2 + \mu_2). \end{aligned}$$

Using the inequality (2), we receive

$$F(t, \mu_2(t), I^{q_2}\mu_2(t)) - F(t, \mu_1(t), I^{q_2}\mu_1(t)) \geq -L(\mu_2 - \mu_1) - MI^{q_2}(\mu_2 - \mu_1).$$

If the expression is plugged into the last inequality, we derive

$$\begin{aligned} {}^C D^{q_1} p(t) &\geq -L(\mu_2 - \mu_1) - MI^{q_2}(\mu_2 - \mu_1) + L(u_1 - \mu_1 - u_2 + \mu_2) + MI^{q_2}(u_1 - \mu_1 - u_2 + \mu_2) \\ &= -Lp(t) - MI^{q_2}p(t). \end{aligned}$$

Also we obtain

$$\begin{aligned} p(0) &= u_2(0) - u_1(0) \\ &= \mu_2(0) - \frac{1}{M}h(\mu_2(0), \mu_2(T)) - \mu_1(0) + \frac{1}{M}h(\mu_1(0), \mu_1(T)) \\ &= \mu_2(0) - \mu_1(0) + \frac{1}{M}(h(\mu_1(0), \mu_1(T)) - h(\mu_2(0), \mu_2(T))) \\ &\geq \mu_2(0) - \mu_1(0) + \frac{1}{M}(h(\mu_1(0), \mu_2(T)) - h(\mu_2(0), \mu_2(T))) \\ &\geq \mu_2(0) - \mu_1(0) + \frac{1}{M}(-M)(\mu_2(0) - \mu_1(0)) \\ &= 0. \end{aligned}$$

Therefore, by applying Corollary 1, we can conclude that $A\mu_2 \geq A\mu_1$.

We now define the sequences $\vartheta_n = A\vartheta_{n-1}$ and $\omega_n = A\omega_{n-1}$ for $n = 1, 2, ...$ Based on the monotonicity argument of the operator, we can infer that

$$\vartheta_0 \leq \vartheta_1 \leq ... \leq \vartheta_n \leq \omega_n \leq ... \leq \omega_1 \leq \omega_0,$$

on $[0, T]$ for all $n \in \mathbb{N}$. These functions correspond to solutions of the following linear equations:

$$\begin{aligned} {}^C D^{q_1} \vartheta_{n+1}(t) &= F(t, \vartheta_n(t), I^{q_2}\vartheta_n) - L(\vartheta_{n+1} - \vartheta_n) - MI^{q_2}(\vartheta_{n+1} - \vartheta_n), && (6) \\ \vartheta_{n+1}(0) &= \vartheta_n(0) - \frac{1}{M}h(\vartheta_n(0), \vartheta_n(T)). && (7) \\ {}^C D^{q_1} \omega_{n+1}(t) &= F(t, \omega_n(t), I^{q_2}\omega_n) - L(\omega_{n+1} - \omega_n) - MI^{q_2}(\omega_{n+1} - \omega_n), && (8) \\ & \omega_{n+1}(0) = \omega_n(0) - \frac{1}{M}h(\omega_n(0), \omega_n(T)). && (9) \end{aligned}$$

Now we have to prove that the monotone sequences $\{\vartheta_n\}$ and $\{\omega_n\}$ converge uniformly. In order to accomplish this, we will utilize the Arzela-Ascoli's theorem once we have revealed that the sequences are equicontinuous and uniformly bounded.

Given that $\vartheta_0, \omega_0 \in C^1[J, \mathbb{R}]$ are bounded on J, a constant $K > 0$ exists, such that $|\vartheta_0(t)| \leq K$ and $|\omega_0(t)| \leq K$ on J. In the light of the fact that $\vartheta_0 \leq \vartheta_n \leq \omega_n \leq \omega_0$, it can be concluded that for all $n \in N$, $|\vartheta_n(t)| \leq K$ and $|\omega_n(t)| \leq K$ on J. As a result, $\{\vartheta_n\}$ and $\{\omega_n\}$ are uniformly bounded on J. Our next objective is to demonstrate that $\{\vartheta_n\}$ is equicontinuous. To do so, let $0 \leq t_1 \leq t_2 \leq T$. Then for $n > 0$,

$$|\vartheta_n(t_1) - \vartheta_n(t_2)| =$$
$$= \Bigg|\vartheta_n(0) + \frac{1}{\Gamma(q_1)} \int_0^{t_1} (t_1 - \sigma)^{q_1 - 1} [F(\sigma, \vartheta_{n-1}(\sigma), I^{q_2}\vartheta_{n-1}(\sigma)) - L(\vartheta_n - \vartheta_{n-1}) - MI^{q_2}(\vartheta_n - \vartheta_{n-1})] d\sigma$$
$$-\vartheta_n(0) - \frac{1}{\Gamma(q_1)} \int_0^{t_2} (t_2 - \sigma)^{q_1 - 1} [F(\sigma, \vartheta_{n-1}(\sigma), I^{q_2}\vartheta_{n-1}(\sigma)) - L(\vartheta_n - \vartheta_{n-1}) - MI^{q_2}(\vartheta_n - \vartheta_{n-1})] d\sigma\Bigg|$$
$$\leq \frac{1}{\Gamma(q_1)} \int_0^{t_1} \left((t_1 - \sigma)^{q_1 - 1} - (t_2 - \sigma)^{q_1 - 1}\right) |F(\sigma, \vartheta_{n-1}(\sigma), I^{q_2}\vartheta_{n-1}(\sigma)) - L(\vartheta_n - \vartheta_{n-1}) - MI^{q_2}(\vartheta_n - \vartheta_{n-1})| d\sigma$$
$$+ \frac{1}{\Gamma(q_1)} \int_{t_1}^{t_2} (t_2 - \sigma)^{q_1 - 1} |F(\sigma, \vartheta_{n-1}(\sigma), I^{q_2}\vartheta_{n-1}(\sigma)) - L(\vartheta_n - \vartheta_{n-1}) - MI^{q_2}(\vartheta_n - \vartheta_{n-1})| d\sigma.$$

Since $\{\vartheta_n\}$, $\{\omega_n\}$, $\{I^{q_2}\vartheta_n\}$ and $\{I^{q_2}\omega_n\}$ are uniformly bounded, there exist a $K_1 > 0$, independent of n, such that

$$|F(t, \vartheta_n(t), I^{q_2}\vartheta_n(t))| \leq K_1,$$
$$|F(t, \omega_n(t), I^{q_2}\omega_n(t))| \leq K_1,$$
$$|I^{q_2}\vartheta_n(t)| \leq K_1,$$

and

$$|I^{q_2}\omega_n(t)| \leq K_1.$$

Thus, if these expressions are substituted into the inequality above, we get

$$\begin{aligned}
&|\vartheta_n(t_1) - \vartheta_n(t_2)| \\
&\leq \frac{K_2}{\Gamma(q_1)} \int_0^{t_1} \left((t_1 - \sigma)^{q_1 - 1} - (t_2 - \sigma)^{q_1 - 1}\right) d\sigma + \frac{K_2}{\Gamma(q_1)} \int_{t_1}^{t_2} (t_2 - \sigma)^{q_1 - 1} d\sigma \\
&= -\frac{K_2}{q_1\Gamma(q_1)} (t_1 - \sigma)^{q_1} \Big|_{\sigma=0}^{\sigma=t_1} + \frac{K_2}{q_1\Gamma(q_1)} (t_2 - \sigma)^{q_1} \Big|_{\sigma=0}^{\sigma=t_1} - \frac{K_2}{q_1\Gamma(q_1)} (t_2 - \sigma)^{q_1} \Big|_{\sigma=t_1}^{\sigma=t_2} \\
&= \frac{K_2}{\Gamma(q_1 + 1)} t_1^{q_1} + \frac{K_2}{\Gamma(q_1 + 1)} (t_2 - t_1)^{q_1} - \frac{K_2}{\Gamma(q_1 + 1)} t_2^{q_1} + \frac{K_2}{\Gamma(q_1 + 1)} (t_2 - t_1)^{q_1} \\
&= \frac{K_2}{\Gamma(q_1 + 1)} [(t_1)^{q_1} - (t_2)^{q_1}] + \frac{2K_2}{\Gamma(q_1 + 1)} (t_2 - t_1)^{q_1} \\
&\leq \frac{2K_2}{\Gamma(q_1 + 1)} (t_2 - t_1)^{q_1} \\
&= \frac{2K_2}{\Gamma(q_1 + 1)} |t_2 - t_1|^{q_1},
\end{aligned}$$

where $K_2 = K_1 + 2LK + 2MK_1$. We conclude, that for given $\epsilon > 0$, there is a $\delta(\epsilon) = \left(\frac{\epsilon\Gamma(q_1+1)}{2K_2}\right)^{\frac{1}{q_1}}$ (which merely depends on ϵ), such that $|t_2 - t_1| < \delta$ imply that $|\vartheta_n(t_1) - \vartheta_n(t_2)| < \epsilon$. Therefore $\{\vartheta_n\}$ is equicontinuous on J and so is $\{\omega_n\}$ in the similar fashion. The use of Arzela-Ascoli's theorem allows us to conclude, that there exist subsequences $\{\vartheta_{n_k}\}$ and $\{\omega_{n_k}\}$ that uniformly converge to ϱ and r respectively. Due to their monotonic nature, the entire sequences $\{\vartheta_n\}$ and $\{\omega_n\}$ converge uniformly to ϱ and r respectively on J.

We can prove that the limit functions (ϱ, r) satisfy the problem (1). To do so, we establish corresponding integral equations to (6)-(7) and (8)-(9), then take limits as $n \longrightarrow \infty$.

Finally, it is required to clarify that (r, ϱ) occurs as the maximal and minimal solutions of (1), respectively. For any given solution u of (1) such that $\vartheta_0(t) \le u(t) \le \omega_0(t)$ on J, we need to check that

$$\vartheta_0(t) \le \varrho(t) \le u(t) \le r(t) \le \omega_0(t),$$

on J. To achieve this, it is sufficient to demonstrate $\vartheta_n(t) \le u(t) \le \omega_n(t)$ on J. This fact is obvious for $n = 0$. By applying induction principle, we claim that for some $k > 0$, the inequality $\vartheta_k(t) \le u(t) \le \omega_k(t)$ on J is true. It is necessary to prove that the following relation holds:

$$\vartheta_{k+1}(t) \le u(t) \le \omega_{k+1}(t),$$

on J. Taking $p(t) = u(t) - \vartheta_{k+1}(t)$ leads to

$$\begin{aligned} {}^C D^{q_1} p(t) &= {}^C D^{q_1} u(t) - {}^C D^{q_1} \vartheta_{k+1}(t) \\ &= F(t, u(t), I^{q_2} u(t)) - [F(t, \vartheta_k(t), I^{q_2}\vartheta_k) - L(\vartheta_{k+1} - \vartheta_k) - MI^{q_2}(\vartheta_{k+1} - \vartheta_k)]. \end{aligned}$$

Since we know that $\vartheta_k(t) \le u(t)$, we can use the inequality (3) to attain

$$F(t, u(t), I^{q_2} u(t)) - F(t, \vartheta_k(t), I^{q_2}\vartheta_k) \ge -L(u - \vartheta_k) - MI^{q_2}(u - \vartheta_k).$$

By inserting the foregoing expression into the equation above, we acquire

$$\begin{aligned} {}^C D^{q_1} p(t) &\ge -L(u - \vartheta_k) - MI^{q_2}(u - \vartheta_k) + L(\vartheta_{k+1} - \vartheta_k) + MI^{q_2}(\vartheta_{k+1} - \vartheta_k) \\ &= -Lp(t) - MI^{q_2} p(t). \end{aligned}$$

Meanwhile, if we recall the characteristics of the function $h(u, v)$, we can deduce

$$\begin{aligned} p(0) &= u(0) - \vartheta_{k+1}(0) \\ &= u(0) - \frac{1}{M} h(u(0), u(T)) - \left[\vartheta_k(0) - \frac{1}{M} h(\vartheta_k(0), \vartheta_k(T))\right] \\ &= u(0) - \vartheta_k(0) - \frac{1}{M}(h(u(0), u(T)) - h(\vartheta_k(0), \vartheta_k(T))) \\ &\ge u(0) - \vartheta_k(0) - \frac{1}{M}(h(u(0), \vartheta_k(T)) - h(\vartheta_k(0), \vartheta_k(T))) \\ &\ge u(0) - \vartheta_k(0) - \frac{1}{M} M(u(0) - \vartheta_k(0)) \\ &= 0. \end{aligned}$$

Owing to Corollary 1, it directly results in $p(0) \ge 0$ on J. As a result, $\vartheta_{k+1}(t) \le u(t)$. In the same manner, we are able to demonstrate that $u(t) \le \omega_{k+1}(t)$ on J. Therefore, for all n, we get

$$\vartheta_n(t) \le u(t) \le \omega_n(t).$$

By taking the limit, as n approaches infinity, we may deduce that

$$\varrho(t) \le u(t) \le r(t),$$

on J, which establishes the validity of the proof.

Theorem 2. Along with the assumptions stated in Theorem 1, further assume that for $L > 0$, $M > 0$

$$F(t, u_1(t), v_1(t)) - F(t, u_2(t), v_2(t)) \leq L(u_1 - u_2) + M(v_1 - v_2),$$

where $\vartheta_0 \leq u_2 \leq u_1 \leq \omega_0$ and $\vartheta_0 \leq v_2 \leq v_1 \leq \omega_0$. Thereafter, a unique solution to equation (1) exists in which $\varrho = u = r$.

Proof. If we continue by keeping the fact $\varrho \leq r$ aside, let $p(t) = r(t) - \varrho(t)$. Then, it follows that

$$\begin{aligned} {}^{C}D^{q_1}p(t) &= {}^{C}D^{q_1}r(t) - {}^{C}D^{q_1}\varrho(t) \\ &= F(t, r(t), I^{q_2}r(t)) - F(t, \varrho(t), I^{q_2}\varrho(t)) \\ &\leq L(r - \varrho) + MI^{q_2}(r - \varrho) \\ &= Lp(t) + MI^{q_2}p(t), \end{aligned}$$

and

$$\begin{aligned} p(0) &= r(0) - \varrho(0) \\ &= 0. \end{aligned}$$

This facts indicate that $p(t) \leq 0$. As a consequence, we arrive at $\varrho = u = r$ meaning that the sequences approach to the same solution of (1).

In the subsequent result, we employ coupled LUSs of type 1 to derive monotone sequences that uniformly and monotonically converge to coupled MMSs of the problem (1).

Theorem 3. Assume that

(B_1) $\vartheta_0, \omega_0 \in C^1[J, \mathbb{R}]$ are coupled LUSs of type 1 of problem (1) with $\vartheta_0(t) \leq \omega_0(t)$ on J;

(B_2) (A_2) holds;

(B_3) $F(t, u, v) \in C[J \times \mathbb{R} \times \mathbb{R}, \mathbb{R}]$ is non-decreasing in u and is non-increasing in v and

$$F(t, u_1(t), v(t)) - F(t, u_2(t), v(t)) \geq -L(u_1 - u_2), \tag{10}$$

$$F(t, u(t), v_1(t)) - F(t, u(t), v_2(t)) \leq M(v_1 - v_2), \tag{11}$$

whenever $u_1 \geq u_2$, $v_1 \geq v_2$ and $L > 0$, $M > 0$.

Then there exist monotone sequences $\{\vartheta_n(t)\}$, $\{\omega_n(t)\}$ converging uniformly and monotonically to the functions ϱ and r on J. It is implied that ϱ and r coupled MMSs of (1), respectively.

Proof. Let $\psi, \xi \in C^1[J, \mathbb{R}]$ such that $\vartheta_0 \leq \psi \leq \omega_0$ and $\vartheta_0 \leq \xi \leq \omega_0$. We set the linear fractional integro-differential initial value problems (IVPs):

$$\begin{aligned} {}^{C}D^{q_1}u(t) &= F(t, \psi(t), I^{q_2}\omega_0(t)) - L(u - \psi), &(12)\\ u(0) &= \psi(0) - \frac{1}{M}h(\psi(0), \psi(T)), &(13)\\ {}^{C}D^{q_1}v(t) &= F(t, \omega_0(t), I^{q_2}\xi(t)) + MI^{q_2}(v - \xi), &(14)\\ v(0) &= \xi(0) - \frac{1}{M}h(\xi(0), \xi(T)). &(15) \end{aligned}$$

Define the mapping A and B by $A\psi = u$ and $B\xi = v$ and use it to construct the sequences $\{\vartheta_n\}$ and $\{\omega_n\}$. We aim to prove that

(i) $\vartheta_0 \leq A\vartheta_0$ and $\omega_0 \geq B\omega_0$;

(ii) the operators A and B are monotone on the sector $[\vartheta_0, \omega_0]$.

To prove (i), take $A\vartheta_0 = \vartheta_1$, where ϑ_1 is the unique solution of (12)-(13) with $\psi = \vartheta_0$. By letting $p(t) = \vartheta_1(t) - \vartheta_0(t)$, we see that

$$\begin{aligned} {}^C D^{q_1} p(t) &= {}^C D^{q_1} \vartheta_1(t) - {}^C D^{q_1} \vartheta_0(t) \\ &\geq F(t, \vartheta_0(t), I^{q_2}\omega_0(t)) - L(\vartheta_1 - \vartheta_0) - F(t, \vartheta_0(t), I^{q_2}\omega_0(t)) \\ &= -Lp(t), \end{aligned}$$

and

$$\begin{aligned} p(0) &= \vartheta_1(0) - \vartheta_0(0) \\ &= \vartheta_0(0) - \frac{1}{M} h(\vartheta_0(0), \vartheta_0(T)) - \vartheta_0(0) \\ &\geq 0. \end{aligned}$$

According to Corollary 1, it appears that $p(t) \geq 0$, which implies $\vartheta_0(t) \leq \vartheta_1(t)$ on J. Similarly, let $B\omega_0 = \omega_1$, where ω_1 is the unique solution of (14)-(15) with $\xi = \omega_0$. Setting $p(t) = \omega_1(t) - \omega_0(t)$, we get

$$\begin{aligned} {}^C D^{q_1} p(t) &= {}^C D^{q_1} \omega_1(t) - {}^C D^{q_1} \omega_0(t) \\ &\leq F(t, \omega_0(t), I^{q_2}\omega_0(t)) + MI^{q_2}(\omega_1 - \omega_0) - F(t, \omega_0(t), I^{q_2}\vartheta_0(t)) \\ &= F(t, \omega_0(t), I^{q_2}\omega_0(t)) - F(t, \omega_0(t), I^{q_2}\vartheta_0(t)) + MI^{q_2}(\omega_1 - \omega_0) \\ &\leq F(t, \omega_0(t), I^{q_2}\vartheta_0(t)) - F(t, \omega_0(t), I^{q_2}\vartheta_0(t)) + MI^{q_2}(\omega_1 - \omega_0) \\ &= MI^{q_2} p(t), \end{aligned}$$

and

$$\begin{aligned} p(0) &= \omega_1(0) - \omega_0(0) \\ &= \omega_0(0) - \frac{1}{M} h(\omega_0(0), \omega_0(T)) - \omega_0(0) \\ &= -\frac{1}{M} h(\omega_0(0), \omega_0(T)) \\ &\leq 0. \end{aligned}$$

By utilizing Corollary 1 yields $\omega_0(t) \geq \omega_1(t)$ on J.

To prove (ii), let $\psi_1, \psi_2 \in [\vartheta_0, \omega_0]$, such that $\psi_1 \leq \psi_2$ and put $A\psi_1 = u_1$ and $A\psi_2 = u_2$. It is enough to define $p(t) = u_2(t) - u_1(t)$ in a manner that

$$\begin{aligned} {}^C D^{q_1} p(t) &= {}^C D^{q_1} u_2(t) - {}^C D^{q_1} u_1(t) \\ &= F(t, \psi_2(t), I^{q_2}\omega_0(t)) - L(u_2 - \psi_2) - F(t, \psi_1(t), I^{q_2}\omega_0(t)) + L(u_1 - \psi_1) \\ &= F(t, \psi_2(t), I^{q_2}\omega_0(t)) - F(t, \psi_1(t), I^{q_2}\omega_0(t)) + L(u_1 - \psi_1 - u_2 + \psi_2). \end{aligned}$$

Using the inequality (10) and recalling the fact that $\psi_1 \leq \psi_2$, we may deduce that

$$F(t, \psi_2(t), I^{q_2}\omega_0(t)) - F(t, \psi_1(t), I^{q_2}\omega_0(t)) \geq -L(\psi_2 - \psi_1).$$

Substituting that expression into previous equation yields

$$\begin{aligned} {}^C D^{q_1} p(t) &= F(t, \psi_2(t), I^{q_2}\omega_0(t)) - F(t, \psi_1(t), I^{q_2}\omega_0(t)) + L(u_1 - \psi_1 - u_2 + \psi_2) \\ &\geq -L(\psi_2 - \psi_1) + L(u_1 - \psi_1 - u_2 + \psi_2) \\ &= -L(\psi_2 - \psi_1 - u_1 + \psi_1 + u_2 - \psi_2) \\ &= -L(u_2 - u_1) \\ &= -Lp(t), \end{aligned}$$

and

$$\begin{aligned} p(0) &= u_2(0) - u_1(0) \\ &= \psi_2(0) - \frac{1}{M} h(\psi_2(0), \psi_2(T)) - \psi_1(0) + \frac{1}{M} h(\psi_1(0), \psi_1(T)) \\ &= \psi_2(0) - \psi_1(0) + \frac{1}{M}(h(\psi_1(0), \psi_1(T)) - h(\psi_2(0), \psi_2(T))) \\ &\geq \psi_2(0) - \psi_1(0) + \frac{1}{M}(h(\psi_1(0), \psi_2(T)) - h(\psi_2(0), \psi_2(T))) \\ &\geq \psi_2(0) - \psi_1(0) + \frac{1}{M}(-M)(\psi_2(0) - \psi_1(0)) \\ &= 0. \end{aligned}$$

It follows that $A\psi_2 \leq A\psi_1$, whenever $\psi_1 \leq \psi_2$ on J.

Similarly, assume that $\xi_1, \xi_2 \in [\vartheta_0, \omega_0]$ such that $\xi_1 \leq \xi_2$. Let $B\xi_1 = v_1$, $B\xi_2 = v_2$ and set $p(t) = v_2(t) - v_1(t)$, so that

$$\begin{aligned} {}^C D^{q_1} p(t) &= {}^C D^{q_1} v_2(t) - {}^C D^{q_1} v_1(t) \\ &= F(t, \omega_0(t), I^{q_2}\xi_2(t)) + M I^{q_2}(v_2 - \xi_2) - F(t, \omega_0(t), I^{q_2}\xi_1(t)) - M I^{q_2}(v_1 - \xi_1). \end{aligned}$$

Furthermore, utilizing the inequality (11), we have

$$F(t, \omega_0(t), I^{q_2}\xi_2(t)) - F(t, \omega_0(t), I^{q_2}\xi_1(t)) \leq M I^{q_2}(\xi_2 - \xi_1).$$

When the last phrase is included into previous relation, it gives

$$\begin{aligned} {}^C D^{q_1} p(t) &\leq M I^{q_2}(\xi_2 - \xi_1) + M I^{q_2}(v_2 - \xi_2) - M I^{q_2}(v_1 - \xi_1) \\ &= M I^{q_2} p(t). \end{aligned}$$

We can figure out that $p(0) \leq 0$ implies $p(t) \leq 0$, based on the implications outlined in Corollary 1.

At this point, one may specify the sequences $\vartheta_n = A\vartheta_{n-1}$ and $\omega_n = B\omega_{n-1}$ for $n = 1, 2, \ldots$. In this case, the monotone sequences $\{\vartheta_n\}$ and $\{\omega_n\}$ can be represented by the following iterative schemes.

$$\begin{aligned} {}^C D^{q_1} \vartheta_{n+1}(t) &= F(t, \vartheta_n(t), I^{q_2}\omega_n) - L(\vartheta_{n+1} - \vartheta_n), && (16) \\ \vartheta_{n+1}(0) &= \vartheta_n(0) - \frac{1}{M} h(\vartheta_n(0), \vartheta_n(T)). && (17) \\ {}^C D^{q_1} \omega_{n+1}(t) &= F(t, \omega_n(t), I^{q_2}\omega_n) + M I^{q_2}(\omega_{n+1} - \omega_n), && (18) \\ \omega_{n+1}(0) &= \omega_n(0) - \frac{1}{M} h(\omega_n(0), \omega_n(T)). && (19) \end{aligned}$$

Suppose that u is an arbitrary solution to the problem (1) satisfying $\vartheta_0(t) \leq u(t) \leq \omega_0(t)$. Then we must demonstrate that $\vartheta_n(t) \leq u(t) \leq \omega_n(t)$ for $n \in \mathbb{N}$. The proof is clear for $n = 0$. Assume that for some k, $\vartheta_k(t) \leq u(t) \leq \omega_k(t)$ is true on J. Thus, we prove the validity of the subsequent relationship

$$\vartheta_{k+1}(t) \leq u(t) \leq \omega_{k+1}(t)$$

on J. In order to verify this, we implement $p(t) = u(t) - \vartheta_{k+1}(t)$ and, have

$$\begin{aligned} {}^C D^{q_1} p(t) &= {}^C D^{q_1} u(t) - {}^C D^{q_1} \vartheta_{k+1}(t) \\ &= F(t, u(t), I^{q_2} u(t)) - [F(t, \vartheta_k(t), I^{q_2}\omega_k) - L(\vartheta_{k+1} - \vartheta_k)] \\ &\geq -L(u - \vartheta_k) + L(\vartheta_{k+1} - \vartheta_k) \\ &= -Lp(t). \end{aligned}$$

Reviewing the fundamental characteristics of the function g, we get

$$\begin{aligned} p(0) &= u(0) - \vartheta_{k+1}(0) \\ &= u(0) - \left[\vartheta_k(0) - \frac{1}{M} h(\vartheta_k(0), \vartheta_k(T))\right] - \frac{1}{M} h(u(0), u(T)) \\ &= u(0) - \vartheta_k(0) + \frac{1}{M}\left(h(\vartheta_k(0), \vartheta_k(T)) - h(u(0), u(T))\right) \\ &\geq u(0) - \vartheta_k(0) + \frac{1}{M}\left(h(\vartheta_k(0), u(T)) - h(u(0), u(T))\right) \\ &\geq u(0) - \vartheta_k(0) + \frac{1}{M}(-M)(u(0) - \vartheta_k(0)) \\ &= 0. \end{aligned}$$

Following Corollary 1, we see that $\vartheta_{k+1}(t) \leq u(t)$ on J. By using a similar approach, we can show that $u(t) \leq \omega_{k+1}(t)$ on J. This result in for all n,

$$\vartheta_0 \leq \vartheta_1 \leq ... \leq \vartheta_n \leq u \leq \omega_n \leq ... \leq \omega_1 \leq \omega_0.$$

By employing standard techniques as in the preceding result, we reveal that the sequences $\{\vartheta_n\}$ and $\{\omega_n\}$ converge uniformly and monotonically to the functions ϱ and r, respectively. To prove that ϱ and r are coupled solutions of the main problem, one can establish the corresponding Volterra integral equations to the problems (16–19) and then taking limits as $n \to \infty$, that is,

$${}^C D^{q_1} \varrho(t) = F(t, \varrho(t), I^{q_2} r(t)), \quad h(\varrho(0), \varrho(T)) = 0,$$

and

$${}^C D^{q_1} r(t) = F(t, r(t), I^{q_2} \varrho(t)), \quad h(r(0), r(T)) = 0.$$

Finally, we need to demonstrate that (ϱ, r) are coupled MMSs of (1), respectively. Let u be any solution of (1) such that $\vartheta_0(t) \leq u(t) \leq \omega_0(t)$ on J. After proving the inequality $\vartheta_n(t) \leq u(t) \leq \omega_n(t)$ with the same approach as before and considering the limit as $n \longrightarrow \infty$, we receive $\varrho(t) \leq u(t) \leq r(t)$, which concludes the proof.

Theorem 4. In addition to conditions of Theorem 3, suppose also

$$F(t, u_1(t), v(t)) - F(t, u_2(t), v(t)) \leq L(u_1 - u_2),$$

$$F(t, u(t), v_1(t)) - F(t, u(t), v_2(t)) \geq -M(v_1 - v_2),$$

whenever $u_1 \geq u_2$, $v_1 \geq v_2$ and $L > 0$, $M > 0$. Then we have unique solution of (1) such that $\varrho = u = r$.

If we utilize coupled LUSs of type 2 of (1), we get also monotone sequences that converge uniformly and monotonically to the extremal solutions of (1) that we state as the next result.

In order to prevent repetition, we shall omit the details of the proofs for the subsequent results.

Theorem 5. Suppose that

(C_1) $\vartheta_0, \omega_0 \in C^1[J, \mathbb{R}]$ are coupled LUSs of type 2 of problem (1) with $\vartheta_0(t) \leq \omega_0(t)$ on J;

(C_2) (A_2) holds;

(C_3) $F(t,u,v) \in C[J \times \mathbb{R} \times \mathbb{R}, \mathbb{R}]$ is non-increasing in u and non-decreasing in v, moreover

$$F(t, u_1(t), v(t)) - F(t, u_2(t), v(t)) \le L(u_1 - u_2),$$

$$F(t, u(t), v_1(t)) - F(t, u(t), v_2(t)) \ge -M(v_1 - v_2),$$

where $u_1 \ge u_2$, $v_1 \ge v_2$ and $L > 0$, $M > 0$.

Then there exist two sequences $\{\vartheta_n(t)\}$, $\{\omega_n(t)\}$ such that $\lim_{n\longrightarrow\infty} \omega_n = r$, $\lim_{n\longrightarrow\infty} \vartheta_n = \varrho$ uniformly and monotonically on J and that (ϱ, r) are coupled MMSs of (1).

Remark 1. Observe that coupled LUSs of type 1 together with increasing and decreasing properties of F in Theorem 3 result in the natural ULSs and coupled ULSs of type 3 separately, hence both yield the coupled LUSs of type 2 at the end. The analogous approach for coupled LUSs of type 2 is true and this can be stated in the opposite manner.

In the following theorem, we take coupled LUSs of type 3 and find the similar conclusion as in Theorem 1.

Theorem 6. Let the following conditions hold:

(D_1) $\vartheta_0, \omega_0 \in C^1[J, \mathbb{R}]$ are coupled LUSs of type 3 of (1) with $\vartheta_0(t) \le \omega_0(t)$ on J;

(D_2) (A_2) holds;

(D_3) the function $F(t,u,v) \in C[J \times \mathbb{R} \times \mathbb{R}, \mathbb{R}]$ is non-increasing in both u, v for each $t \in J$ and

$$F(t, u_1(t), v_1(t)) - F(t, u_2(t), v_2(t)) \le L(u_1 - u_2) + M(v_1 - v_2),$$

where $\vartheta_0 \le u_2 \le u_1 \le \omega_0$ and $\vartheta_0 \le v_2 \le v_1 \le \omega_0$ and $L > 0$, $M > 0$.

Then we obtain the sequences $\{\vartheta_n(t)\}$, $\{\omega_n(t)\}$ such that $\vartheta_n \longrightarrow \varrho$ and $\omega_n \longrightarrow r$ as $n \longrightarrow \infty$ uniformly and monotonically on J and ϱ and r are the MMSs of (1), respectively.

Remark 2. Note that the assumption (D_1) with the non-increasing property of $F(t,u,v)$ in both u and v for each $t \in J$ implies the natural LUSs of (1) for the functions ϑ_0, ω_0.

3 Conclusion

We have considered the boundary value problem of a Caputo fractional integro-differential equation to analyze the existence and uniqueness of the problem. We employ the monotone iterative technique generating monotone sequences that converge uniformly to the extremal solutions of the main problem. It would be valuable to explore extensions and refinements of the monotone iterative technique for solving more general classes of FIDEs, as well as to investigate its applicability to practical problems arising in real-world applications. Additionally, the development of computational algorithms based on the theoretical results could lead to the implementation of efficient numerical solvers for FIDEs with boundary conditions.

Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

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Operator-pencil treatment of multi-interval Sturm-Liouville equation with boundary-transmission conditions

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This paper is devoted to a new type of boundary-value problems for Sturm-Liouville equations defined on three disjoint intervals $(-\pi, -\pi + d), (-\pi + d, \pi - d)$ and $(\pi - d, \pi)$ together with eigenparameter dependent boundary conditions and with additional transmission conditions specified at the common end points $-\pi + d$ and $\pi - d$, where $0 < d < \pi$. The considered problem cannot be treated by known techniques within the usual framework of classical Sturm-Liouville theory. To establish some important spectral characteristics we introduced the polynomial-operator formulation of the problem. Moreover, we develop a new modification of the Rayleigh method to obtain lower bound of eigenvalues.

Keywords: boundary-value-transmission problems, eigenvalues, generalized eigenfunctions, lower bound estimation, Rayleigh's method, transmission conditions.

2020 Mathematics Subject Classification: 34L10, 34B24.

Introduction

This work is motivated by the problem of understanding the nature of the spectral characteristics of the class of boundary-value problems (BVPs) for Sturm-Liouville equations (SLEs) defined of finite number of nonintersecting intervals together with additional interaction conditions specified at the common endpoints of these intervals. Moreover, the spectral parameter appears linearly in both differential equation and boundary conditions (BCs). Such type of BVPs (the so-called many-interval boundary value transmission problems (MIBVTPs)) are encountered in solving various transfer problems of mathematical physics. For example, some MIBVTPs arise in heat transfer problems, mass transfer problems, diffraction problems, seismic behavior of the Earth's, waves in the atmosphere, etc. (see, [1–7]). Its solutions are determined by different special functions, such as Bessel functions, Chebyshev polynomials, Legendre polynomials, Hypergeometric functions etc. Important studies have been carried out recently regarding MIBVTPs [8–24].

The aim of this work is to investigate the following MIBVTP, consisting of three-interval SLE

$$- g''(x) + q(x)g(x) = \lambda r(x)g(x) \tag{1}$$

defined on three-interval $(-\pi, -\pi + d) \cup (-\pi + d, \pi - d) \cup (\pi - d, \pi)$, together with the λ-dependent BCs given by

$$\cos\varphi \; g(-\pi + d) \; + \; \sin\varphi \; g'(-\pi + d) \; = \; 0 \; , \; 0 \; < \; \varphi \; < \; \pi, \tag{2}$$

$$\alpha g(\pi) \; - \; \alpha' g'(\pi) \; + \; \lambda \left(\beta g(\pi) \; - \; \beta' g'(\pi)\right) \; = \; 0 \tag{3}$$

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and with the additional transmission conditions (TCs) at the points of interaction $-\pi + d$ and $\pi - d$ given by

$$T_{-\pi+d}(g) \ = \ 0 \ , \ T_{-\pi+d}(g') \ = \ \theta_1 \ g(-\pi + d), \tag{4}$$

$$T_{\pi-d}(g) \ = \ 0 \ , \ T_{\pi-d}(g') \ = \ \theta_2 \ g(\pi - d), \tag{5}$$

where $0 < d < \pi$, $T_x(g)$ is the linear form defined by $T_x(g) \ = \ \lim_{\delta\to 0} g(x + |\delta|) - \lim_{\delta\to 0} g(x - |\delta|)$, $\alpha, \alpha^{'}, \beta, \beta^{'}, \theta_1, \theta_2$ are real numbers, $q(x)$ is a real-valued function, $q \in L_2(-\pi, \pi)$. Everywhere we shall assume that

$$\theta_3 := \begin{vmatrix} \alpha^{'} & \alpha \\ \beta^{'} & \beta \end{vmatrix} \ > \ 0.$$

To study some important spectral characteristic of the considered MIBVTP (1)-(5) we introduced a corresponding operator-polynomial in appropriate Hilbert space. Note that, MIBVTPs have been an important research in recent years [25–31].

1 Operator-pencil treatment of the problem

To study some spectral characteristics of the MIBVTP (1)–(5) we shall use the operator-pencil theory and Rayleigh theory. Let us formulate some definitions and facts, which is needed for further consideration.

Let $k \geq 0$ be an integer. The Sobolev space $W_2^k(a, b)$ is defined to be the linear space consisting of all functions $g \in L_2(a, b)$ having generalized derivatives $g', g'', ..., g^{(k)} \in L_2(a, b)$ equipped with the inner product

$$\langle g, h\rangle_{W_2^k(a,b)} := \sum_{j=0}^{k} \langle g^{(j)}, \overline{h}^{(j)}\rangle_{L_2(a,b)}$$

and corresponding norm $\|g\|^2_{W_2^k(a,b)} \ = \ \langle g, g\rangle_{W_2^k(a,b)}$. Here, $L_2(a, b)$ denotes the space of all complex-valued functions g, such that $\int_a^b |g^2(x)|dx < \infty$, equipped with the inner product

$$\langle g, h\rangle_{L_2(a,b)} := \int_a^b g(x) \ \overline{h}(x)dx.$$

Denote $\Omega_1 = (-\pi, -\pi+d)$, $\Omega_2 = (-\pi+d, \pi-d)$, $\Omega_3 = (\pi-d, \pi)$ and $\Omega = \Omega_1 \cup \Omega_2 \cup \Omega_3$. For investigation of the BVTP (1)–(5) we shall use the discret sum space $\oplus L_2 := L_2(\Omega_1) \oplus L_2(\Omega_2) \oplus L_2(\Omega_3)$ with the inner-product

$$\langle g, h\rangle_0 := \sum_{i=1}^{3} \int_{\Omega_i} g(x)\overline{h}(x)dx$$

and direct sum space

$$\oplus W_2^1 = \left\{ \begin{array}{c} g \in \oplus L_2 \left| g \in W_2^1(\Omega_i)(i = 1, 2, 3), g(-\pi + d + 0) = g(-\pi + d - 0), \right. \\ g(\pi - d + 0) = g(\pi - d - 0) \end{array} \right\}$$

with the inner-product

$$\langle g, h\rangle_1 := \sum_{i=1}^{3} \int_{\Omega_i} \Big(g'(x)\overline{h}'(x) \ + \ g(x)\overline{h}(x)\Big) \, dx.$$

We can show that the inner-product spaces $\oplus L_2$ and $\oplus W_2^1$ are Hilbert spaces.

In the Hilbert space $\oplus W_2^1$ we define a new inner-product by

$$\langle g,h\rangle_2 \ := \ \sum_{i=1}^{3}\int_{\Omega_i}\left\{g'(x)\overline{h}'(x)+q(x)g(x)\overline{h}(x)\right\}dx$$

with the corresponding norm $\|g\|_2^2 \ = \ \langle g,g\rangle_2$. Obviously, there are positive constants m and M, such that

$$m\ \|g\|_1<\|g\|_2<M\ \|g\|_1$$

for all $g\in\oplus W_2^1$.

Using the well-known embedding properties for Sobolev spaces (see [20]) we can show that

$$|g(x_j)|^2\leq \ell\ ||g'||_0^2+\frac{2}{\ell}\ ||g||_0^2, \tag{6}$$

$$|g(\xi)| \ \ \leq \ \ C(\xi)\ ||g||_2 \tag{7}$$

for any $g\in\oplus W_2^1$ where $j=1,2,3,4,\ x_1=-\pi,\ x_2=-\pi+d\mp0,\ x_3=\pi-d\mp0,\ x_4=\pi,\ \ell$ is a positive number (small enough), $\xi\in\Omega$, the constant $C(\xi)$ is independent of the function g and dependent only of ξ. Let us introduce to the consideration the Hilbert space $\mathbb{H}$, consisting of all vector-functions $\big(\ \chi(x),\chi_1\ \big)\in\oplus W_2^1\oplus\mathbb{C}\ :=\ \mathbb{H}$ equipped with the inner product

$$\langle\Gamma,\Psi\rangle_{\mathbb{H}} \ := \ \langle\chi,\varphi\rangle_1+\chi_1\ \overline{\varphi_1},$$

where $\Gamma=(\chi,\ \chi_1)$ and $\Psi=(\varphi,\ \varphi_1)\in\mathbb{H}$.

The concept of weak eigenfunction is based on the weak solutions of the problem (1)–(5), which we shall define by the following procedure. By multiplying the differential equation (1) by the conjugate of an arbitrary $h\in\oplus W_2^1$ satisfying the conditions $h(\pi-d+0)=h(\pi-d-0)$ and $h(-\pi+d+0)=h(-\pi+d-0)$ and then integrating by parts over the intervals Ω_i $(i=1,2,3)$ we have

$$\sum_{i=1}^{3}\int_{\Omega_i}\left\{g'(x)\overline{h}'(x)+q(x)g(x)\overline{h}(x)\right\}dx-\frac{\beta}{\beta'}g(\pi)\overline{h}(\pi)-\frac{\cos\varphi}{\sin\varphi}g(-\pi)\overline{h}(-\pi)+$$
$$+\ \ \theta_1 g(-\pi+d)\overline{h}(-\pi+d)\ +\ \theta_2 g(\pi-d)\overline{h}(\pi-d)+\frac{\kappa}{\beta'}\overline{h}(\pi)\ =\ \lambda\sum_{i=1}^{3}\int_{\Omega_i}g\overline{h}dx, \tag{8}$$

and

$$\frac{g(\pi)}{\beta'}-\frac{\alpha'}{\beta'}\frac{\kappa}{\theta_3}=\lambda\frac{\kappa}{\theta_3}, \tag{9}$$

where $\kappa\ :=\ \beta g(\pi)-\beta' g'(\pi)$. Thus the BVTP (1)–(5) is transformed into the system of equalities (8) and (9), all terms of which are defined for the $g,h\in\oplus W_2^1$.

Definition 1. The element $\Gamma=(g(x),\ \kappa)\in\oplus W_2^1$ is said to be a weak solution of the BVTP (1)–(5) if the equations (8)-(9) are satisfied for any $h\in\oplus W_2^1$.

Let us introduce to the consideration the following bilinear forms:

$$\tau_0(g,h) \ := \ -\frac{\beta}{\beta'}g(\pi)\overline{h}(\pi)-\frac{\cos\varphi}{\sin\varphi}g(-\pi)\overline{h}(-\pi)+\theta_1 g(-\pi+d)\overline{h}(-\pi+d)+$$
$$+\ \ \theta_2 g(\pi-d)\overline{h}(\pi-d), \tag{10}$$

$$\tau_1(g,h) := \sum_{i=1}^{3} \int_{\Omega_i} r(x)g(x)\, \overline{h}(x)dx, \tag{11}$$

and

$$\tau_2(\kappa,h) := \frac{\kappa}{\beta'}\overline{h}(\pi). \tag{12}$$

The reduction of identities (8)-(9) to an operator equation is based on the following result.

Theorem 1. There are bounded linear operators $S_0, S_1 : \oplus W_2^1 \to \oplus W_2^1$ and $S_2 : \mathbb{C} \to \oplus W_2^1$ such that

$$\begin{aligned} \tau_n(g,h) &= \langle S_n g, h\rangle_2 \text{ for } n = 0,1 \text{ and} \\ \tau_n(\kappa,h) &= \langle S_n \kappa, h\rangle_2 \text{ for } n = 2. \end{aligned} \tag{13}$$

Proof. $\tau_n(g,h),\ n = 0,1,$ are linear functionals in $h \in \oplus W_2^1$ for any given $g \in \oplus W_2^1$ and that $\tau_2(\kappa,h)$ is a linear functional in $h \in \oplus W_2^1$ for any given $\kappa \in \mathbb{C}$.

Let $g \in \oplus W_2^1$ be any function. From (10)–(12), it follows immediately that

$$|\tau_0(g,h)| \le C_1 \{|g(\pi)||h(\pi)| + |g(-\pi)||h(-\pi)| + |g(-\pi+d)||h(-\pi+d)| +$$

$$+|g(\pi-d)||h(\pi-d)|\},$$

$$|\tau_1(g,h)| \le C_2\|g\|\|h\|,$$

$$|\tau_2(\kappa,h)| \le C_3|\kappa|\ |h(\pi)|.$$

Here and below, the symbols C_k, for $k = 1,2,\dots$ denote different positive constants whose exact values are not important for the proof.

The interpolation inequalities (6)-(7) imply

$$\|g\| \le C_4\|g\|_2 \quad \text{and} \quad |g(\xi)| \le C_5\|g\|_2 \ \text{ for any } \xi \in \Omega.$$

Hence, the functionals τ_n $(n = 0,1,2)$ allow the following estimates:

$$\begin{aligned} |\tau_0(g,h)| &\le C_6\ \|g\|_2\ \|h\|_2, \\ |\tau_1(g,h)| &\le C_7\ \|g\|_2\ \|h\|_2, \\ |\tau_2(\kappa,h)| &\le C_8\ |\kappa|\ \|h\|_2. \end{aligned}$$

Therefore, τ_n $(n = 0,1,2)$ are linear continuous functionals in $h \in \oplus W_2^1$ for any given $g \in \oplus W_2^1$, $n = 0,1,$ and $\kappa \in \mathbb{C}$, $n = 2$, respectively. Then, the existence of linear bounded operators S_0, S_1 and S_2 follows immediately from the well-known Riesz representation theorem (see, for example, [25]).

Theorem 2. The operators $S_0, S_1 : \oplus W_2^1 \to \oplus W_2^1$ are self-adjoint and the operator S_1 is positive.

Proof. Let $g,h \in \oplus W_2^1$ be arbitrary functions. By (10) and (13), we have that

$$\langle g, S_0 h\rangle_{\oplus W_2^1} = \overline{\langle S_0 h, g\rangle}_{\oplus W_2^1} = \overline{\tau_0(h,g)} = \tau_0(g,h) = \langle S_0 g, h\rangle_{\oplus W_2^1}.$$

Hence, the operator S_0 is self-adjoint in $\oplus W_2^1$. The proof of the self-adjointness of S_1 is totaly similar. The positivity of S_1 follows immediately from the fact that the function $r(x)$ is positive definitely.

Theorem 3. The operators $S_i : \oplus W_2^1 \to \oplus W_2^1$ $(i = 0,1)$, $S_2 : \mathbb{C} \to \oplus W_2^1$ and $S_2^* : \oplus W_2^1 \to \mathbb{C}$ are compact, where S_2^* is the adjoint of S_2.

Proof. To prove the compactness of the operator S_0 it is sufficient to show that any weakly convergent sequence $\{g_k\}(k = 1, 2, ...)$ in $\oplus W_2^1$ is transformed by S_0 into a strongly convergent sequence $\{S_0 g_k\}$ in the same space. The boundedness of S_0 implies the weakly convergence of $\{S_0 g_k\}$ to $S_0 g$ in $\oplus W_2^1$, where $g(x)$ is the weak limit of $\{g_k\}$. Since the embedding operator $J : \oplus W_2^1 \hookrightarrow \oplus L_2$ is compact [20], the sequences (g_k) and $(S_0 g_k)$ converge strongly to g and $S_0 g$ in $\oplus L_2$ respectively. In addition, since for each bounded interval $I \subset \mathbb{R}$ the embedding operator $J : W_2^1(I) \hookrightarrow C(I)$ is compact and the sequences $\{g_k\}$ and $\{S_0 g_k\}$ are bounded in $\oplus W_2^1$ it follows that these sequences converge in $C(\Omega_1) \oplus C(\Omega_2) \oplus C(\Omega_3)$.

Further, the compactness of the embedding operator $J : \oplus W_2^1 \hookrightarrow C(\Omega_1) \oplus C(\Omega_2) \oplus C(\Omega_3)$ (see, for example, [20]) implies that the sequences $\{g_k(d_i)\}$ and $\{(S_0 g_k)(d_i)\}$ converge in $\mathbb{C}$ to $g(d_i)$ and $(S_0 g)(d_i)$ $(i = 1, 2, 3, 4)$ with $d_1 = -\pi$ or $d_2 = -\pi + d \mp 0$ or $d_3 = \pi - d \mp 0$ or $d_4 = \pi$, respectively. The representations (10)–(12) and inequalities (6) imply

$$\begin{aligned}
\| \quad & S_0(g_k - g_m)\|_2^2 = \langle S_0(g_k - g_m), S_0(g_k - g_m)\rangle_2 = \tau_0\big(g_k - g_m, S_0(g_k - g_m)\big) \\
\leq \quad & C_1 \left\{|(g_k(\pi) - g_m(\pi))| + |(g_k(-\pi) - g_m(-\pi))|\right\} \\
+ \quad & C_1 \left\{|(g_k(-\pi + d + 0) - g_m(-\pi + d - 0))| + |(g_k(\pi - d + 0) - g_m(\pi - d - 0))|\right\}.
\end{aligned}$$

Therefore, $\|S_0(g_k - g_m)\|_2 \to 0$ as $k,\ m \to \infty$. Hence, the sequence $\{S_0 g_k\}$ is the Cauchy sequence in the space $\oplus W_2^1$ and therefore converges strongly in $\oplus W_2^1$. Thus the compactness of the operator S_0 is proven. The proof of the compactness of the operator S_1 is totally similar.

It is easy to show that the adjoint operator S_2^* is defined by the equality $S_2^* g = \frac{g(\pi)}{\beta'}$, from which it follows that this operator is compact. Then by virtue of well-known theorem of Functional Analysis the operator S_2 is also compact. The proof is complete.

2 Positiveness of the operator-pencil

It is evident that the BVTP (1)–(5) can be written as the operator-pencil equation in $\mathbb{H}$, given by

$$\mathcal{A}(\lambda)\ \Gamma = 0\ , \qquad \mathcal{A}(\lambda) = \Delta - \lambda\ \Lambda, \tag{14}$$

where the operators Δ and Λ are defined by

$$\Delta(g, \kappa) = \left(g + S_0 g + S_2 \kappa\ , \quad S_2^* g - \frac{\alpha'}{\beta'} \frac{\kappa}{\theta_3} \right), \tag{15}$$

$$\Lambda(g, \kappa) = \left(S_1 g\ ,\ \frac{\kappa}{\theta_3} \right), \tag{16}$$

respectively.

Lemma 1. For all real λ_0, the operator $\mathcal{A}(-\lambda_0) = \Delta + \lambda_0\ \Lambda$ is self-adjoint in the Hilbert space $\mathbb{H}$.

Proof. Using Theorem 2, it is easy to show that the linear operators Δ and Λ are self-adjoint. Therefore, the operator-pencil $\mathcal{A}(-\lambda_0) = \Delta + \lambda_0\ \Lambda$ is also self-adjoint in the Hilbert space $\mathbb{H}$.

Lemma 2. The operator-polynomial $\mathcal{A}(-\lambda_0)$ is positive definite for sufficiently large positive values of λ_0.

Proof. Taking in view the equality

$$\mathcal{A}(-\lambda_0)\Gamma = \Big(\ g(x) + S_0 g(x) + S_2\kappa + \lambda_0 S_1 g(x)\ ,\ S_2^* g(x) - \tfrac{\alpha'}{\beta'} \tfrac{\kappa}{\theta_3}\ +\ \lambda_0 \tfrac{\kappa}{\theta_3}\ \Big)$$

for $\Gamma = (g(x), \kappa)$, we get

$$\langle \mathcal{A}(-\lambda_0)\Gamma, \Gamma\rangle_{\mathbb{H}} = \langle g(x), g(x)\rangle_2 + \langle S_0 g(x), g(x)\rangle_2 + \langle S_2\kappa, g(x)\rangle_2 + (S_2^* g(x))\overline{\kappa} - \\ - \frac{\alpha'}{\beta'\ \theta_3}|\kappa|^2 + \lambda_0 \left\{ \langle S_1 g(x), g(x)\rangle_2 + \frac{1}{\theta_3}|\kappa|^2 \right\}. \tag{17}$$

Let us define the following functionals

$$P(g) := \langle g', g'\rangle_0, \quad Q(g) := \langle qg, g\rangle_0, \quad R(g) := \langle rg, g\rangle_0. \tag{18}$$

From the well-known embedding theorems for Sobolev spaces it follows easily that the inequalities

$$|g(x_j)|^2 \le C_{j1}\epsilon_j P(g) + \frac{C_{j2}}{\epsilon_j} Q(g) \tag{19}$$

hold for sufficiently small positive ϵ_j, where $g \in \oplus W_2^1$ $(j = 1,2,3,4)$, C_{jk} $(k = 1,2)$ are positive constants; $x_1 = -\pi,\ x_2 = -\pi + d \mp 0,\ x_3 = \pi - d \mp 0,\ x_4 = \pi$.

Using (18) and (19) and applying the well-known Young inequality, we have the following estimates

$$\begin{aligned} \langle S_0 g(x), g(x)\rangle_2 &= -\frac{\beta}{\beta'}\ |g(\pi)|^2 - \frac{\cos\varphi}{\sin\varphi}\ |g(-\pi)|^2 + \theta_1|g(-\pi+d)|^2 + \theta_2|g(\pi-d)|^2 \\ &\ge \left(-\frac{\cos\varphi}{\sin\varphi}C_{11}\epsilon_1 + \theta_1 C_{21}\epsilon_2 + \theta_2 C_{31}\epsilon_3 - \frac{\beta}{\beta'}C_{41}\epsilon_4\right) P(g) \\ &+ \left(-\frac{\cos\varphi}{\sin\varphi}\frac{C_{12}}{\epsilon_1} + \theta_1\frac{C_{22}}{\epsilon_2} + \theta_2\frac{C_{32}}{\epsilon_3} - \frac{\beta}{\beta'}\frac{C_{42}}{\epsilon_4}\right) Q(g). \end{aligned} \tag{20}$$

$$\begin{aligned} \langle S_2\kappa, g(x)\rangle_2 &+ (S_2^* g(x))\overline{\kappa} = \frac{2}{\beta'}\ Re(\kappa\ \overline{g}(\pi)) \\ &\ge -\frac{1}{|\beta'|\ \gamma}\ |g(\pi)|^2\ - \frac{\gamma}{|\beta'|}|\kappa|^2 \\ &\ge -\frac{1}{|\beta'|\ \gamma}\left\{C_{41}\epsilon_4 P(g) + \frac{C_{42}}{\epsilon_4}Q(g)\right\} \\ &- \frac{\gamma}{|\beta'|}|\kappa|^2 \end{aligned} \tag{21}$$

for arbitrary $\gamma\ > 0$. It is easy to see that,

$$\langle S_1 g, g\rangle_2 = R(g) \ge M_1 Q(g) \tag{22}$$

for some $M_1 > 0$.

Taking in view the equality

$$\|g\|_2^2 = P(g) + Q(g)\ , g \in \oplus W_2^1 \tag{23}$$

and substituting (20)–(23) into (17) we have

$$\langle \mathcal{A}(-\lambda_0)\Gamma, \Gamma\rangle_{\mathbb{H}} \quad \ge \quad \Phi_1 P(g)\ +\ \Phi_2(\lambda_0)Q(g)\ + \Phi_3(\lambda_0)|\kappa|^2, \tag{24}$$

where

$$\Phi_1 \ := \ 1-\left|\frac{\cos\varphi}{\sin\varphi}\right| C_{11}\epsilon_1+\theta_1\ C_{21}\ \epsilon_2+\theta_2\ C_{31}\ \epsilon_3 - \left(\left|\frac{\beta}{\beta'}\right|+\frac{1}{\gamma\ |\beta'|}\right) C_{41}\epsilon_4, \tag{25}$$

$$\Phi_2(\lambda_0) \ := \ 1-\left|\frac{\cos\varphi}{\sin\varphi}\right| \frac{C_{12}}{\epsilon_1}+\theta_1\frac{C_{22}}{\epsilon_2}+\theta_2\frac{C_{32}}{\epsilon_3} - \left(\left|\frac{\beta}{\beta'}\right|+\frac{1}{\gamma|\beta'|}\right)\frac{C_{42}}{\epsilon_4}+\lambda_0 M, \tag{26}$$

$$\Phi_3(\lambda_0) \ = \ -\left|\frac{\alpha^{'}}{\beta'}\right|\frac{1}{\theta_3}-\frac{\gamma}{|\beta'|}+\frac{\lambda_0}{\theta_3}. \tag{27}$$

Since $\theta_3 > 0$, it is possible to choose the positive parameters $\gamma, \epsilon_1, \epsilon_2, \ \epsilon_3$ and ϵ_4 so small and the positive parameter λ_0 so large that $\Phi_1 > 0, \ \Phi_2(\lambda_0) > 0, \ \Phi_3(\lambda_0) > 0$. Now denoting

$$\Phi(\lambda_0) := \min\left(\Phi_1\ ,\ \Phi_2(\lambda_0)\ ,\ \Phi_3(\lambda_0)\right),$$

we have

$$\langle \mathcal{A}(-\ \lambda_0)\Gamma, \Gamma\rangle_{\mathbb{H}} \ \geq \ \Phi(\lambda_0)\ \|\Gamma\|^2_{\mathbb{H}}$$

for all $\Gamma \in \mathbb{H}$. Consequently the operator pencil $\mathcal{A}(-\lambda_0)$ is positive definite for sufficiently large $\lambda_0 > 0$. The proof is complete.

3 Modified Rayleigh quotient and estimation of the eigenvalues

For finding lower bound estimation for eigenvalues we shall introduce a new spectral parameter $\mu = \lambda + \lambda_0$, where λ_0 is the parameter from Lemma 2. Then the operator pencil equation $\mathcal{A}(\lambda)\ \Gamma = 0$ is transformed to the spectral problem

$$\mathcal{A}(-\lambda_0)\ \Gamma \ - \ \mu\Lambda\Gamma \ = \ 0 \tag{28}$$

with the new spectral parameter μ. This problem can be rewritten as

$$\mu \ = \ \frac{\langle\ \big(\Delta+\lambda_0\ \Lambda\big)\ \Gamma\ ,\ \Gamma\rangle_{\mathbb{H}}}{\langle\Lambda\Gamma\ ,\ \Gamma\rangle_{\mathbb{H}}}. \tag{29}$$

Let $h = g$ in (8). Then equation (8) is converted into the form

$$\langle g, g\rangle_2+\langle S_0 g, g\rangle_2+\langle S_2\kappa, g\rangle_2 \ = \ \lambda\langle S_1 g, g\rangle_2. \tag{30}$$

Using (30), we have the following Rayleigh quotient

$$\mu \ = \ \frac{\langle g, g\rangle_2+\langle S_0 g, g\rangle_2+\langle S_2\kappa, g\rangle_2+(S_2^* g)\kappa-\frac{\alpha^{'}}{\beta'\theta_3}|\kappa|^2+\lambda_0\big\{\langle S_1 g, g\rangle_2+\frac{1}{\theta_3}|\kappa|^2\big\}}{\langle S_1 g, g\rangle_2+\frac{1}{\theta_3}|\kappa|^2}. \tag{31}$$

Using (14)–(16), (20)–(27) and (28)–(31) we have the following inequality

$$\mu \ \geq \ \frac{\Phi_1 P(g)+\Phi_2(\lambda_0)Q(g)+\lambda_0 R(g)+\Phi_3(\lambda_0)|\kappa|^2}{|\kappa|^2+\frac{1}{\theta_3}|\kappa|^2}. \tag{32}$$

It is easy to show that there are $M_2 > 0$ and $M_3 > 0$, such that

$$R(g) \leq M_2 Q(g) \ \leq \ M_3 \|g\|^2$$

for all g.

Then from inequality (32) we get

$$\mu \ \geq \ \min \left(M_2 \Phi_2(\lambda_0) \ + \ \lambda_0 \ , \ \theta_3 \Phi_3(\lambda_0) \right).$$

Thus, we have the lower bound estimation for eigenvalues of the BVTP (1)–(5) given by

$$\lambda_k \ \geq \ -\lambda_0 \ + \ \min \left(M_2 \Phi_2(\lambda_0) \ + \ \lambda_0 \ , \ \theta_3 \Phi_3(\lambda_0) \right).$$

Conclusion

In this work, we investigated a new type of boundary value problems (BVPs) for Sturm-Liouville equations. The problem addressed in our study is different from standard Sturm-Liouville problems in the sense that the differential equation is defined on three non-overlapping intervals $(-\pi, -\pi + d)$, $(-\pi + d, \pi - d)$ and $(\pi - d, \pi)$ and the boundary conditions are included four additional conditions at the interaction points $x = -\pi + d$ and $x = \pi - d$, so-called transmission conditions. Spectral analysis, such type of multi-interval boundary value transmission problems (MIBVTPs), is much more complicated to analyze than BVPs. It is not obvious how to apply the known classical methods to such MIBVTPs. To establish some important spectral characteristics, we introduced a new type polynomial-operator formulation of the considered MIBVTP. We then proved that this polynomial-operator is self-adjoint and positive definite for sufficiently large positive values of the spectral parameter λ. Moreover, we have been developed a new modification of the Rayleigh method to obtain a lower bound for the eigenvalues.

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Author Contributions

All the authors equally contributed to this work. They all read and approved the final version of the paper.

Conflict of Interest

The authors declare no conflict of interest.

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Source identification problems for the neutron transport equations

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In this study, the time-dependent source identification problem for the two-dimensional neutron transport equation was studied. For the approximate solution of this problem a first order of accuracy difference scheme was presented. Stability estimates for the solution of these differential and difference problems were established. Numerical results were given.

Keywords: identification problem, neutron transport equation, difference scheme, differential equation, stability inequality.

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Introduction

The neutron transport equation describes the distribution of neutrons in terms of their positions in space and time, their energies and their travel directions. The various neutron transport equations are studied by many researchers (see, [1–4] and the references given therein). Identification problems play an important role in applied sciences and engineering applications and have been investigated in various papers (see, e.g., [5–27] and the references given therein). In the present paper, we consider the time-dependent source identification problem for two dimensional neutron transport equation

$$\begin{cases} \frac{\partial u(t,x,y)}{\partial t} = \frac{\partial u(t,x,y)}{\partial x} + \frac{\partial u(t,x,y)}{\partial y} + p\left(t\right)q\left(x,y\right) + f\left(t,x,y\right), \\ t \in (0,T),\ x,\ y \in (0,L), \\ u\left(0,x,y\right) = \varphi\left(x,y\right), \quad x,\ y \in [0,L], \\ u\left(t,0,y\right) = 0, \quad u\left(t,x,0\right) = 0, \quad t \in [0,T],\ x,\ y \in [0,L], \\ u(t,l,y) = \alpha\left(t,y\right), \quad t \in [0,T], \quad y \in [0,L], \quad l \in (0,L]. \end{cases} \tag{1}$$

Here, $u\left(t,x,y\right)$ and $p\left(t\right)$ are unknown functions, $f\left(t,x,y\right)$, $q\left(x,y\right)$, $\varphi\left(x,y\right)$, and $\alpha\left(t,y\right)$ are given sufficiently smooth functions and all compatibility conditions are satisfied.

In the rest of paper, the theorem on the stability of differential problem (1) is established. For the approximate solution of problem (1), a first order of accuracy difference scheme is proposed. The theorem on stability of this difference scheme is established. Some results of numerical experiment are presented.

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1 Stability of differential equation

To formulate our results, we introduce the Banach space $C(E) = C([0,T], E)$ of all abstract continuous functions $\phi(t)$ defined on $[0,T]$ with values in E equipped with the norm

$$\|\phi\|_{C(E)} = \max_{0\le t\le T} \|\phi(t)\|_E .$$

Let $E = C_{[0,L]\times[0,L]}$ be the space of all continuous functions $\psi(x,y)$ defined on $[0,L]\times[0,L]$ equipped with norm

$$\|\psi\|_{C_{[0,L]\times[0,L]}} = \max_{0\le x,y\le L} |\psi(x,y)|$$

and $C^{(1)}_{[0,L]\times[0,L]}$ be the space of all continuously differentiable functions $\psi(x,y)$ defined on $[0,L]\times[0,L]$ equipped with norm

$$\|\psi\|_{C^{(1)}_{[0,L]\times[0,L]}} = \|\psi\|_{C_{[0,L]\times[0,L]}} + \max_{0<x,y<L} |\psi_x(x,y)| + \max_{0<x,y<L} |\psi_y(x,y)| .$$

We introduce the positive operator A, defined by formula

$$Au = -\left(\frac{\partial u(x,y)}{\partial x} + \frac{\partial u(x,y)}{\partial y}\right)$$

with the domain

$$D(A) = \left\{u : u, u_x, u_y \in C_{[0,L]\times[0,L]},\ u(0,y) = u(x,0) = 0,\ 0\le x,y\le L\right\}.$$

Throughout the present paper, M denotes positive constants, which may differ in time and thus are not a subject of precision. However, we will use $M(\alpha,\ \beta,\ \gamma,...)$ to stress the fact that the constant depends only on $\alpha,\ \beta,\ \gamma,...$.

We have the following theorem on the stability of problem (1):

Theorem 1. Assume that $\varphi \in C^{(1)}_{[0,L]\times[0,L]}$, $f(t,x,y)$ is a continuously differentiable function in t and continuous in x and y, and $\alpha(t,y)$ is a continuously differentiable function in t and continuous in y. Then, for the solution of problem (1) the following stability estimates hold:

$$\left\|\frac{\partial u}{\partial t}\right\|_{C\left(C_{[0,L]\times[0,L]}\right)} + \|u\|_{C\left(C^{(1)}_{[0,L]\times[0,L]}\right)} + \|p\|_{C[0,T]} \le M(q)\left[\|\varphi\|_{C^{(1)}_{[0,L]\times[0,L]}} + \right.$$

$$\left. + \|f(0,.)\|_{C_{[0,L]\times[0,L]}} + \left\|\frac{\partial f}{\partial t}\right\|_{C\left(C_{[0,L]\times[0,L]}\right)} + \|\alpha(0,\cdot)\|_{C[0,L]} + \|\alpha_t\|_{C(C[0,L])}\right].$$

Proof. We will use the following substitution

$$u(t,x,y) = w(t,x,y) + \eta(t)\, q(x,y),$$

where $\eta(t)$ is the function defined by formula

$$\eta(t) = \int_0^t p(s)\,ds,\ \eta(0) = 0. \tag{2}$$

It is clear that $w(t,x,y)$ is the solution of the following initial boundary value problem

$$\begin{cases} \frac{\partial w(t,x,y)}{\partial t} = \frac{\partial w(t,x,y)}{\partial x} + \frac{\partial w(t,x,y)}{\partial y} + \eta\left(t\right)\left(q_x\left(x,y\right) + q_y\left(x,y\right)\right) + f\left(t,x,y\right), \\ t \in \left(0,T\right),\ x,y \in \left(0,L\right), \\ w\left(0,x,y\right) = \varphi\left(x,y\right),\ x,y \in \left[0,L\right], \\ w\left(t,0,y\right) = 0,\ t \in \left[0,T\right],\ y \in \left[0,L\right], \\ w\left(t,x,0\right) = 0,\ t \in \left[0,T\right],\ x \in \left[0,L\right], \\ q\left(x,0\right) = 0,\ q\left(0,y\right) = 0,\ q\left(l,y\right) \neq 0, \\ w\left(t,l,y\right) = \alpha\left(t,y\right) - \eta\left(t\right) q\left(l,y\right),\ t \in \left[0,T\right],\ y \in \left[0,L\right]. \end{cases} \tag{3}$$

Applying the over determined condition $u\left(t,l,y\right) = \alpha\left(t,y\right)$ at substitution (2), we get

$$w\left(t,l,y\right) + \eta\left(t\right) q\left(l,y\right) = \alpha\left(t,y\right),$$

$$\eta\left(t\right) = \frac{\alpha\left(t,y\right) - w\left(t,l,y\right)}{q\left(l,y\right)}.$$

From that and $p\left(t\right) = \eta'\left(t\right)$, it follows

$$p\left(t\right) = \frac{\alpha_t\left(t,y\right) - w_t\left(t,l,y\right)}{q\left(l,y\right)}. \tag{4}$$

From identity (4) and the triangle inequality, we get the estimate

$$\left|p\left(t\right)\right| = \left|\frac{\alpha_t\left(t,y\right) - w_t\left(t,l,y\right)}{q\left(l,y\right)}\right| \leq M\left(q\right)\left[\left|\alpha_t\left(t,y\right)\right| + \left|w_t\left(t,l,y\right)\right|\right] \leq$$

$$\leq M\left(q\right)\left[\max_{0\leq t\leq T}\left|\alpha_t\left(t,y\right)\right| + \max_{0\leq t\leq T}\max_{0\leq y\leq L}\left|w_t\left(t,l,y\right)\right|\right].$$

From that it follows

$$\left\|p\right\|_{C[0,T]} \leq M\left(q\right)\left[\left\|\alpha_t\right\|_{C(C[0,T],\ C[0,L])} + \left\|w_t\right\|_{C(C[0,T],\ C[0,L])}\right]. \tag{5}$$

Using operator A with the domain $D\left(A\right)$ we can rewrite problem (3) in the abstract form as an initial value problem

$$\begin{cases} \frac{dw}{\partial t} + Aw = -\frac{\alpha(t,.) - w(t,l,.)}{q(l,.)} Aq + f\left(t\right), \\ w\left(0\right) = \varphi. \end{cases}$$

By the Cauchy formula, the solution can be written as

$$w\left(t\right) = e^{-tA}\varphi + \int_0^t e^{-(t-s)A}\left\{-\frac{\alpha\left(s,.\right) - w\left(s,l,.\right)}{q\left(l,.\right)} Aq + f\left(s\right)\right\} ds.$$

Taking derivative with respect to t and using Leibniz integral rule, we obtain

$$w_t(t) = -Ae^{-tA}\varphi + \left\{-\frac{\alpha(t,.)-w(t,l,.)}{q(l,.)}Aq + f(t)\right\} + \int_0^t -Ae^{-(t-s)A}\left\{-\frac{\alpha(s,.)-w(s,l,.)}{q(l,.)}Aq + f(s)\right\}ds.$$

Applying the integration by parts formula, we get

$$w_t(t) = -Ae^{-tA}\varphi + e^{-tA}\left\{-\frac{\alpha(0,.)-w(0,l,.)}{q(l,.)}Aq + f(0)\right\} +$$

$$+\int_0^t e^{-(t-s)A}\left\{-\frac{\alpha_s(s,.)-w_s(s,l,.)}{q(l,.)}Aq + f'(s)\right\}ds = \sum_{k=1}^{3} G_k(t),$$

where

$$G_1(t) = -Ae^{-tA}\varphi,$$

$$G_2(t) = e^{-tA}\left\{-\frac{\alpha(0,.)-w(0,l,.)}{q(l,.)}Aq + f(0)\right\},$$

$$G_3(t) = \int_0^t e^{-(t-s)A}\left\{-\frac{\alpha_s(s,.)-w_s(s,l,.)}{q(l,.)}Aq + f'(s)\right\}ds.$$

Now, we estimate, G_1, G_2, and G_3, separately. Using the triangle inequality, we obtain

$$\|w_t\|_E \le \|G_1(t)\|_E + \|G_2(t)\|_E + \|G_3(t)\|_E.$$

It is known (see [20]) that for any $t \in [0,T]$,

$$\left\|e^{-tA}\right\|_{E\to E} \le Me^{-\delta t},\ M>0,\ \delta>0. \tag{6}$$

Applying the definition of norm of the spaces E and estimate (6), we get

$$\|G_1(t)\|_E = \left\|-Ae^{-tA}\varphi\right\|_E \le \left\|e^{-tA}\right\|_{E\to E}\|A\varphi\|_E \le M_1(\delta)\|A\varphi\|_E. \tag{7}$$

Let us estimate $G_2(t)$. Using the triangle inequality, we get

$$\|G_2(t)\|_E = \left\|e^{-tA}\left\{-\frac{\alpha(0,.)-w(0,l,.)}{q(l,.)}Aq + f(0)\right\}\right\|_E \le$$

$$\le \left\|e^{-tA}\right\|_{E\to E}\left[\left[\left|\frac{\alpha(0,.)}{q(l,.)}\right| + \left|\frac{w(0,l,.)}{q(l,.)}\right|\right]\|Aq\|_E + \|f(0)\|_E\right],$$

$$\|G_2(t)\|_E \le \left\|e^{-tA}\right\|_{E\to E}\left\{\|Aq\|_{E\to E}\frac{\|\alpha(0,.)\|_E + \|w(0,l,.)\|_E}{\min\limits_{0\le y\le L}|q(l,.)|} + \|f(0)\|_E\right\}.$$

Hence,

$$\|G_2(t)\|_E \le M_2(\delta,q)\left[\|\alpha(0,.)\|_E + \|\varphi\|_E + \|f(0)\|_E\right] \tag{8}$$

for any t, $t\in[0,T]$.

Let us estimate $G_3(t)$. Using the triangle inequality, we get

$$\|G_3(t)\|_E \le \int_0^t \left\|e^{-(t-s)A}\right\|_{E\to E}\left\{\frac{\max\limits_{0\le s\le T}|\alpha_s(s,.)|_{E'} + \|w_s(s,.)\|_E}{\min\limits_{0\le y\le L}|q(l,.)|}\|Aq\|_E + \|f'(s)\|_E\right\}ds \le$$

$$\le M_3(\delta,q)\int_0^t\left[\max_{0\le s\le T}\|f'(s)\|_E + \max_{0\le s\le T}|\alpha_s(s,.)|\right]ds + \int_0^t M_4(\delta,q)\|w_s(s)\|_E\,ds, \tag{9}$$

where $E^{'} \subset E$.

Combining estimates (7), (8), and (9), we get

$$\|w_t\|_E \leq M_1(\delta)\|A\varphi\|_E + M_2(\delta,q)\left[\|\alpha(0,.)\|_E + \|\varphi\|_E + \|f(0)\|_E\right] +$$

$$+M_3(\delta,q)\int_0^t\left[\max_{0\leq s\leq T}\|f'(s)\|_E + \max_{0\leq s\leq T}\|\alpha_s(s,.)\|_{E'}\right]ds + \int_0^t M_4(\delta,q)\|w_s(s)\|_E\,ds.$$

Using Grönwall's inequality, we can write

$$\|w_t\|_E \leq M_5 e^{M_4(\delta,q)T},$$

where

$$M_5 = M_6(\delta,q)\left[\|A\varphi\|_E + \|\alpha(0,.)\|_E + \|\varphi\|_E + \|f(0)\|_E + \max_{0\leq s\leq T}\|f'(s)\|_E + \max_{0\leq s\leq T}|\alpha_s(s,.)|\right]. \tag{10}$$

Finally, combining estimates (10) and (5) it completes the proof of Theorem 1.

2 Stability of difference scheme

For the approximate solution of problem (1) we present the first order of accuracy difference scheme

$$\begin{cases} \frac{u_{n,m}^k - u_{n,m}^{k-1}}{\tau} = \frac{u_{n+1,m+1}^k - u_{n,m+1}^k}{h} + \frac{u_{n,m+1}^k - u_{n,m}^k}{h} + p^k q_{n,m} + f_{n,m}^k, \\ f_{n,m}^k = f(t_k,x_n,y_m),\ q_{n,m} = q(x_n,y_m),\ x_n = nh,\ y_m = mh, \\ t_k = k\tau,\ 1\leq k\leq N,\ \ 1\leq n,m\leq M-1,\ Mh = L,\ N\tau = T, \\ u_{n,m}^0 = \varphi(x_n,y_m),\ \ 0\leq n,m\leq M, \\ u_{0,m}^k = 0,\ u_{n,0}^k = 0,\ 0\leq k\leq N,\ 0\leq n,m\leq M, \\ u_{s,m}^k = \alpha(t_k,y_m),\ \ 0\leq k\leq N,\ 0\leq m\leq M,\ s = \left\lfloor \frac{l}{h}\right\rfloor. \end{cases} \tag{11}$$

To formulate the results on difference problem, we introduce the Banach space

$$C_\tau(E) = C([0,T]_\tau, E)$$

of all grid functions

$$\phi^\tau = \{\phi(t_k)\}_{k=0}^N$$

defined on

$$[0,T]_\tau = \{t_k :\ t_k = k\tau,\ 0\leq k\leq N,\ N\tau = T\}$$

with values in E equipped with the norm

$$\|\phi^\tau\|_{C_\tau(E)} = \max_{0\leq k\leq N}\|\phi(t_k)\|_E.$$

Let $C_h = C_{[0,L]_h\times[0,L]_h}$ and $C_h^{(1)} = C_{[0,L]_h\times[0,L]_h}^{(1)}$ be spaces of all grid functions $\psi^h = \{\psi_{n,m}\}_{m,n=1}^M$ defined on $[0,L]_h\times[0,L]_h = \{x_n = nh,\ y_m = mh,\ 0\leq n,m\leq M\}$ equipped with the norms

$$\left\|\psi^h\right\|_{C_h} = \max_{0\leq n,m\leq M}|\psi_{n,m}|,$$

$$\left\|\psi^h\right\|_{C_h^{(1)}} = \left\|\psi^h\right\|_{C_h} + \frac{1}{h}\max_{0\le n\le M}\max_{1\le m\le M}\left|\psi_{n,m}-\psi_{n,m-1}\right| + \frac{1}{h}\max_{1\le n\le M}\max_{0\le m\le M}\left|\psi_{n,m}-\psi_{n-1,m}\right|,$$

respectively.

Moreover, we introduce difference neutron transport operator A_h

$$A_h u^h = -\left\{\frac{u_{n+1,m+1}-u_{n,m+1}}{h} + \frac{u_{n,m+1}-u_{n,m}}{h}\right\}_{n,m=1}^{M-1}$$

acting in the space of grid functions $u^h = \{u_{n,m}\}_{n,m=1}^{M}$, $u_{0,m}=0,\ u_{n,0}=0,\ 0\le n,m\le M.$

Then, the following theorem on stability of problem (11) is established.

Theorem 2. For the solution of problem (11), the following stability estimates hold

$$\left\|\left\{\left\{\frac{u_{n,m}^k-u_{n,m}^{k-1}}{\tau}\right\}_{k=1}^{N}\right\}_{n,m=0}^{M}\right\|_{C_\tau(C_h)} + \left\|\left\{\left\{u_{n,m}^k\right\}_{k=1}^{N}\right\}_{n,m=0}^{M}\right\|_{C_\tau\left(C_h^{(1)}\right)} + \left\|p^\tau\right\|_{C_\tau} \le$$

$$\le M_1(q)\left[\left\|\varphi^h\right\|_{C_h^{(1)}} + \left\|f^{1,h}\right\|_{C_h} + \left\|\left\{\left\{\frac{f_{n,m}^k-f_{n,m}^{k-1}}{\tau}\right\}_{k=2}^{N}\right\}_{n,m=0}^{M}\right\|_{C_\tau(C_h)} + \right.$$

$$\left. + \left\|\left\{\left\{\frac{\alpha_{s,m}^k-\alpha_{s,m}^{k-1}}{\tau}\right\}_{k=2}^{N}\right\}_{m=0}^{M}\right\|_{C_\tau\left(C[0,L]_h\right)} + \left\|\left\{\alpha_m^1\right\}_{m=0}^{M}\right\|_{C[0,L]_h}\right].$$

Proof. For the solution of difference scheme (11), we consider substitution

$$u_{n,m}^k = \eta^k q_{n,m} + w_{n,m}^k, \tag{12}$$

where

$$q_{n,m} = q(x_n, y_m),$$

and η^k is the grid function determined by

$$\eta^k = \sum_{i=1}^{k} p_i\,\tau,\ \eta^0 = 0,\ p_k = \frac{\eta^k-\eta^{k-1}}{\tau},\ 0\le k\le N.$$

It is easy to see that grid function $\left\{\left\{w_{n,m}^k\right\}_{k=1}^{N}\right\}_{n,m=0}^{M}$ is the solution of difference scheme

$$\begin{cases} \frac{w_{n,m}^k-w_{n,m}^{k-1}}{\tau} = \frac{w_{n+1,m+1}^k-w_{n,m+1}^k}{h} + \frac{w_{n,m+1}^k-w_{n,m}^k}{h} \\ +\eta^k\left[\frac{q_{n+1,m+1}-q_{n,m+1}}{h} + \frac{q_{n,m+1}-q_{n,m}}{h}\right] + f(t_k,x_n,y_m), \\ f_{n,m}^k = f(t_k,x_n,y_m),\ q_{n,m} = q(x_n,y_m),\ x_n = nh,\ y_m = mh, \\ t_k = k\tau,\ 1\le k\le N,\ \ 1\le n,m\le M-1,\ Mh=L,\ N\tau=T, \\ w_{n,m}^0 = \varphi(x_n,y_m),\ \ 0\le n,m\le M, \\ w_{0,m}^k = 0,\ u_{n,0}^k = 0,\ 0\le k\le N,\ 0\le n,m\le M, \\ w_{s,m}^k = \alpha(t_k,y_m),\ \ 0\le k\le N,\ 0\le m\le M,\ s = \left\lfloor\frac{l}{h}\right\rfloor. \end{cases} \tag{13}$$

Difference derivative of (12) can be written as

$$\frac{u_{n,m}^{k}-u_{n,m}^{k-1}}{\tau}=\frac{\eta^{k}-\eta^{k-1}}{\tau}q_{n,m}+\frac{w_{n,m}^{k}-w_{n,m}^{k-1}}{\tau}=p_{k}q_{n,m}+\frac{w_{n,m}^{k}-w_{n,m}^{k-1}}{\tau}. \tag{14}$$

Hence,

$$p_{k}=\frac{\frac{u_{n,m}^{k}-u_{n,m}^{k-1}}{\tau}-\frac{w_{n,m}^{k}-w_{n,m}^{k-1}}{\tau}}{q_{n,m}} \tag{15}$$

for n,m and k, $1\leq n,m\leq M-1$ and $1\leq k\leq N$. Applying the overdetermined condition $u_{s,m}^{k}$ in (15), we obtain that

$$p_{k}=\frac{\frac{u_{s,m}^{k}-u_{s,m}^{k-1}}{\tau}-\frac{w_{s,m}^{k}-w_{s,m}^{k-1}}{\tau}}{q_{s,m}}.$$

Using the triangle inequality, we obtain

$$|p_{k}|\leq M_{7}\left(q\right)\left[\left|\frac{u_{s,m}^{k}-u_{s,m}^{k-1}}{\tau}\right|+\left|\frac{w_{s,m}^{k}-w_{s,m}^{k-1}}{\tau}\right|\right]$$

for all $0\leq k\leq N$. From that it follows,

$$\begin{aligned}\left\|\{p_{k}\}_{k=1}^{N}\right\|_{C[0,T]_{\tau}}\leq M_{7}\left(q\right)&\left[\left\|\left\{\frac{u_{s,m}^{k}-u_{s,m}^{k-1}}{\tau}\right\}_{k=1}^{N}\right\|_{C[0,T]_{\tau}}+\right.\\&\left.+\left\|\left\{\frac{w_{s,m}^{k}-w_{s,m}^{k-1}}{\tau}\right\}_{k=1}^{N}\right\|_{C_{\tau}\left(C\left([0,L]_{h}\times[0,L]_{h},\ E\right)\right)}\right].\end{aligned} \tag{16}$$

Now using substitution (14) we get

$$\frac{u_{n,m}^{k}-u_{n,m}^{k-1}}{\tau}=\frac{w_{n,m}^{k}-w_{n,m}^{k-1}}{\tau}+p_{k}q_{n,m}.$$

Applying the triangle inequality, we obtain

$$\begin{aligned}\left\|\left\{\frac{u_{n,m}^{k}-u_{n,m}^{k-1}}{\tau}\right\}_{k=1}^{N}\right\|_{C[0,T]_{\tau}}&\leq\left\|\left\{\frac{w_{n,m}^{k}-w_{n,m}^{k-1}}{\tau}\right\}_{k=1}^{N}\right\|_{C_{\tau}\left(C\left([0,L]_{h}\times[0,L]_{h}\right)\right)}+\\&+\left\|\{p_{k}\}_{k=1}^{N}\right\|_{C[0,T]_{\tau}}\left\|\left\{\{q_{n,m}\}_{n=1}^{M}\right\}_{m=1}^{M}\right\|_{C\left([0,L]_{h}\times[0,L]_{h}\right)}\end{aligned} \tag{17}$$

for all $0\leq k\leq N$. We can rewrite difference scheme (13) in the abstract form as

$$\begin{cases}\frac{w_{h}^{k}-w_{h}^{k-1}}{\tau}+A_{h}w_{h}^{k}+\eta^{k}Aq=f^{h}\left(t_{k}\right),\\ w_{h}^{0}=\varphi^{h},\ \eta^{0}=0,\ t_{k}=k\tau,\ 1\leq k\leq N,\ N\tau=T\end{cases} \tag{18}$$

in a Banach space $C_{\tau}\left(E\right)=C\left([0,T]_{\tau},\ E\right)$ with the positive operator A_{h} defined by

$$A_{h}u^{h}=-\left\{\frac{u_{n+1,m+1}-u_{n,m+1}}{h}+\frac{u_{n,m+1}-u_{n,m}}{h}\right\}_{n,m=1}^{M-1},$$

acting on grid functions u^h such that satisfies the condition $u^h = \{u_{n,m}\}_{n,m=1}^{M}, u_{0,m} = 0,\ \ u_{n,0} = 0$, $0 \leq n, m \leq M$.

For equation (18) we have that

$$w_h^k = Rw_h^{k-1} + R\tau \left(Aq \frac{\alpha(t_k) - w_s^k}{q_s} + f^h(t_k) \right),$$

for all k, $1 \leq k \leq N$, where $R = (I + \tau A_h)^{-1}$. By recurrence relations, we get

$$w_h^k = R^k \varphi^h + \sum_{i=1}^{k} R^{k-i+1} \tfrac{\tau}{q_s} \alpha(t_i) Aq - \sum_{i=1}^{k} R^{k-i+1} \tfrac{\tau}{q_s} w_s^i Aq + \sum_{i=1}^{k} R^{k-i+1} \tau f^h(t_i)$$

for any k, $1 \leq k \leq N$. Taking the difference derivative of both sides, we obtain that

$$\tfrac{w_h^k - w_h^{k-1}}{\tau} = \tfrac{R^k - R^{k-1}}{\tau} \varphi^h + \tfrac{1}{q_s} \alpha(t_k) Aq + \sum_{i=1}^{k} \left(R^{k-i+1} - R^{k-i}\right) \tfrac{1}{q_s} \alpha(t_i) Aq -$$

$$-\tfrac{1}{q_s} w_s^k Aq - \sum_{i=1}^{k} \left(R^{k-i+1} - R^{k-i}\right) \tfrac{1}{q_s} w_s^i Aq + f^h(t_k) + \sum_{i=1}^{k} \left(R^{k-i+1} - R^{k-i}\right) f^h(t_i).$$

Applying the formula,

$$\sum_{i=1}^{k} \left(R^{k-i+1} - R^{k-i}\right) w_s^i = \sum_{i=1}^{k} \left(R^{k-i+1} - R^{k-i}\right) \varphi(x_s, y_m) +$$

$$+ \sum_{i=1}^{k} \left(R^{k-i+1} - R^{k-i}\right) \sum_{j=1}^{i} \tfrac{w_s^j - w_s^{j-1}}{\tau} \tau$$

and changing the order of summation, we get

$$\sum_{i=1}^{k} \left(R^{k-i+1} - R^{k-i}\right) w_s^i = \sum_{i=1}^{k} \left(R^{k-i+1} - R^{k-i}\right) \varphi(x_s, y_m) +$$

$$+ \sum_{j=1}^{k} \sum_{i=j}^{k} \left(R^{k-i+1} - R^{k-i}\right) \tfrac{w_s^j - w_s^{j-1}}{\tau} \tau.$$

Consequently, we obtain the following presentation for the solution of equation (13)

$$\tfrac{w_h^k - w_h^{k-1}}{\tau} = \tfrac{R^k - R^{k-1}}{\tau} \varphi^h + \tfrac{1}{q_s} \alpha(t_k) Aq + \sum_{i=1}^{k} \left(R^{k-i+1} - R^{k-i}\right) \tfrac{1}{q_s} \alpha(t_i) Aq -$$

$$-\tfrac{1}{q_s} w_s^k Aq - \sum_{i=1}^{k} \left(R^{k-i+1} - R^{k-i}\right) \tfrac{1}{q_s} Aq \varphi^h(x_s, y_m) - \sum_{j=1}^{k} \sum_{i=j}^{k} \left(R^{k-i+1} - R^{k-i}\right) \tfrac{w_s^j - w_s^{j-1}}{\tau} \tau +$$

$$+ f^h(t_k) + \sum_{i=1}^{k} \left(R^{k-i+1} - R^{k-i}\right) f^h(t_i).$$

Applying the definition of norm of the spaces $C_\tau(E) = C([0, T]_\tau,\ E)$ and methods of monograph [20], we can write,

$$\left\|\left\{\frac{w_{n,m}^{k}-w_{n,m}^{k-1}}{\tau}\right\}_{k=1}^{N}\right\|_{C_{\tau}\left(C\left([0,L]_{h}\times[0,L]_{h}\right)\right)}\leq M_{8}\left(q\right)\left[\left\|\varphi^{h}\right\|_{C_{h}^{(1)}}+\left\|\left\{\left\{\frac{f_{n,m}^{k}-f_{n,m}^{k-1}}{\tau}\right\}_{k=2}^{N}\right\}_{n,m=0}^{M}\right\|_{C_{\tau}(C_{h})}+\right.$$

$$\left.+\left\|f^{1,h}\right\|_{C_{h}}+\left\|\left\{\left\{\frac{\alpha_{s,m}^{k}-\alpha_{s,m}^{k-1}}{\tau}\right\}_{k=2}^{N}\right\}_{m=0}^{M}\right\|_{C_{\tau}\left(C[0,L]_{h}\right)}+\left\|\left\{\alpha_{m}^{1}\right\}_{m=0}^{M}\right\|_{C[0,L]_{h}}\right]. \tag{19}$$

Finally, combining estimates (16), (17), and (19), it completes the proof of Theorem 2.

3 Numerical experiments

In this section, we study the numerical solution of the neutron transport identification problem with initial condition

$$\begin{cases} \frac{\partial u(t,x,y)}{\partial t}=\frac{\partial u(t,x,y)}{\partial x}+\frac{\partial u(t,x,y)}{\partial y}+p\left(t\right)\sin\pi x\sin\pi y+f\left(t,x,y\right), \\ f\left(t,x,y\right)=-e^{-2t}(3\sin\pi x\sin\pi y+\pi\cos\pi x\sin\pi y+\pi\sin\pi x\cos\pi y), \\ t\in\left(0,1\right],\ x,y\in\left(0,1\right], \\ u\left(0,x,y\right)=\sin\pi x\sin\pi y,\ x,y\in\left[0,1\right], \\ u\left(t,0,y\right)=0,\ t\in\left[0,1\right],\ y\in\left[0,1\right], \\ u\left(t,x,0\right)=0,\ t\in\left[0,1\right],\ x\in\left[0,1\right], \\ u(t,\frac{1}{2},y)=e^{-2t}\sin\pi y,\quad t\in\left[0,1\right],\ y\in\left[0,1\right]. \end{cases} \tag{20}$$

The exact solution of problem is $u(t,x,y)=e^{-2t}\sin\pi x\sin\pi y$ and for the control parameter $p\left(t\right)=e^{-2t}$.

For the approximate solution of problem (20), we get the following first order of accuracy difference scheme

$$\begin{cases} \frac{u_{n,m}^{k}-u_{n,m}^{k-1}}{\tau}=\frac{u_{n+1,m+1}^{k}-u_{n,m+1}^{k}}{h}+\frac{u_{n,m+1}^{k}-u_{n,m}^{k}}{h}+p^{k}q_{n,m}+f_{n,m}^{k}, \\ f_{n,m}^{k}=-e^{-2t_{k}}(3\sin\pi x_{n}\sin\pi y_{m}+\pi\cos\pi x_{n}\sin\pi y_{m}+\pi\sin\pi x_{n}\cos\pi y_{m}), \\ q_{n,m}=\sin\pi x_{n}\sin\pi y_{m},\ x_{n}=nh,\ y_{m}=mh,\ t_{k}=k\tau, \\ 1\leq k\leq N,\quad 0\leq n,m\leq M-1,\ Mh=1,\quad N\tau=1, \\ u_{n,m}^{0}=\sin\pi x_{n}\sin\pi y_{m},\ 0\leq n,m\leq M, \\ u_{0,m}^{k}=0,\ u_{n,0}^{k}=0,\quad 0\leq k\leq N,\ 0\leq n,m\leq M, \\ u_{s,m}^{k}=e^{-2t_{k}}\sin\pi y_{m},\quad 0\leq k\leq N,\ 0\leq m\leq M,s=\left\lfloor\frac{M}{2}\right\rfloor. \end{cases} \tag{21}$$

For the solution of difference scheme (21), we consider the substitution

$$u_{n,m}^k = \eta^k q_{n,m} + w_{n,m}^k, \tag{22}$$

where

$$\eta^k = \sum_{i=1}^{k} p_i \ \tau, \ \eta^0 = 0, \tag{23}$$

$w_{n,m}^k$ is the solution of difference scheme

$$\begin{cases} \frac{w_{n,m}^k - w_{n,m}^{k-1}}{\tau} = \frac{w_{n+1,m+1}^k - w_{n,m+1}^k}{h} + \frac{w_{n,m+1}^k - w_{n,m}^k}{h} + \\ +\eta^k \left[\frac{q_{n+1,m+1} - q_{n,m+1}}{h} + \frac{q_{n,m+1} - q_{n,m}}{h}\right] + f_{n,m}^k, \\ 1 \le k \le N, \quad 1 \le n,m \le M, \\ w_{n,m}^0 = \sin \pi x_n \sin \pi y_m, \quad 0 \le n,m \le M, \\ w_{0,m}^k = 0, \ w_{n,0}^k = 0, \ 0 \le k \le N, \ 0 \le n,m \le M. \end{cases} \tag{24}$$

Applying (21) and formulas (22), (23), we get

$$\eta^k = \frac{u_{s,m}^k - w_{s,m}^k}{q_{s,m}} = \frac{e^{-2t_k} \sin \pi y_m - w_{s,m}^k}{q_{s,m}}, \tag{25}$$

$$p^k = \frac{1}{\tau} \left[\frac{(e^{-2t_k} - e^{-2t_{k-1}}) \sin \pi y_m - (w_{s,m}^k - w_{s,m}^{k-1})}{\sin \pi x_s \sin \pi y_m}\right] \tag{26}$$

for any k, $1 \le k \le N$.

It is easy to see that (24) and (25) can be written in the matrix form

$$A \ w^k + B \ w^{k-1} = \varphi^k, \quad 1 \le k \le N, \quad w^0 = \{\sin \pi x_n \sin \pi y_m\}_{n,m=0}^M ,$$

where

$$\begin{cases} \varphi_{n,m}^k = e^{-2t_k} \left[\frac{\sin \pi x_{n+1} \sin \pi y_{m+1} - \sin \pi x_n \sin \pi y_{m+1}}{h} + \frac{\sin \pi x_n \sin \pi y_{m+1} - \sin \pi x_n \sin \pi y_m}{h}\right] - \\ -e^{-2t_k}(3 \sin \pi x_n \sin \pi y_m + \pi \cos \pi x_n \sin \pi y_m + \pi \sin \pi x_n \cos \pi y_m), \\ 1 \le n,m \le M, \ \varphi_{0,m}^k = 0, \ \varphi_{n,0}^k = 0, \ 1 \le n,m \le M. \end{cases}$$

Here A and B are $(M+1) \times (M+1) \times (N+1)$ square matrices, w^k and φ^k are $(M+1) \times (M+1) \times 1$ column matrices. First, we obtain w^k by formula

$$w^k = -A^{-1} B w^{k-1} + A^{-1} \varphi^k, \ 1 \le k \le N, w^0 = \{\sin \pi x_n \sin \pi y_m\}_{n,m=0}^M .$$

Second, applying formulas (22) and (26), we get p^k and u^k.

4 Error analysis

Now, we will give the results of the numerical analysis. In order to get the solution of (21), we used MATLAB program. The errors are computed by

$$E_M^N u = \max_{0\leq k\leq N} \max_{0\leq n,m\leq M} \left| u(t_k, x_n, y_m) - u_{n,m}^k \right|, \quad E^N p = \max_{1\leq k\leq N} \left| p(t_k) - p^k \right|$$

of the numerical solutions for different values of M and N, where $u(t_k, x_n, y_m)$ represents the exact solution, $u_{n,m}^k$ represents the numerical solution at (t_k, x_n, y_m), $p(t_k)$ represents the exact solution, and p^k represents the numerical solution at t_k. Now, let us give the obtained numerical results (Table).

Table

Error analysis of first order DS

Error	$N=M=10$	$N=M=20$	$N=M=40$	$N=M=80$
$E_M^N u$	0.1813	0.0952	0.0488	0.0247
$E^N p$	0.0698	0.0481	0.0264	0.0137

The obtained results indicate that when the numerical parameters N and M are multiplied by two, the errors in the solution for first order difference scheme (21) decrease by approximately half.

Conclusion

In this study, we consider an inverse problem related to the two-dimensional neutron transport equation with a time-dependent source control parameter. For the approximate solution of this problem, a first-order accuracy difference scheme is constructed. A finite difference scheme is presented for identifying the control parameter. Stability inequalities for the solution of this problem are established. The results of a numerical experiment are presented, and the accuracy of the solution for this inverse problem is discussed.

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Conflict of Interest

The author declare no conflict of interest.

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https://doi.org/10.31489/2024M3/150-157 *Research article*

Spectrum and resolvent of multi-channel systems with internal energies and common boundary conditions

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In the article the spectrum and resolvent of the so-called multichannel systems with nonzero internal energies were investigated. The spectrum and resolvent of multichannel Sturm-Liouville systems with nonzero internal energies m_i^2 and general boundary conditions were investigated. These systems describe the propagation of partial waves in the theory of quantum physics. The importance of studying the spectral characteristics of these systems is presented in the well-known books of the theory of quantum physics. The finiteness of the number of eigenvalues was proved, the multiplicity of positive eigenvalues was investigated, and as well as the resolvent kernel of the system was found.

Keywords: operator, eigenvalues, edge problem, Wronskian, transformation operator, asymptotics, continuous spectrum, resolvent, multi-channel systems, internal energy, quantum physics.

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Introduction

The spectrum and resolvent of $0 = m_1^2 \leq m_2^2 \leq \ldots \leq m_n^2 = m^2$ multilayer systems with non-zero internal energies are studied in the present article. These kinds of systems are described by differential equations

$$-y_i^{''} + \sum_{j=1}^{n} q_{ij}(x)y_j + m_i^2 y_i = \lambda^2 y_i,\ i = \overline{1,n},\ 0 \leq x < \infty$$

and boundary conditions

$$y_i^{'}(0) - \sum_{j=1}^{n} h_{ij} y_j(0) = 0,\ i = \overline{1,n}.$$

This system can be rewritten in the next form

$$-y^{''} + Q(x)y + My = \lambda^2 y,\ 0 \leq x < +\infty, \tag{1}$$

$$y^{'}(0) - Hy(0) = 0, \tag{2}$$

where $Q(x) = \{q_{ij}(x)\}$ $(i,j = \overline{1,n},\ 0 \leq x < +\infty)$ is a semi-continuous matrix-function, $M = \{\delta_{ij} m_i^2\ \}_1^n$ is diagonally constant matrix, H is a self-constructed constant matrix, $y(x)$ is a column vector-function. Assume that the Euclidean norm of the matrix function $Q(x)$ satisfies the following condition

$$\int_0^{+\infty} x e^{mx} \|Q(x)\| \, dx < +\infty. \tag{3}$$

Boundary value problem (1)-(2) occurs in the theory of dispersion multichannel particles with nonzero inner energy $m_i^2, i = \overline{1,n}$ and it describes the spread of partial waves.

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Notice that for the condition $y(0) = 0$ this problem was investigated in the works [1], [2]. In recent years, many studies have been conducted on this issue [3–7].

Let us consider the diagonal matrix

$$K(\lambda) = \left(\lambda^2 I - M\right)^{\frac{1}{2}} = \{\delta_{ij} K_j(\lambda)\}, K_j(\lambda) = \sqrt{\lambda^2 - m_j^2},$$

where $\delta_{ii} = 1$ and $\delta_{ij} = 0$ for $i \neq j$.

Here we define branches so that at $\operatorname{Im}\lambda K_j(\lambda) > 0$, and at $\operatorname{Im}\lambda = 0$, $K_j(\lambda) = \lim_{\varepsilon\to+0} k_j(\lambda + i\varepsilon)$. Hence, if $\operatorname{Im}\lambda > 0$, we have

$$K_j(\lambda) = \begin{cases} \sqrt{m_j^2 - \lambda^2}, \ if \ |\lambda| \leq m_j, \\ \lambda\sqrt{1 - \frac{m_j^2}{\lambda^2}} \ if \ |\lambda| \geq m_j. \end{cases} \tag{4}$$

Note that for $|\lambda| \leq m_j$ the functions $K_j(\lambda)$ are even, and for $|\lambda| \geq m_j$ they are odd. It is not difficult to verify that, when $\operatorname{Im}\lambda > 0$, $i < j$

$$0 \leq \operatorname{Im} K_j(\lambda) - \operatorname{Im} K_i(\lambda) \leq \sqrt{m_j^2 - m_i^2} \leq m.$$

Therefore,

$$\|\exp(iK(\lambda)x)\| = \exp(-\operatorname{Im}\lambda x), \ \|\exp(-iK(\lambda)x)\| = \exp(\operatorname{Im} K_n(\lambda)x).$$

Denote by $L_2((0,\infty); E_n)$ the Hilbert space of the column vector-functions $y(x) = \{y_i(x)\}, i = \overline{1,n}$ of the quadratic integrable on the semiaxis $(0, +\infty)$ of all the components, in which the inner product is defined by the formula

$$\langle y, z\rangle = \sum_{k=1}^{n} y_k(x)\, \overline{z}_{k(x)} dx = \int_0^{+\infty} z^*(x)\, y(x)\, dx.$$

Denote by $[y(x), z(x)]$ the Wronskian of the differentiable matrix functions which is defined by the formula, where the transposition of the matrix means $y(x)$ and $z(x)$,

$$[y(x), z(x)] = \widetilde{y}(x)\, z'(x) - \widetilde{y}'(x)\ z(x),$$

where $\widetilde{y}$ is the transpose of the matrix y.

1 *Solutions at* $F(x,\lambda), \Phi(x,\lambda)$ *and connections between them at* $\operatorname{Im}\lambda = 0$, $|\lambda| > m$

Consider the matrix equation

$$-y'' + Q(x)yMy = \lambda^2 y, \ 0 \leq x < +\infty. \tag{5}$$

In the case of $Q(x) = 0$, this equation has a solution

$$e(x,\lambda) = \exp(iK(\lambda)x) = \left\{\delta_{\alpha j} e^{ikj(\lambda)x}\right\}, \ \alpha, j = \overline{1,n}.$$

In [1] it is proved that if the condition (3) is satisfied, then the equation (5) will have analytic solution λ in the upper half-scope $\operatorname{Im}\lambda > 0$ and the continuous up to the real axis $\operatorname{Im}\lambda = 0$, and the solution is $F(x,\lambda)$ satisfying the condition

$$\lim_{x\to+\infty} e(x,\lambda)\, F(x,\lambda) = I,$$

and there is a core $K(x,\lambda)$ of conversion operator that is

$$F(x,\lambda) = e(x,\lambda) + \int_x^{+\infty} K(x,t)\, e(t,\lambda)\, dt. \tag{6}$$

Further, the following assessments are valid (see [2]):

$$\|F(x,\lambda) - \exp(iK(x))\| \le C_1 \exp(-(m + \operatorname{Im}\lambda)x),$$

$$\|F(x,\lambda) - \exp(iK(\lambda)x)\| \le C_2 \left\|K^{-1}(\lambda)\right\| \exp(-(m + \operatorname{Im}\lambda)x),$$

where $C_{1,}C_2$ are constants.

Let us denote by $\Phi(x,\lambda)$ the solution of differential equation (5) satisfying initial conditions

$$\Phi(0,\lambda) = I, \Phi'(0,\lambda) = H. \tag{7}$$

It is known that the solution $\Phi(x,\lambda)$ of the boundary value problem (5)–(7) is an integer function of the parameter λ. Obviously, $\Phi(x,\lambda)$ is an even function of parameter λ.

If $\operatorname{Im}\lambda = 0$, $|\lambda|>m$, then solutions $F(x,\lambda)$ and $F(x,-\lambda)$ of equation (5) are linearly independent (see [2]) and

$$[F(x,\lambda), F(x,-\lambda)] = 2iK(\lambda).$$

Then, there are $A(\lambda), B(\lambda)$ matrices which independent of x such that

$$\Phi(x,\lambda) = F(x,\lambda)A(\lambda) + F(x,-\lambda)B(\lambda) \tag{8}$$

at $\operatorname{Im}\lambda = 0$, $|\lambda|>m$. Hence,

$$[F(x,\lambda), \Phi(x,\lambda)] = -2iK(\lambda)B(\lambda),$$

$$[F(x,-\lambda), \Phi(x,\lambda)] = 2iK(\lambda)A(\lambda).$$

Since

$$[F(x,\lambda), \Phi(x,\lambda)] = \widetilde{F}(0,\lambda)\Phi'(0,\lambda) - \widetilde{F'}(0,\lambda)\Phi(0,\lambda) = \widetilde{F}(0,\lambda)H - \widetilde{F'}(0,\lambda),$$

we get

$$A(\lambda) = \frac{1}{2i}K^{-1}(\lambda) - \widetilde{W}(-\lambda),\ B(\lambda) = -\frac{1}{2i}K^{-1}(\lambda) - \widetilde{W}(\lambda),$$

where

$$W(\lambda) = HF(0,\lambda) - F'(0,\lambda). \tag{9}$$

Substituting these expressions for matrices $A(\lambda)$ and $B(\lambda)$ into formula (8), we obtain

$$\Phi(x,\lambda) = \frac{1}{2i}(F(x,\lambda)K^{-1}(\lambda)\widetilde{W}(-\lambda) - F(x,-\lambda)K^{-1}(\lambda)\widetilde{W}(\lambda)) \tag{10}$$

for the case $\operatorname{Im}\lambda = 0$, $|\lambda| > m$.

Lemma 1. If $\operatorname{Im}\lambda = 0$, $|\lambda| > m$ then the matrix $W(\lambda)$ is non-singular.

Proof. From the condition $\Phi(0,\lambda) = 0$ and equality (10), it follows that

$$F(0,\lambda)\ K^{-1}(\lambda)\widetilde{W}(-\lambda) - F(0,-\lambda)K^{-1}(\lambda)\widetilde{W}(\lambda) = 2iI. \tag{11}$$

Due to (4) we have $e(x,-\lambda) = \overline{e(x,\lambda)}$. Similarly, from the equations (6) and (9) we have $W(-\lambda) = \overline{W(-\lambda)}$. Assume that there is a vector $\vec{a}$, such that $\widetilde{W}(\lambda)\vec{a} = 0$. Then, $\widetilde{W}(-\lambda)\vec{a} = 0$

and from (11) it follows that $2i\ I\ \vec{a}=0,\ \vec{a}=0$. The matrix is consistent $\widetilde{W}(\lambda)$ and it means that $W(\lambda)$ is not singular.

From formula (10) and boundary conditions (7) it follows that

$$F(0,\lambda)\,K^{-1}(\lambda)\,\widetilde{W}(-\lambda)-F(0,-\lambda)\,K^{-1}(\lambda)\,\widetilde{W}(\lambda)=2iI,$$

$$F^{'}(0,\lambda)\,K^{-1}(\lambda)\,\widetilde{W}(-\lambda)-F^{'}(0,-\lambda)\,K^{-1}(\lambda)\,\widetilde{W}(\lambda)=2i.$$

Multiplying the first equation by H and subtracting the second equation, we obtain the next formula

$$W(\lambda)\,K^{-1}(\lambda)\,\widetilde{W}(-\lambda)=W(-\lambda)\,K^{-1}(\lambda)\,\widetilde{W}(\lambda)\,. \tag{12}$$

Multiplying equation (12) by $W^{-1}(\lambda)$ from the left and by $\widetilde{W^{-1}}(-\lambda)$ from the right, we get the formula

$$W^{-1}(-\lambda)\,W(\lambda)\,K^{-1}(\lambda)=K^{-1}(\lambda)\,\widetilde{W}(\lambda)\,\widetilde{W^{-1}}(-\lambda)\,. \tag{13}$$

Under conditions $\operatorname{Im}\lambda=0,|\lambda|>m$, by using equations (10) and (13), the solution $\Phi(x,\lambda)$ is presented as follows

$$\Phi(x,\lambda)=\tfrac{1}{2i}(F(x,\lambda)\,K^{-1}(\lambda)-F(x,-\lambda)\,K^{-1}(\lambda)\,\widetilde{W}(\lambda)\,\widetilde{W^{-1}}(-\lambda))\widetilde{W}(-\lambda)$$

$$=\tfrac{1}{2i}\left(F(x,\lambda)\,K^{-1}(\lambda)-F(x,-\lambda)\,W^{-1}(-\lambda)\,W(\lambda)\,K^{-1}(\lambda)\right)\widetilde{W}(-\lambda)\,,$$

or

$$\Phi(x,\lambda)=\frac{1}{2i}\left(F(x,\lambda)-F(x,-\lambda)\,W^{-1}(-\lambda)\,W(\lambda)\right)K^{-1}(\lambda)\,\widetilde{W}(-\lambda)\,.$$

In the future, we will need asymptotic behavior of solution $\Phi(x,\lambda)$, when $\lambda\to\infty$ in the case $\operatorname{Im}\lambda\geq 0$. Denote by $E(x,\lambda)$, the solution of equation (5) whose asymptotic, when $\lambda\to\infty$. In the case $\operatorname{Im}\lambda\geq 0$ it is derived by (see [2])

$$E(x,\lambda)=\exp(-iK(x)\lambda)\,(I+O(1))\,.$$

Since

$$[F(x,\lambda)\,,E(x,\lambda)]=\lim_{x\to\infty}[c]=-2iK(\lambda)\,,$$

the solutions $F(x,\lambda)$ and $E(x,\lambda)$ are linearly independent in the case $\operatorname{Im}\lambda\geq 0,\ \lambda\neq m_j^2,\ j=\overline{1,n}$. Thus it is easy to get

$$\Phi(x,\lambda)=F(x,\lambda)\,A+E(x,\lambda)\,B, \tag{14}$$

where

$$A=\frac{1}{2i}K^{-1}(\lambda)\ \left(\widetilde{E}(0,\lambda)\,H-\widetilde{E}'(0,\lambda)\right),\ B\ =\ \frac{1}{2i}K^{-1}(\lambda)\ \widetilde{W}(\lambda)$$

with

$$\widetilde{E}(0,\lambda)\,H-\widetilde{E}'(0,\lambda)=\{C_{ij}(\lambda)\}\,,\ i,j=\overline{1,n},\ \widetilde{W}(\lambda)=\{w_{ij}(\lambda)\}\,,\ i,j=\overline{1,n}.$$

Using asymptotic formula (14) for solutions $F(x,\lambda)$ and $E(x,\lambda)$ when $x\to\infty$, we obtain the following asymptotic formulas

$$\varphi_{\alpha j}(x,\lambda)=\frac{1}{2iK_\alpha(\lambda)}\left(C_{j\alpha}(\lambda)\,e^{iK_\alpha(\lambda)x}-w_{j\alpha}(\lambda)\,e^{-iK_\alpha(\lambda)x}\right)+O(1) \tag{15}$$

for elements of the matrix $\Phi(x,\lambda)=\{\varphi_{\alpha j}(x,\lambda)\}$, when $x\to\infty$.

2 On the resolvent of problem (1),(2)

In the space $L_2(0,\infty;E_n)$, the boundary value problem (1), (2) defines the differential operator by

$$l(y) = -y'' + Q(x)y + My,\ 0 \le x < +\infty$$

for y which satisfies the condition

$$y'(0) - Hy(0) = 0.$$

The domain D_L of the operator L contains vector-functions $y(x) \in L_2(0,\infty;E_n)$, satisfying the following conditions:

1. $y'(x)$ exists and absolutely continuous at finite interval $[0,a]$,
2. $y'(0) - Hy(0) = 0$,
3. $l(y) \in L_2((0,\infty);E_n)$.

It is not difficult to verify that the operator L self adjoint. Let us define the core $R_z(x,t)$ of resolvent $R_z = (L - ZI)^{-1}$. Let us solve the boundary value problem

$$-y'' + Q(x)y + My = \lambda^2 y + f(x), \tag{16}$$

$$y'(0) - Hy(0) = 0, \tag{17}$$

where $f(x)$ is an arbitrary vector functions in $L_2((0,\infty);E_n)$.

If

$$[F(x,\lambda),\Phi(x,\lambda)] = [F(x,\lambda),\Phi(x,\lambda)]_{x=0} = \widetilde{F}(0,\lambda)H - \widetilde{F}'(0,\lambda) = \widetilde{W}(\lambda),$$

then the solutions of the homogeneous equation (1) are linearly independent in the case $W(\lambda) \neq 0$. We seek the solution $y(x,\lambda)$ of the problem (16), (17) in the form

$$y(x,\lambda) = F(x,\lambda)C_1(x,\lambda) + \Phi(x,\lambda)C_2(x,\lambda), \tag{18}$$

where $C_1(x,\lambda)$ and $C_2(x,\lambda)$ are some vector-functions. Applying constant variation method, we get a system of equations

$$F(x,\lambda)C_1'(x,\lambda) + \Phi(x,\lambda)C_2'(x,\lambda) = 0,$$

$$F'(x,\lambda)C_1'(x,\lambda) + \Phi'(x,\lambda)C_2'(x,\lambda) = -f(x).$$

Solving it, we have

$$C_1'(x,\lambda) = W(\lambda)\widetilde{\Phi}(x,\lambda)f(x), \tag{19}$$

$$C_2'(x,\lambda) = -\widetilde{W^{-1}}(\lambda)\widetilde{F}(x,\lambda)f(x). \tag{20}$$

From the asympotic equation (15) it follows that the elements of the matrix function $\Phi(x,\lambda)$ do not belong to space $L_2(0,\infty)$. Consequently, from condition $y(x,\lambda) \in L_2((0,\infty);E_n)$ and equality (20) follows that

$$C_2(+\infty,\lambda) = \lim_{x\to+\infty} C_2(x,\lambda) = 0.$$

Since

$$y'(x,\lambda) = F'(x,\lambda)C_1(x,\lambda) + \Phi'(x,\lambda)C_2(x,\lambda),$$

by using condition (19), we have

$$\left[F'(0,\lambda) - HF(0,\lambda)\right]C_1(0,\lambda) = 0.$$

So $W(\lambda)C_1(0,\lambda) = 0$. Since $detW(\lambda) \neq 0$, we obtain $C_1(x,\lambda) = 0$. Now, by integrating equality (19) from 0 to x, we get

$$C_1(x,\lambda) = W^{-1}(\lambda)\int_0^x \widetilde{\Phi}(t,\lambda) f(t)dt.$$

By using (20), we get

$$C_2(x,\lambda) = W^{-1}(\lambda)\int_x^{+\infty} \widetilde{F}(t,\lambda) f(t)dt.$$

Substituting these expressions in equality (18), we have

$$y(x,\lambda) = R_Z f = \int_0^{+\infty} R_z(x,\lambda) f(t)dt, Z = \lambda^2,$$

where

$$R_z(x,\lambda) = \begin{cases} F(x,\lambda) W^{-1}(\lambda)\widetilde{\Phi}(t,\lambda), t \le x, \\ \Phi(x,\lambda)\widetilde{W^{-1}}(\lambda)\widetilde{F}(t,\lambda), t \ge x. \end{cases} \tag{21}$$

This is the resolvent of operator L.

3 Spectrum of boundary value problem (1),(2)

Since boundary value problem (1),(2) is self-adjoint, from the expression (21) of the kernel R_z(x,t) the resolvents follow that the eigenvalues of the problem are squares of the scalar function $\omega(\lambda) = detw(\lambda)$ and has no other eigenvalues. Since the eigenvalues of the problem (1),(2) are real, the function, $\omega(\lambda) = detw(\lambda)$ can only be zeros on the real and the imaginary axis of the complex plane.

Theorem 1. The boundary value problem (1), (2) has

a) only the finite number of simple negative eigenvalues $-\varkappa_1^2, -\varkappa_{2,}^2 \ldots -\varkappa_q^2$,

b) the finite number of positive eigenvalues $\lambda_1^2, \lambda_{2,}^2 \ldots \lambda_r^2$ from the interval $[0, m^2]$, the multiplicity of eigenvalue λ_j^2 from the interval (m_p^2, m_{p+1}^2), $p = \overline{1,n}$, is not greater than $n-p$ and coincides with the rank of the matrix $W(\lambda, j)$.

Boundary value problem (1), (2) does not have its own values $Z > m^2$ and the continuous spectrum fills the semi-axiss.

Proof. Let $\lambda_{k,}^2 = -\varkappa_k^2$ be the eigenvalue of problem (1), (2) , i.e. $\omega(\varkappa_k) = detW(i\varkappa_k) = 0$. Then, it has a vector such that

$$W(i\varkappa_k)\overrightarrow{a}^{(k)} = HF(0,i\varkappa_k)\overrightarrow{a}^{(k)} - F'(0,i\varkappa_k)\overrightarrow{a}^{(k)} = 0.$$

From that it follows that the vector function $y_k(x) = F(x,i\varkappa_k)\overrightarrow{a}^{(k)}$ is a solution of problem (1)-(2). On the other hand, the elements of the matrix function $F(i\varkappa_k)$ belong to space $L_2(0,+\infty)$. Therefore, the vector-function $y_k(x)$ is the eigenfunction of the edge problem (1)-(2) corresponding to the eigenvalue $-\varkappa_k^2$. Without loss of generality, we will assume that the first component of $\overrightarrow{a}^{(k)}$equals to one and $m_i \neq m_j$ for $i \neq j$. From the asymptotic solution $F(x,\lambda)$, when $x \to +\infty$, it follows that

$$y_k(x) = e^{-\varkappa_k x}\omega_k(x), \lim_{x\to+\infty}\omega_k(x) = (1,0,\ldots,0) \tag{22}$$

uniformly along k.

Denoting by δ an exact lower bound of distances between two neighboring negative eigenvalues and we will prove that $\delta > 0$. Let us $\delta = 0$. Then, we can isolate the sequence of negative eigenvalues $\left\{-\varkappa_k^2\right\}$ and $\left\{-\widehat{\varkappa}_k^2\right\}$ such that $\lim\limits_{k\to+\infty}(\widehat{\varkappa}_k - \varkappa_k) = 0, \widehat{H}_k > H_k \geq 0$. Asymptotics $F(x,\lambda)$, implies that the set of zeros of the function $\omega(\lambda)$ is bounded. So $max_k\{\varkappa_k\} < A$. Later

$$\int_x^{+\infty} y_k(t)\,\widehat{y}_k(t)\,dt = \int_x^{+\infty} \widetilde{\omega}_k(t)\,\widehat{\omega}_k(t)\,e^{-(\varkappa_k+\widehat{\varkappa}_k)}dt, \tag{23}$$

where $\widehat{y}_k(x) = F(x, i\widehat{\varkappa}_k)\,\overrightarrow{b}^{(k)}$ the eigenfunction of the boundary problem (1),(2) is corresponding to the eigenvalue $-\widehat{\varkappa}_k^2$ and $\widehat{y}_k(x) = e^{-\widehat{\varkappa}_k x}\,x\,\widehat{\omega}_k(x)$. The condition (21) implies that, if $x > x_o$ is sufficient uniformly along k, then $\omega_k(x)\,\widehat{\omega}_k(x) > \frac{1}{2}$. Now, from (21) it follows that

$$\int_x^{+\infty} y_k(t)\,\widehat{y}_k(t)\,dt > \frac{1}{2}\int_{x_0}^{+\infty} e^{-(\varkappa_k+\widehat{\varkappa}_k)}dt = \frac{e^{-(\varkappa_k+\widehat{\varkappa}_k)}x_0}{2(\varkappa_k+\widehat{\varkappa}_k)} > \frac{e^{-Ax_0}}{4A}.$$

Since the boundary value problem (1)-(2) is self-adjoint, the vector-functions $y_k(x)$ and $\widehat{y}_k(x)$. Moreover

$0 = \int_x^{+\infty} y_k(t)\,\widehat{y}_k(t)\,dt = \int_0^{x_0}((y_k(t) - \widehat{y}_k(t))\widehat{y}_k(t)\,dt + \int_0^{x_0} y_k(t)\,\widehat{y}_k(t)\,dt + \int_x^{+\infty} y_k(t)\,\widehat{y}_k(t)\,dt.$
Passing to limit to the limit $k \to +\infty$, we find

$$0 = \lim_{k\to+\infty}\int_0^{x_0} y_k(t)\,\widehat{y}_k(t)\,dt + \lim_{k\to+\infty}\int_{x_0}^{+\infty} y_k(t)\,\widehat{y}_k(t)\,dt.$$

Thus,

$$\lim_{k\to+\infty}\int_{x_0}^{+\infty} y_k(t)\,,\widehat{y}_k(t)\,dt \leq 0.$$

Inequalities (22) and (23) lead to contradiction. Hence, $\delta > 0$, and it means, that the number of negative eigenvalues are finite. Now, let $\lambda^2 \in (m_p^2, m_{p+1}^2)$ be the eigenvalue of problem (1), (2). The corresponding eigenfunction has the form $\varphi(\lambda) = F(x,\lambda)\,\overrightarrow{a}$, $\overrightarrow{a} \neq a$. For $\lambda^2 \in (m_p^2, m_{p+1}^2)$ from formula (4) it follows that

$$ik_j(\lambda) = \begin{cases} i\lambda\sqrt{1 - \frac{m_j^2}{\lambda^2}}, if \quad j = 1, 2, \ldots, p, \\ -\sqrt{m_j^2 - \lambda^2}, if \quad j = p+1, \ldots, n. \end{cases}$$

Therefore, the elements of the first columns of the matrix function $F(x,\lambda)$ do not belong to space $L_2(0,\infty)$ and elements of the last $n-p$ columns belong to the space $L_2(0,+\infty)$. It is, because the eigenfunctions $\varphi(x)\,\epsilon L_2((0,+\infty); E_n)$ of the first p coordinates of the vector $\overrightarrow{a} = (a_1, a_2, \ldots, a_n)$ are zero, i.e. $a_1 = a_2 = \ldots = a_p = 0$. On the other hand, the eigenvector function $\varphi(x) = F(x,\lambda)\,\overrightarrow{a}$ satisfies the condition (2). Therefore, we have

$$a_{p+1}\omega_{j(p+1)}(\lambda) + \ldots + a_n\omega_{jn}(\lambda) = 0, j = 1, 2, \ldots, n,$$

where at least one of the numbers $a_{p+1}, \ldots, a_n$ is not zero. Therefore, the last $n-r$ columns of the matrix $W(\lambda) = \{\omega_{ij}(\lambda)\}_1^n$ are linearly independent, and therefore the multiplicity of eigenvalues $\lambda^2\epsilon(m_p^2, m_{p+1}^2)$ coincides with the rank of the matrix $W(\lambda)$. The finiteness of the number of eigenvalues from the interval $\left[0, m^2\right]$ is proved similarly to the case of negative eigenvalues. According to $\lambda^2\epsilon(m_1^2, +\infty)$ by Lemma 1 we get $detW(\lambda) \neq 0$. (1)-(2) does not have eigenvalues $(m_1^2, +\infty)$ from the interval. In the complex plane Z, the cut along the positive part of the real axis is a feature of the matrix function $W(\sqrt{z})$ and means the resolvent R_z by the formula (21). Hence, the half-axis $[0, +\infty]$ is the continuous spectrum of the boundary value problem (1)-(2).

Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

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