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Recent advances in PDE and their applications
Preface

This issue is a collection of 15 selected papers of foreign and national scientists. All these have been accepted after peer-reviewing and contain numerous new results in the fields of construction and investigation of solutions of well-posed and ill-posed boundary value problems for partial differential equations and their related applications. The authors of the selected papers are from different countries: Turkey, Kazakhstan, USA, Russian Federation, Azerbaijan, Kirgizistan, Uzbekistan, Turkmenistan, Pakistan and Nijerya. We are especially pleased with the fact that many articles are written by co-authors who work in different universities around the world.

Keywords: partial differential equations, hyperbolic-parabolic equations, integro-differential equations, boundary value problem, Dirichelt problem, well-posedness, regular solutions, numerical methods and solutions, difference scheme, involution, stability.

Guest-Editors: *A. Ashyralyev* and *M. Sadybekov*

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Parabolic time dependent source identification problem with involution and Neumann condition

A time dependent source identification problem for parabolic equation with involution and Neumann condition is studied. The well-posedness theorem on the differential equation of the source identification parabolic problem is established. The stable difference scheme for the approximate solution of this problem and its stability estimates are presented. Numerical results are given.

Keywords: well-posedness, coercive stability, source identification, exact estimates, boundary value problem.

Introduction

The theory and applications of source identification problems (SIPs) for partial differential equations have been studied and for references we refer to articles [1–9] and the references given therein. Also, numerous source identification problems for hyperbolic-parabolic equations and their applications have been investigated too (see, e.g., [10–13] and the references given therein). In the last decade partial differential equations with involutions were investigated by several authors including Ashyralyev and Sarsenbi [14–17]. However, source identification problems for parabolic equations with involution still need more investigating.

The present paper is devoted to the study of a time SIP for parabolic equation with involution and Neumann condition. The stability theorem on the differential equation of the source identification parabolic problem is proved. The stable difference scheme (DS) for the approximate solution of this problem is constructed. Furthermore, stability estimates for the DS of the time source identification parabolic problem are established. Numerical results are provided.

Stability and coercive stability of the differential problem

We consider the time SIP

$$\begin{cases} u_t(t,x) - (a(x)u_x(t,x))_x - \beta\left(a(-x)u_x(t,-x)\right)_x + \delta u(t,x) \\ = p(t)q(x) + g(t,x),\ -l < x < l,\ 0 < t < T, \\ u(0,x) = \varphi(x),\ -l \le x \le l, \\ u_x(t,-l) = u_x(t,l) = 0,\ \displaystyle\int_0^l u(t,x)dx = \gamma(t),\ 0 \le t \le T \end{cases} \tag{1}$$

for the one dimensional parabolic differential equation with involution and Neumann boundary condition. Throughout this paper, we assume that the following conditions hold

$$\overline{a} \ge a(x) = a(-x) \ge \underline{a} > 0,\ x \in (-\ell,\ell),\ \underline{a} - \overline{a}|\beta| \ge 0,\ \delta \ge 0,$$

$$q'(-l) = q'(l) = 0,\ \int_0^l q(x)\,dx \ne 0.$$

Under compatibility conditions, identification problem (1) has a unique solution $(u(t,x),p(t))$ for the smooth functions $g(t,x),\ (t,x) \in (0,T)\times(-l,l),\ a(x), q(x), x \in (-l,l)$ and $\gamma(t), t \in [0,T],\ \varphi(x),\ x \in [-l,l]$.

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Assume that H is a Hilbert space and A is the self-adjoint positive-definite operator defined by the formula

$$Az=-\frac{d}{dx}\left(a(x)\frac{dz(x)}{dx}\right)-\beta\frac{d}{dx}\left(a(-x)\frac{dz(-x)}{dx}\right)+\delta z(x) \tag{2}$$

with domain

$$D(A)=\{z:z,z^{''}\in L_2[-l,l],z^{'}(-l)=z^{'}(l)=0\}.$$

Here and in the rest of this paper, $C_0^{\alpha}\left([0,T],H\right)(0<\alpha<1)$ stands for Banach spaces of all abstract continuous functions $\varphi(t)$ defined on $[0,T]$ with values in H satisfying a Hölder condition with weight t^{α} for which the following norm is finite

$$\|\varphi\|_{C_0^{\alpha}([0,T],H)}=\|\varphi\|_{C([0,T],H)}+\sup_{0\leq t<t+\tau\leq T}\frac{(t+\tau)^{\alpha}\|\varphi(t+\tau)-\varphi(t)\|_H}{\tau^{\alpha}}.$$

Here, $C\left([0,T],H\right)$ stands for the Banach space of all abstract continuous functions $\varphi(t)$ defined on $[0,T]$ with values in H equipped with the norm

$$\|\varphi\|_{C([0,T],H)}=\max_{0\leq t\leq T}\|\varphi(t)\|_H.$$

Moreover, let the Sobolev space $W_2^2[-\ell,\ell]$ be defined as the set of all functions $v(x)$ defined on $[-\ell,\ell]$ such that both $v(x)$ and $v''(x)$ are locally integrable in $L_2[-\ell,\ell]$, equipped with the norm

$$\|v\|_{W_2^2[-\ell,\ell]}=\left(\int_{-\ell}^{\ell}|v(x)|^2\,dx\right)^{1/2}+\left(\int_{-\ell}^{\ell}|v''(x)|^2\,dx\right)^{1/2}.$$

Theorem 1. Assume that $f(t,x)$ and $\zeta\left(t\right)$ are continuously differentiable functions. Then the SIP (1) has a unique solution $u\in C\left(L_2\left[-l,l\right]\right)$ and $p\in C\left[0,T\right]$, and for the solution of SIP (1) the following stability estimates hold

$$\|u_t\|_{C(L_2[-l,l])}+\|u\|_{C\left(W_2^2[-l,l]\right)}+\|p\|_{C[0,T]}\leqslant M\left(q,\delta\right)\Big[\|\varphi\|_{W_2^2[-l,l]}$$
$$+\|g\left(0,\cdot\right)\|_{L_2[-l,l]}+\|g_t\|_{C(L_2[-l,l])}+\|\zeta_t\|_{C[0,T]}\Big].$$

Theorem 2. Assume that $g(t,x)$ and $\zeta\left(t\right)$ are continuously differentiable functions and $\zeta_t\left(t\right)$ is satisfying a Hölder condition with the ; weight t^{α}. Then the SIP (1) has a unique solution $u\in C_0^{\alpha}\left([0,T],\ L_2\left[-l,l\right]\right)$ and $p\in C_0^{\alpha}\left[0,T\right]$. For the solution of SIP (1) the following coercive stability estimates hold:

$$\|u_t\|_{C_0^{\alpha}([0,T],L_2[-l,l])}+\|u\|_{C_0^{\alpha}\left([0,T],W_2^2[-l,l]\right)}+\|p\|_{C_0^{\alpha}[0,T]}\leqslant M\left(q,\delta\right)\Big[\|\varphi\|_{W_2^2[-l,l]}$$
$$+\tfrac{1}{\alpha(1-\alpha)}\|g\|_{C_0^{\alpha}([0,T],L_2[-l,l])}+\|\zeta_t\|_{C_0^{\alpha}[0,T]}\Big].$$

Proof. Denoted as

$$u\left(t,x\right)=w\left(t,x\right)+\eta\left(t\right)q\left(x\right), \tag{3}$$

where

$$\eta\left(t\right)=\int_0^t p\left(s\right)ds,\quad \eta\left(0\right)=0 \tag{4}$$

and $w\left(t,x\right)$ is the solution of the following problem

$$\begin{cases} w_t\left(t,x\right)-\left(a\left(x\right)w_x\left(t,x\right)\right)_x+\delta w(t,x) \\ =g\left(t,x\right)+\eta\left(t\right)\left[\left(a\left(x\right)q_x\left(x\right)\right)_x-\delta q\left(x\right)\right], \\ x\in\left(-l,l\right),t\in\left(0,T\right), \\ w\left(0,x\right)=\varphi\left(x\right),x\in\left[-l,l\right], \\ w_x\left(t,-l\right)=w_x\left(t,l\right)=0,t\in\left[0,T\right]. \end{cases} \tag{5}$$

Applying the condition

$$\int_0^l u\left(t,x\right)dx=\zeta\left(t\right)$$

and formula (3), we can write

$$\eta\left(t\right)=q\left(-\zeta\left(t\right)+\int_0^l w\left(t,x\right)dx\right), \tag{6}$$

where

$$q=\frac{1}{\int_0^l q\left(x\right)dx}.$$

Applying formulas (4) and (6), we get

$$p\left(t\right)=q\left(-\zeta'\left(t\right)+\int_0^l w_t\left(t,x\right)dx\right). \tag{7}$$

Applying $\int_0^l q(x)dx\neq 0$, we get the estimate

$$|p(t)|\leqslant K_1\left(q\right)\left[\left|\zeta_t\left(t\right)\right|+||w_t(t,\cdot)||_{L_2[-l,l]}\right] \tag{8}$$

for each $t\in\left[0,T\right]$. From (7) and (8) follows it

$$\|p\|_{C([0,T])}\leq K_1\left(q\right)\left[\|\zeta_t\|_{C[0,T]}+\|w_t\|_{C([0,T],L_2[-l,l])}\right], \tag{9}$$

$$\|p\|_{C_0^\alpha[0,T]}\leq K_2\left(q\right)\left[\|\zeta_t\|_{C_0^\alpha[0,T])}+\|w_t\|_{C_0^\alpha([0,T],L_2[-l,l])}\right]. \tag{10}$$

Applying (3), we get

$$u_t(t,x)=w_t(t,x)+p(t)q(x)$$

and

$$\|u_t\|_{C([0,T],L_2[-l,l])}\leqslant\|w_t\|_{C([0,T],L_2[-l,l])}+\|p\|_{C[0,T])}\|q\|_{L_2[-l,l])},$$
$$\|u_t\|_{C_0^\alpha([0,T],L_2[-l,l])}\leqslant\|w_t\|_{C_0^\alpha([0,T],L_2[-l,l])}+\|p\|_{C_0^\alpha[0,T])}\|q\|_{L_2[-l,l])}.$$

Therefore, the following theorems will complete the proof of Theorem 1 and 2.

Theorem 3. Under assumptions of Theorem 1, in $C\left(\left[0,T\right],L_2\left[-l,l\right]\right)$ the problem (5) has a unique solution and the following stability estimate is satisfied:

$$\|w_t\|_{C([0,T],L_2[-l,l])}\leqslant K_2\left(q,\delta\right)\left[\|\varphi\|_{W_2^2[-l,l]}+\left|\zeta\left(0\right)\right|\right.$$
$$\left.+\left\|g\left(0,\cdot\right)\right\|_{L_2[-l,l]}+\|g_t\|_{C([0,T],L_2[-l,l])}+\|\zeta_t\|_{C[0,T]}\right].$$

Theorem 4. Under assumptions of Theorem 2, in $C_0^\alpha\left(\left[0,T\right],L_2[-l,l]\right)$ the problem (5) has a unique solution and the following coercive stability estimate is satisfied:

$$\|w_t\|_{C_0^\alpha([0,T],L_2[-l,l])}\leqslant K_2\left(q,\delta\right)\left[\|\varphi\|_{W_2^2[-l,l]}\right.$$
$$\left.+\tfrac{1}{\alpha(1-\alpha)}\|g\|_{C_0^\alpha([0,T],L_2[-l,l])}+\|\zeta_t\|_{C_0^\alpha[0,T]}\right].$$

Proof. Problem (5) can be written in the following abstract form

$$\begin{cases} w'(t)+Aw(t)=-\eta(t)Aq+g(t),\ 0<t<T,\\ w(0)=\varphi \end{cases} \tag{11}$$

in a Hilbert space $H=L_2[-\ell,\ell]$ with the space operator $A=A^x$ defined by the formula (2). Here $g(t)=g(t,x)$ is given abstract function, $w(t)=w(t,x)$ is unknown function, and $q=q(x)$ is the unknown element of $L_2[-\ell,\ell]$. The proofs of Theorems 3 and 4 are based on estimates (8), (9) and (10), theorems on stability and coercive stability of the abstract problem (11) [9], the integral inequality and the self-adjointness and positive definiteness of the space operator A^x defined by formula (2) [15].

Stability and coercive stability of DS

Let $\alpha \in (0,1)$ be a given number and $C_\tau^\alpha(H) = C_0^\alpha([0,T]_\tau, H]), C_\tau(H) = C([0,T]_\tau, H)$ be Banach spaces of all H-valued mesh functions $w_\tau = \{w_k\}_{k=0}^N$ defined on

$$[0,T]_\tau = \{t_k = k\tau, 0 \leqslant k \leqslant N, N\tau = T\}$$

with the corresponding norms

$$\|w_\tau\|_{C_\tau(H)} = \max_{0\le k\le N} \|w_k\|_H,$$

$$\|w_\tau\|_{C_\tau^\alpha(H)} = \sup_{1\le k<k+n\le N} (N-n)^{-\alpha}(k)^\alpha \|w_{k+n} - w_k\|_H + \|w_\tau\|_{C_\tau(H)}.$$

Moreover, let $L_{2h} = L_2[-l,l]_h$ and $W_{2h}^2 = W_2^2[-l,l]_h$ be normed spaces of all mesh functions $\gamma^h(x) = \{\gamma_n\}_{n=-M}^M$ defined on

$$[-l,l]_h = \{x_n = nh, -M \leqslant n \leqslant M, Mh = l\}$$

equipped with norms

$$\|\gamma^h\|_{L_{2h}} = \left(\sum_{x\in[-l,l]_h} |\gamma^h(x)|^2 h\right)^{1/2}$$

and

$$\|\gamma^h\|_{W_{2h}^2} = \|\gamma^h\|_{L_{2h}} + \left(\sum_{x\in[-l,l]_h} \left|(\gamma^h)_{x\overline{x},j}\right|^2 h\right)^{1/2},$$

respectively. Moreover, we introduce the difference operator A_h^x defined by the formula

$$A_h^x u^h(x) = \{-(a(x)u_{\overline{x}}(x))_{x,r} - \beta\,(a(-x)u_{\overline{x}}(-x))_{x,r} + \delta u_r\}_{-M+1}^{M-1}, \tag{12}$$

acting in the space of mesh functions $u^h(x) = \{u_n\}_{n=-M}^M$ defined on $[-l,l]_h$ satisfying the conditions $u_M - u_{M-1} = u_{-M} - u_{-M+1} = 0$. For the numerical solution $\{u_k^h(x)\}_{k=0}^N$ of SIP (1) we present DS of the first order of approximation

$$\begin{cases} \frac{u_n^k - u_n^{k-1}}{\tau} - \frac{1}{h}\left(a_{n+1}\frac{u_{n+1}^k - u_n^k}{h} - a_n\frac{u_n^k - u_{n-1}^k}{h}\right) - \frac{\beta}{h}\left(a_{-n+1}\frac{u_{-n+1}^k - u_{-n}^k}{h} - a_{-n}\frac{u_{-n}^k - u_{-n-1}^k}{h}\right) \\ +\delta u_n^k = p_k q_n + g_n^k, g_n^k = g(t_k, x_n), t_k \in [0,T]_\tau, x_n \in [-l,l]_h, \quad k \in \overline{1,N}, \quad n \in \overline{1,M-1}, \\ u_n^0 = \varphi_n, \varphi_n = \varphi(x_n), n \in \overline{0,M}, \\ u_M^k - u_{M-1}^k = u_{-M+1}^k - u_{-M}^k = 0, \sum_{i=1}^M u_i^k h = \zeta_k, \zeta_k = \zeta(t_k), \quad k \in \overline{0,N}. \end{cases} \tag{13}$$

Here it is assumed that $q_M - q_{M-1} = q_{-M} - q_{-M+1} = 0$, and $\sum_{m=1}^M q_m h \neq 0$. Let us give the following results on the stability of DS (13).

Theorem 5. For the solution of DS (13), the stability estimate

$$\left\|\left\{\frac{1}{\tau}\left(u_k^h - u_{k-1}^h\right)\right\}_{k=1}^N\right\|_{C_\tau(L_{2h})} + \left\|\{u_k^h\}_{k=1}^N\right\|_{C_\tau(W_{2h}^2)} + \left\|\{p_k\}_{k=1}^N\right\|_{C[0,T]_\tau}$$

$$\leqslant K(q)\left[\|\varphi^h\|_{W_{2h}^2} + \|g_1^h\|_{L_{2h}} + |\zeta_0|\right.$$

$$\left. + \left\|\left\{\frac{1}{\tau}\left(g_k^h - g_{k-1}^h\right)\right\}_{k=2}^N\right\|_{C_\tau(L_{2h})} + \left\|\left\{\frac{1}{\tau}\left(\zeta_k - \zeta_{k-1}\right)\right\}_{k=1}^N\right\|_{C[0,T]_\tau}\right],$$

and coercive stability estimate

$$\left\|\left\{\frac{1}{\tau}\left(u_k^h-u_{k-1}^h\right)\right\}_{k=1}^N\right\|_{C_\tau^\alpha(L_{2h})}+\left\|\left\{u_k^h\right\}_{k=1}^N\right\|_{C_\tau^\alpha\left(W_{2h}^2\right)}+\left\|\left\{p_k\right\}_{k=1}^N\right\|_{C_0^\alpha[0,T]_\tau}$$

$$\leqslant K\left(q\right)\left[\left\|\varphi^h\right\|_{W_{2h}^2}\right.$$

$$\left.+\frac{1}{\alpha\left(1-\alpha\right)}\left\|\left\{g_k^h\right\}_{k=1}^N\right\|_{C_\tau^\alpha(L_{2h})}+\left\|\left\{\frac{1}{\tau}\left(\zeta_k-\zeta_{k-1}\right)\right\}_{k=1}^N\right\|_{C_0^\alpha[0,T]_\tau}\right]$$

hold.

Proof. We will use

$$u_n^k=w_n^k+\eta_k q_n, \tag{14}$$

where

$$q_n=q(x_n), \eta_k=\sum_{m=1}^k p_m\tau.$$

It is easy to use $\left\{w_k^h\left(x\right)\right\}_{k=0}^N$ as the solution of the following DS

$$\begin{cases}\frac{w_n^k-w_n^{k-1}}{\tau}-\left[\frac{1}{h}\left(a_{n+1}\frac{w_{n+1}^k-w_n^k}{h}-a_n\frac{w_n^k-w_{n-1}^k}{h}\right)\right.\\ -\frac{\beta}{h}\left(a_{-n+1}\frac{w_{-n+1}^k-w_{-n}^k}{h}-a_{-n}\frac{w_{-n}^k-w_{-n-1}^k}{h}\right)+\delta w_n^k\\ =-\left[-\frac{1}{h}\left(q_{n+1}\frac{w_{n+1}^k-w_n^k}{h}-q_n\frac{w_n^k-w_{n-1}^k}{h}\right)\right.\\ \left.-\frac{\beta}{h}\left(q_{-n+1}\frac{w_{-n+1}^k-w_{-n}^k}{h}-q_{-n}\frac{w_{-n}^k-w_{-n-1}^k}{h}\right)+\delta q_n^k\eta_k\right]+g_n^k,\\ k\in\overline{1,N}, n\in\overline{1,M-1},\\ w_n^0=\varphi_n, n\in\overline{0,M}\quad,\\ w_M^k-w_{M-1}^k=w_{-M+1}^k-w_{-M}^k=0, k\in k\in\overline{0,N}.\end{cases} \tag{15}$$

Now we estimate $|p_k|$. Using the condition $\sum_{m=1}^M u_m^k h=\zeta_k$ and (14), we obtain

$$\eta_k=b_1\left(\zeta_k-\sum_{m=1}^M w_m^k h\right),$$

where

$$b_1=\frac{1}{\sum_{m=1}^M q_m h}.$$

Then,

$$p_k=\frac{b_1}{\tau}\left(\zeta_k-\zeta_{k-1}-\sum_{m=1}^M(w_m^k-w_m^{k-1})h\right). \tag{16}$$

Applying the Cauchy-Schwartz inequality, we get

$$|p_k|\leqslant|b_1|\,[|\frac{\zeta_k-\zeta_{k-1}}{\tau}|+\sum_{m=1}^M|\frac{w_m^k-w_m^{k-1}}{\tau}|h]$$

$$\leqslant K(b)\left[\left|\frac{\zeta_k-\zeta_{k-1}}{\tau}\right|+\left\|\frac{w_k^h-w_{k-1}^h}{\tau}\right\|_{L_{2h}}\right] \tag{17}$$

for every $1\leqslant k\leqslant N$, and

$$\left\|\left\{p_k\right\}_{k=1}^N\right\|_{C[0,T]_\tau}$$

$$\leqslant K(b)\left[\left\|\left\{\frac{\zeta_k-\zeta_{k-1}}{\tau}\right\}_{k=1}^{N}\right\|_{C[0,T]_\tau}+\left\|\left\{\frac{w_k^h-w_{k-1}^h}{\tau}\right\}_{k=1}^{N}\right\|_{C_\tau(L_{2h})}\right].$$

Moreover, using (16), we can write

$$\left\|\{p_k\}_{k=1}^N\right\|_{C_0^\alpha[0,T]_\tau}$$

$$\leqslant K(b)\left[\left\|\left\{\frac{\zeta_k-\zeta_{k-1}}{\tau}\right\}_{k=1}^{N}\right\|_{C_0^\alpha[0,T]_\tau}+\left\|\left\{\frac{w_k^h-w_{k-1}^h}{\tau}\right\}_{k=1}^{N}\right\|_{C_\tau^\alpha(L_{2h})}\right]. \tag{18}$$

Applying (14), we obtain

$$\frac{u_n^k-u_n^{k-1}}{\tau}=\frac{w_n^k-w_n^{k-1}}{\tau}+p_kq_n.$$

From that it follows

$$\left\|\left\{\frac{1}{\tau}\left(u_k^h-u_{k-1}^h\right)\right\}_{k=1}^{N}\right\|_{C_\tau(L_{2h})}$$

$$\leqslant\left\|\left\{\frac{1}{\tau}\left(w_k^h-w_{k-1}^h\right)\right\}_{k=1}^{N}\right\|_{C_\tau(L_{2h})}+\left\|\{p_k\}_{k=1}^N\right\|_{C[0,T]_\tau}\left\|q^h\right\|_{L_{2h}} \tag{19}$$

and

$$\left\|\left\{\frac{1}{\tau}\left(u_k^h-u_{k-1}^h\right)\right\}_{k=1}^{N}\right\|_{C_\tau^\alpha(L_{2h})}$$

$$\leqslant\left\|\left\{\frac{1}{\tau}\left(w_k^h-w_{k-1}^h\right)\right\}_{k=1}^{N}\right\|_{C_\tau^\alpha(L_{2h})}+\left\|\{p_k\}_{k=1}^N\right\|_{C_0^\alpha[0,T]_\tau}\left\|q^h\right\|_{L_{2h}}.$$

Therefore, the following theorem will complete the proof of Theorem 5.

Theorem 6. For the solution of DS (15), the stability estimate

$$\left\|\left\{\frac{1}{\tau}\left(w_k^h-w_{k-1}^h\right)\right\}_{k=1}^{N}\right\|_{C_\tau(L_{2h})}\leqslant K_3(a)\left[\left\|\varphi^h\right\|_{W_{2h}^2}+\left\|g_1^h\right\|_{L_{2h}}+|\zeta_0|\right.$$

$$\left.+\left\|\left\{\frac{1}{\tau}\left(g_k^h-g_{k-1}^h\right)\right\}_{k=2}^{N}\right\|_{C_\tau(L_{2h})}+\left\|\left\{\frac{\zeta_k-\zeta_{k-1}}{\tau}\right\}_{k=1}^{N}\right\|_{C[0,T]_\tau}\right]$$

and coercive stability estimate

$$\left\|\left\{\frac{1}{\tau}\left(w_k^h-w_{k-1}^h\right)\right\}_{k=1}^{N}\right\|_{C_\tau^\alpha(L_{2h})}$$

$$\leqslant K_3(q)\left[\left\|\varphi^h\right\|_{W_{2h}^2}+\frac{1}{\alpha(1-\alpha)}\left\|\{g_k^h\}_{k=1}^N\right\|_{C_\tau^\alpha}+\left\|\left\{\frac{\zeta_k-\zeta_{k-1}}{\tau}\right\}_{k=1}^{N}\right\|_{C_0^\alpha[0,T]_\tau}\right]$$

hold.

Proof. Problem (15) can be written in the following abstract form

$$\begin{cases}\frac{w_k^h-w_{k-1}^h}{\tau}+A^hw_k^h=g_k^h-A^hq^h\eta_k,\\ t_k=k\tau,1\leqslant k\leqslant N,w_0^h=\varphi^h\end{cases} \tag{20}$$

in a Hilbert space $H=L_{2h}$ with the space operator $A^h=A_h^x$ defined by the formula (12). Here, $g_k^h=g_k^h(x)$ is given abstract mesh function, $w_k^h=w_k^h(x)$ is unknown mesh function and $q^h=q^h(x)$ is the unknown element of L_{2h}. The proof of Theorem 6 is based on estimates (17), (18) and (19), theorems on stability and coercive stability of the abstract problem (20) (see [9]), difference analogy of integral inequalities, and the self-adjointness and positive definiteness of the difference operator A_h^x defined by the formula (12) [15].

Numerical experiment

In this section a numerical computation to approximate solution of a time dependent source identification problem with involution and Neumann conditions is considered to support the theoretical results. We use the first order of accuracy difference schemes. The error analysis is given.

We consider

$$\begin{cases} u_t\left(t,x\right)-u_{xx}\left(t,x\right)-\frac{1}{2}u_{xx}\left(t,-x\right)+u(t,x) \\ =p(t)\left(1+\cos x\right)+\left(\dfrac{\cos x}{2}-1\right)e^{-t}, \quad x\in\left(-\pi,\pi\right),t\in\left(0,\pi\right), \\ u\left(0,x\right)=1+\cos x, x\in\left[-\pi,\pi\right], \\ u_x\left(t,-\pi\right)=u_x\left(t,\pi\right)=0,\ t\in\left[0,\pi\right], \\ \int_0^\pi u\left(t,s\right)ds=\pi e^{-t},\ t\in\left[0,\pi\right] \end{cases} \tag{21}$$

for parabolic equation with involution and Neumann condition. The integral condition is given as an overdetermined condition. The exact solution of this problem is

$$\begin{aligned} u\left(t,x\right) &= \left(1+\cos x\right)e^{-t}, -\pi\leq x\leq\pi, 0\leq t\leq\pi, \\ p\left(t\right) &= e^{-t},\ t\in\left[0,\pi\right]. \end{aligned}$$

Here we denote the set $[0,\pi]_\tau\times[-\pi,\pi]_h$ of all grid points

$$[0,\pi]_\tau\times[-\pi,\pi]_h=\{(t_k,x_n): t_k=k\tau, 0\leq k\leq N,$$

$$N\tau=\pi, x_n=nh, -M\leq n\leq M, Mh=\pi\}.$$

For obtaining the solution to problem (21) we apply the substitution

$$u\left(t,x\right)=w\left(t,x\right)+\eta\left(t\right)\left(1+\cos x\right),$$

where

$$\eta\left(t\right)=\int_0^t p\left(s\right)ds,\ \eta\left(0\right)=0.$$

One can show that after the substitution, problem (21) turns to

$$\begin{cases} w_t\left(t,x\right)-w_{xx}\left(t,x\right)-\frac{1}{2}w_{xx}\left(t,-x\right)+w(t,x) \\ =-\left(\dfrac{5\cos x}{2}+1\right)\eta\left(t\right)+\left(\dfrac{\cos x}{2}-1\right)e^{-t}, \quad x\in\left(-\pi,\pi\right),\ t\in\left(0,\pi\right), \\ w\left(0,x\right)=1+\cos x,\ x\in\left[-\pi,\pi\right], \\ w_x\left(t,-\pi\right)=w_x\left(t,\pi\right)=0,\ t\in\left[0,\pi\right]. \end{cases} \tag{22}$$

Moreover, using the overdetermined condition we can write

$$\int_0^\pi u\left(t,s\right)ds=\int_0^\pi w\left(t,s\right)ds+\eta\left(t\right)\int_0^\pi\left(1+\cos s\right)ds=\pi e^{-t}$$

and

$$\eta\left(t\right)=\frac{\pi e^{-t}-\int_0^\pi w\left(t,s\right)ds}{\pi}.$$

For the numerical solution of (22), we present the first order of accuracy difference scheme

$$\begin{cases} \tau^{-1}\left(w_n^k-w_n^{k-1}\right)-h^{-2}\left(w_{n+1}^k-2w_n^k+w_{n-1}^k\right) \\ -\frac{1}{2}h^{-2}\left(w_{-n+1}^k-2w_{-n}^k+w_{-n-1}^k\right)+w_n^k \\ -\frac{1}{\pi}\left(\dfrac{5\cos x_n}{2}+1\right)\sum_{n=0}^{M-1}w_n^k\ h \\ =-\left(2+2\cos x_n\right)e^{-t_k}, \\ 1\leq k\leq N,\ -M+1\leq n\leq M-1, \\ w_n^0=1+\cos x_n,\ -M\leq n\leq M, \\ w_{-M+1}^k-w_{-M}^k=w_M^k-w_{M-1}^k=0,\ 0\leq k\leq N. \end{cases} \tag{23}$$

For obtaining the solution of difference scheme (23), we rewrite it in the matrix form

$$A\,W^k + B\,W^{k-1} = R\varphi^k, \quad 1 \le k \le N,\ W^0 = \varphi\ , \tag{24}$$

where

$$A = \begin{bmatrix} 1 & -1 & 0 & 0 & . & . & . & . & . & 0 & 0 & 0 & 0 \\ a & b & a & 0 & . & . & d_{-M+1} & . & . & d_{-M+1} & d_{-M+1}+c & d_{-M+1}-a & c \\ 0 & a & b & a & . & . & d_{-M+2} & . & . & d_{-M+2}+c & d_{-M+2}-a & d_{-M+2}+c & 0 \\ . & . & . & . & . & . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & . & a+c & b-a+d_0 & a+c+d_0 & . & d_0 & d_0 & d_0 & 0 \\ . & . & . & . & . & . & . & . & . & . & . & . & . \\ 0 & c & -a & c & . & . & d_{M-2} & . & . & a+d_{M-2} & b+d_{M-2} & a+d_{M-2} & 0 \\ c & -a & c & 0 & . & . & d_{M-1} & . & . & d_{M-1} & d_{M-1} & d_{M-1} & a \\ 0 & 0 & 0 & 0 & . & . & . & . & . & 0 & 0 & 1 & -1 \end{bmatrix},$$

$$B = \begin{bmatrix} 0 & 0 & 0 & 0 & . & 0 & 0 & 0 \\ 0 & e & 0 & 0 & . & 0 & 0 & 0 \\ 0 & 0 & e & 0 & . & 0 & 0 & 0 \\ . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & . & e & 0 & 0 \\ 0 & 0 & 0 & 0 & . & 0 & e & 0 \\ 0 & 0 & 0 & 0 & . & 0 & 0 & 0 \end{bmatrix}$$

and

$$a = -\frac{1}{h^2},\ b = \frac{1}{\tau} + \frac{2}{h^2} + 1,\ c = -\frac{1}{2h^2},\ e = \frac{1}{\tau},$$

$$d_i = -\frac{h}{\pi}\left(\frac{5\cos x_i}{2} + 1\right), i = -M+1, -M+2, ...M-1,$$

$$W^s = \begin{bmatrix} W^s_{-M} \\ \vdots \\ W^s_M \end{bmatrix} \text{for } s = k, k-1,$$

$$R = \begin{bmatrix} 1 & 0 & . & 0 \\ 0 & 1 & . & 0 \\ . & . & . & . \\ 0 & 0 & . & 1 \end{bmatrix}, \varphi^k = \begin{bmatrix} 0 \\ \varphi^k_{-M+1} \\ \vdots \\ \varphi^k_{M-1} \\ 0 \end{bmatrix}.$$

So, we have a first order difference equation with respect to k with matrix coefficients. From (24) it follows that

$$W^k = -A^{-1}BW^{k-1} + A^{-1}R\varphi^k \ \ k = 1, \cdots, N.$$

In the second step, using the formulas

$$u^k_n = w^k_n + \eta_k\left(1 + \cos x_n\right),\ 0 \le k \le N,\ -M \le n \le M,$$

$$\begin{aligned} \eta_k &= \frac{\pi e^{-t_k} - \sum_{n=0}^{M-1} w^k_n\, h}{\pi},\ 1 \le k \le N,\ \eta_0 = 0, \\ p_k &= \frac{\eta_k - \eta_{k-1}}{\tau},\ 1 \le k \le N, \end{aligned}$$

we can find the approximate solutions for $u(t,x)$ and $p(t)$.

We compute the error between the exact solution and numerical solution by

$$\begin{cases} \|E_u\|_\infty = \max\limits_{0\le k\le N, -M\le n\le M}\left|u\left(t_k, x_n\right) - u^k_n\right|, \\ \|E_\eta\|_\infty = \max\limits_{1<k<N}\left|\eta\left(t_k\right) - \eta_k\right|, \\ \|E_p\|_\infty = \max\limits_{1<k<N}\left|p\left(t_k\right) - p_k\right|, \end{cases}$$

where $u(t,x)$, $p(t)$, $\eta(t)$ represent the exact solutions, u_n^k represents the numerical solution at (t_k, x_n), and p_k and η_k represent the numerical solutions at t_k. The numerical results are given in Table 1.

Table 1

$Errors$	$\|E_u\|_\infty$	$\|E_\eta\|_\infty$	$\|E_p\|_\infty$
$N=M=30$	$6.6666 \cdot 10^{-2}$	$7.3894 \cdot 10^{-2}$	$1.1917 \cdot 10^{-1}$
$N=M=60$	$3.3333 \cdot 10^{-2}$	$3.6655 \cdot 10^{-2}$	$6.6532 \cdot 10^{-2}$
$N=M=120$	$1.6667 \cdot 10^{-2}$	$1.8262 \cdot 10^{-2}$	$3.5167 \cdot 10^{-2}$
$N=M=240$	$8.3333 \cdot 10^{-3}$	$9.1165 \cdot 10^{-3}$	$1.8081 \cdot 10^{-2}$

Conclusion

In this paper we considered a time dependent source of identification problem for parabolic equation with involution and Neumann condition. The theoretical considerations that prove well-posedness theorem on the differential equation of the source identification parabolic problem and stability estimates for the difference scheme of the source identification parabolic problem were given. To support the theoretical results by a numerical experiment we constructed a stable difference scheme for the approximate solution of the problem. Obtained results given in Table 1 support the theoretical results.

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Инволюциямен және Дирихле шартымен сәйкестендірудің параболалық мәселесі туралы ескерту

Инволюция және Дирихле шарты бар параболалық теңдеу үшін дереккөзді анықтаудың кеңістіктік есептері зерттелді. Параболалық дифференциалдық теңдеу үшін дереккөзді анықтау есебінің дұрыстығы теоремасы анықталды. Бұл есептің жуық шешімін табу үшін тұрақты айырымдық схема берілген. Сонымен қатар, дереккөзді сәйкестендірудің параболалық есебінің айырымдық схемасының тұрақтылығының бағалаулары ұсынылған. Сандық нәтижелер келтірілген.

Кілт сөздер: корректілік, эллипстік теңдеулер, коэрцитивті тұрақтылық, дереккөзді идентификациялау, дәл бағалаулар, шеттік есеп.

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Замечание о параболической проблеме идентификации с инволюцией и условием Дирихле

Исследованы пространственные задачи идентификации источника для параболического уравнения с инволюцией и условием Дирихле. Установлена теорема корректности задачи идентификации источника для параболического дифференциального уравнения. Представлена устойчивая разностная схема для приближенного решения этой задачи. Кроме того, даны оценки устойчивости разностной схемы параболической задачи идентификации источника. Приведены численные результаты.

Ключевые слова: корректность, эллиптические уравнения, положительность, коэрцитивная устойчивость, идентификация источника, точные оценки, краевая задача.

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On the boundedness of solution of the second order ordinary differential equation with damping term and involution

In the present paper the initial value problem for the second order ordinary differential equation with damping term and involution is investigated. We obtain equivalent initial value problem for the fourth order ordinary differential equations to the initial value problem for second order linear differential equations with damping term and involution. Theorem on stability estimates for the solution of the initial value problem for the second order ordinary linear differential equation with damping term and involution is proved. Theorem on existence and uniqueness of bounded solution of initial value problem for second order ordinary nonlinear differential equation with damping term and involution is established.

Keywords: differential equation with damping term and involution, stability, boundedness, existence and uniqueness.

Introduction

Differential equations with involution appear in mathematical models of ecology, biology, and population dynamics (see, e.g, [1–6] and the reference given therein).

Our goal in this paper is to investigate the boundedness of the solution of the initial value problem for the second order ordinary differential equation with damping term and involution

$$y''(t) = f(t, y(t), y'(t), y(u(t))),\ t \in I = (-\infty, \infty),\ y(t_0) = y_0,\ y'(t_0) = y_0'. \tag{1}$$

Here and in future $u(t)$ is involution function, that is $u(u(t)) = t$, and t_0 is a fixed point of u. Problem (1) does not seem to yield directly to any techniques that can be used for ordinary differential equations without involution term [1, 2]. Therefore, we consider the second order linear differential equations with damping term and involution. We obtain equivalent initial value problem for the fourth order ordinary differential equations to the initial value problem for second order linear differential equations with damping term and involution. Theorem on stability estimates for the solution of the initial value problem for the second order ordinary linear differential equation with damping term and involution is proved. Finally, theorem on existence and uniqueness of bounded solution of initial value problem for the second order nonlinear ordinary differential equation with damping term and involution is established. Note that some of the results of this work was presented, without proof, in [7].

Linear ordinary differential equation with damping term and involution

Let $C^{\infty}[I]$ be the set of all differentiable functions for all degrees.

Theorem 1. Let $a(t)$, $b(t), \alpha(t)$ be functions of class C^{∞} on I, such that $b(t)$ does not vanish on the interval I, then the problem

$$y^{''}(t) + \alpha(t)y'(t) = a(t)y(t) + b(t)y(-t) + f(t),\ t \in I,\ y(0) = \varphi,\ y^{'}(0) = \psi$$

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is equivalent to the following problem for the fourth order ordinary differential equation

$$\begin{cases} y^{(4)}(t) = p(t)y(t) + q(t)y^{'}(t) + r(t)y^{''}(t) + s(t)y^{'''}(t) + F(t),\ t \in I, \\ y(0) = \varphi, y^{'}(0) = \psi, \\ y^{''}(0) = a(0)\varphi + b(0)\varphi - \alpha(0)\psi + f(0), \\ y^{'''}(0) = \left[-\alpha(0)\left[a(0) + b(0)\right] + a^{'}(0) + b^{'}(0)\right]\varphi \\ + \left[-\alpha^{'}(0) + \alpha^2(0) + a(0) - b(0)\right]\psi + f^{'}(0) - \alpha(0)f(0), \end{cases}$$

where

$$p(t) = a^{''}(t) + b(-t)b(t) - \left[2b^{'}(t) + b(t)\alpha(-t)\right]\frac{1}{b(t)}a^{'}(t)$$

$$- \left[b^{''}(t) + b(t)a(-t) - \left[2b^{'}(t) + b(t)\alpha(-t)\right]\frac{1}{b(t)}b^{'}(t)\right]\frac{1}{b(t)}a(t),$$

$$q(t) = -\alpha^{''}(t) + 2a'(t) + \left[2b^{'}(t) + b(t)\alpha(-t)\right]\frac{1}{b(t)}\left[\alpha^{'}(t) - a(t)\right]$$

$$- \left[b^{''}(t) + b(t)a(-t) - \left[2b^{'}(t) + b(t)\alpha(-t)\right]\frac{1}{b(t)}b^{'}(t)\right]\frac{1}{b(t)}\alpha(t),$$

$$r(t) = -2\alpha^{'}(t) + a(t) + \left[2b^{'}(t) + b(t)\alpha(-t)\right]\frac{1}{b(t)}\alpha(t)$$

$$+ \left[b^{''}(t) + b(t)a(-t) - \left[2b^{'}(t) + b(t)\alpha(-t)\right]\frac{1}{b(t)}b^{'}(t)\right]\frac{1}{b(t)},$$

$$s(t) = -\alpha(t) + \left[2b^{'}(t) + b(t)\alpha(-t)\right]\frac{1}{b(t)},$$

and

$$F(t) = -\left[b^{''}(t) + b(t)a(-t) - \left[2b^{'}(t) + b(t)\alpha(-t)\right]\frac{1}{b(t)}b^{'}(t)\right]\frac{1}{b(t)}f(t)$$

$$- \left[2b^{'}(t) + b(t)\alpha(-t)\right]\frac{1}{b(t)}f^{'}(t) + b(t)f(-t) + f^{''}(t).$$

The proof of Theorem 1 is based on approaches of proof of Theorem 1 of paper [1] on the first order linear differential equation with involution.

Now, we consider the initial value problem

$$y^{''}(t) + \alpha y^{'}(t) = by(-t) + ay(t) + f(t),\ t \in\ I,\ y(0) = \varphi,\ y'(0) = \psi \tag{2}$$

for the second order involutory ordinary differential equation with damping term. We are interested in studying the stability of problem (2) on I. In general cases of α, a and b the solution of (2) is not bounded on I. Applying Theorem 1, we get the equivalent initial value problem

$$\begin{cases} y^{(4)}(t) + (a^2 - b^2)y(t) - \left(2a + \alpha^2\right)y^{''}(t) = F(t), \\ F(t) = -af(t) + bf(-t) - \alpha f'(t) + f^{''}(t),\ t \in I, \\ y(0) = \varphi, y^{'}(0) = \psi, y^{''}(0) = (b + a)\varphi - \alpha\psi + f(0), \\ y^{'''}(0) = -\alpha\left(b + a\right)\varphi + \left(-b + a + \alpha^2\right)\psi + f'(0) - \alpha f(0) \end{cases} \tag{3}$$

for the fourth order ordinary differential equation. We will obtain the solution of problem (3). Assume that $|b| < |a|$, $a \in \left(-\left(\frac{\alpha^2}{4} + \frac{b^2}{\alpha^2}\right), -\frac{\alpha^2}{2}\right)$. Then, it is easy to see that

$$\frac{d^4y(t)}{dt^4} - \left(2a + \alpha^2\right)\frac{d^2y(t)}{dt^2} + \left(a^2 - b^2\right)y(t)$$

$$= \left(\frac{d^2}{dt^2} - \left(a + \frac{\alpha^2}{2} + \sqrt{a\alpha^2 + \frac{\alpha^4}{4} + b^2}\right)\right)\left(\frac{d^2}{dt^2} - \left(a + \frac{\alpha^2}{2} - \sqrt{a\alpha^2 + \frac{\alpha^4}{4} + b^2}\right)\right)y(t).$$

Therefore problem (3) can be written as initial value problem

$$\begin{cases} \left(\frac{d^2}{dt^2} - \left(a + \frac{\alpha^2}{2} + \sqrt{a\alpha^2 + \frac{\alpha^4}{4} + b^2}\right)\right) y\left(t\right) = v(t), \\ y(0) = \varphi,\ y'(0) = \psi, \\ \left(\frac{d^2}{dt^2} - \left(a + \frac{\alpha^2}{2} - \sqrt{a\alpha^2 + \frac{\alpha^4}{4} + b^2}\right)\right) v(t) = F(t), \\ F(t) = -af(t) + bf(-t) - \alpha f'(t) + f^{''}(t),\ t \in I, \\ v(0) = \left(b - \frac{\alpha^2}{2} - \sqrt{a\alpha^2 + \frac{\alpha^4}{4} + b^2}\right) \varphi - \alpha\psi + f(0), \\ v'(0) = -\alpha\left(b+a\right)\varphi + \left(-b + \frac{\alpha^2}{2} - \sqrt{a\alpha^2 + \frac{\alpha^4}{4} + b^2}\right)\psi \\ +f'(0) - \alpha f(0) \end{cases}$$

for the system of second order differential equations. Applying the d'Alembert's formula, we get

$$y\left(t\right) = \cos\left(mt\right)\varphi + \frac{\sin\left(mt\right)}{m}\psi + \int\limits_0^t \frac{\sin\left(m\left(t-s\right)\right)}{m} v(s)ds, \tag{4}$$

$$v\left(t\right) = \cos\left(nt\right)\left[\left(b - \frac{\alpha^2}{2} - \sqrt{a\alpha^2 + \frac{\alpha^4}{4} + b^2}\right)\varphi - \alpha\psi + f(0)\right]$$
$$+\frac{\sin\left(nt\right)}{n}\left[-\alpha\left(b+a\right)\varphi + \left(-b + \frac{\alpha^2}{2} - \sqrt{a\alpha^2 + \frac{\alpha^4}{4} + b^2}\right)\psi + f'(0) - \alpha f(0)\right]$$
$$+\int\limits_0^t \frac{\sin\left(n\left(t-s\right)\right)}{n} F(s)ds,$$

where

$$m = \sqrt{-\left(a + \frac{\alpha^2}{2} + \sqrt{a\alpha^2 + \frac{\alpha^4}{4} + b^2}\right)}, n = \sqrt{-\left(a + \frac{\alpha^2}{2} - \sqrt{a\alpha^2 + \frac{\alpha^4}{4} + b^2}\right)}.$$

Since $F(t) = -af(t) + bf(-t) - \alpha f'(t) + f^{''}(t)$ and

$$\int\limits_0^t \frac{\sin\left(n\left(t-s\right)\right)}{n} f^{'}\left(s\right)ds = -\frac{\sin nt}{n} f(0) + \int\limits_0^t \cos\left(n\left(t-s\right)\right) f(s)ds,$$

$$\int\limits_0^t \frac{\sin\left(n\left(t-s\right)\right)}{n} f^{''}\left(s\right)ds = -\frac{\sin nt}{n} f'(0) - \cos nt f(0) + f(t) - \int\limits_0^t n\sin\left(n\left(t-s\right)\right) f(s)ds,$$

we can write

$$v\left(t\right) = \cos\left(nt\right)\left[\left(b - \frac{\alpha^2}{2} - \sqrt{a\alpha^2 + \frac{\alpha^4}{4} + b^2}\right)\varphi - \alpha\psi\right] \tag{5}$$
$$+\frac{\sin\left(nt\right)}{n}\left[-\alpha\left(b+a\right)\varphi + \left(-b + \frac{\alpha^2}{2} - \sqrt{a\alpha^2 + \frac{\alpha^4}{4} + b^2}\right)\psi\right]$$

$$-a\int\limits_0^t \frac{\sin\left(n\left(t-s\right)\right)}{n}f(s)ds$$

$$+b\int\limits_{-t}^0 \frac{\sin\left(n\left(t+s\right)\right)}{n}f(s)ds-\alpha\int\limits_0^t \cos\left(n\left(t-s\right)\right)f(s)ds$$

$$+f(t)-\int\limits_0^t n\sin\left(n\left(t-s\right)\right)f(s)ds.$$

Applying formulas (4) and (5), we get

$$y\left(t\right)=\cos\left(mt\right)\varphi+\frac{\sin\left(mt\right)}{m}\psi \tag{6}$$

$$+\frac{\cos\left(nt\right)-\cos\left(mt\right)}{m^2-n^2}\left[\left(b-\frac{\alpha^2}{2}-\sqrt{a\alpha^2+\frac{\alpha^4}{4}+b^2}\right)\varphi-\alpha\psi\right]$$

$$+\frac{\frac{1}{n}\sin\left(nt\right)-\frac{1}{m}\sin\left(mt\right)}{m^2-n^2}\left[-\alpha\left(b+a\right)\varphi+\left(-b+\frac{\alpha^2}{2}-\sqrt{a\alpha^2+\frac{\alpha^4}{4}+b^2}\right)\psi\right]$$

$$+\frac{1}{m^2-n^2}\int\limits_0^t\left[-n\sin\left(n\left(t-s\right)\right)+m\sin\left(m\left(t-s\right)\right)\right]f(s)ds$$

$$+\frac{\alpha}{m^2-n^2}\int\limits_0^t\left[\cos\left(n\left(t-s\right)\right)-\cos\left(m\left(t-s\right)\right)\right]f(s)ds$$

$$+\frac{a}{m^2-n^2}\int\limits_0^t\left[-n\sin\left(n\left(t-s\right)\right)+m\sin\left(m\left(t-s\right)\right)\right]f(s)ds$$

$$-\frac{b}{m^2-n^2}\int\limits_{-t}^0\left[-\frac{1}{n}\sin\left(n\left(t+s\right)\right)+\frac{1}{m}\sin\left(m\left(t+s\right)\right)\right]f(s)ds.$$

Theorem 2. Assume that $|b|<|a|$, $a\in\left(-\left(\frac{\alpha^2}{4}+\frac{b^2}{\alpha^2}\right),-\frac{\alpha^2}{2}\right)$. Then problem (2) is stable and the following stability estimate holds

$$\sup_{t\in I}|y(t)|\leq M(a,b,\alpha)\left[|\varphi|+|\psi|+\int\limits_{-\infty}^{\infty}|f(s)|\,ds\right].$$

The proof is based on formula (6) and the triangle inequality.

Nonlinear ordinary differential equation with involution

We consider the initial value problem

$$y^{''}(t)+\alpha y^{'}(t)=by(-t)+ay(t)+f(t,y(t),\ y'(t)),\ t\in\ I,\ y(0)=\varphi,y'(0)=\psi \tag{7}$$

for the second order nonlinear involutory ordinary differential equation. We are interested in studying the existence and uniqueness of bounded solution of problem (7) on I. In general cases of α, a and b the solution of (7) is not bounded on I. We will apply a fixed point theorem.

Let $C^{(1)}(I)$ be the metric space of all continuously differentiable functions defined on the interval I with the metric d defined by

$$d(x,y)=\sup_{t\in I}|x(t)-y(t)|+\sup_{t\in I}\left|\frac{dx\left(t\right)}{dt}-\frac{dy\left(t\right)}{dt}\right|.$$

Note that $C^{(1)}(I)$ is the complete space. This is first condition of a fixed point theorem in metric space (see [9]).

Theorem 3. Assume that $|b|<|a|$, $a\in\left(-\left(\frac{\alpha^2}{4}+\frac{b^2}{\alpha^2}\right),-\frac{\alpha^2}{2}\right)$, and f is continuous and bounded function on the region

$$P=\{(t,x,y):-\infty<t<\infty,\ \ |x-\varphi|<M,\ \ |y-\psi|<M\}.$$

Suppose that f satisfies a Lipschitz condition on P with respect to its second and third arguments, that is, there is a constant l such that for $(t,x,u)\,,\,(t,y,v)\in P$

$$|f(t,x,u)-f(t,y,v)|\le l\left(|x-y|+|u-v|\right). \tag{8}$$

Then, initial value problem (7) has a unique solution $y\in C^{(1)}(I)$.

Proof. The procedure of proving theorem on the existence and uniqueness of a bounded solution of problem (7) is based on reducing this problem to an integral equation

$$y(t)=Ty(t), \tag{9}$$

where

$$\begin{aligned}
Ty\left(t\right)&=\cos\left(mt\right)\varphi+\frac{\sin\left(mt\right)}{m}\psi\\
&+\frac{\cos\left(nt\right)-\cos\left(mt\right)}{m^2-n^2}\left[\left(b-\frac{\alpha^2}{2}-\sqrt{a\alpha^2+\frac{\alpha^4}{4}+b^2}\right)\varphi-\alpha\psi\right]\\
&+\frac{\frac{1}{n}\sin\left(nt\right)-\frac{1}{m}\sin\left(mt\right)}{m^2-n^2}\left[-\alpha\left(b+a\right)\varphi-\left(b-\frac{\alpha^2}{2}+\sqrt{a\alpha^2+\frac{\alpha^4}{4}+b^2}\right)\psi\right]\\
&+\frac{1}{m^2-n^2}\int_0^t\left[-n\sin\left(n\left(t-s\right)\right)+m\sin\left(m\left(t-s\right)\right)\right]f(s,y(s),y'(s))ds\\
&+\frac{\alpha}{m^2-n^2}\int_0^t\left[\cos\left(n\left(t-s\right)\right)-\cos\left(m\left(t-s\right)\right)\right]f(s,y(s),y'(s))ds\\
&+\frac{a}{m^2-n^2}\int_0^t\left[-n\sin\left(n\left(t-s\right)\right)+m\sin\left(m\left(t-s\right)\right)\right]f(s,y(s),y'(s))ds\\
&-\frac{b}{m^2-n^2}\int_{-t}^0\left[-\frac{1}{n}\sin\left(n\left(t+s\right)\right)+\frac{1}{m}\sin\left(m\left(t+s\right)\right)\right]f(s,y(s),y'(s))ds.
\end{aligned}$$

The proof of equation (9) is based on the formula (6). Note that integral form is a Volterra type integro-differential equation of the second kind. Therefore, the recursive formula for the solution of problem (7) is

$$\begin{aligned}
y_0(t)&=\cos\left(mt\right)\varphi+\frac{\sin\left(mt\right)}{m}\psi\\
&+\frac{\cos\left(nt\right)-\cos\left(mt\right)}{m^2-n^2}\left[\left(b-\frac{\alpha^2}{2}-\sqrt{a\alpha^2+\frac{\alpha^4}{4}+b^2}\right)\varphi-\alpha\psi\right]\\
&+\frac{\frac{1}{n}\sin\left(nt\right)-\frac{1}{m}\sin\left(mt\right)}{m^2-n^2}\left[-\alpha\left(b+a\right)\varphi-\left(b-\frac{\alpha^2}{2}+\sqrt{a\alpha^2+\frac{\alpha^4}{4}+b^2}\right)\psi\right],\\
y_j(t)&=y_0(t)+\frac{1}{m^2-n^2}
\end{aligned} \tag{10}$$

$$\times \int_0^t \left[-n\sin\left(n\left(t-s\right)\right)+m\sin\left(m\left(t-s\right)\right)\right]f(s,y_{j-1}(s)),y'_{j-1}(s))ds$$

$$+\frac{\alpha}{m^2-n^2}\int_0^t \left[\cos\left(n\left(t-s\right)\right)-\cos\left(m\left(t-s\right)\right)\right]f(s,y_{j-1}(s)),y'_{j-1}(s))ds$$

$$+\frac{a}{m^2-n^2}\int_0^t \left[-n\sin\left(n\left(t-s\right)\right)+m\sin\left(m\left(t-s\right)\right)\right]f(s,y_{j-1}(s)),y'_{j-1}(s))ds$$

$$-\frac{b}{m^2-n^2}\int_{-t}^0 \left[-\frac{1}{n}\sin\left(n\left(t+s\right)\right)+\frac{1}{m}\sin\left(m\left(t+s\right)\right)\right]$$

$$\times f(s,y_{j-1}(s)),y'_{j-1}(s))ds,\ j\geq 1.$$

According to the method of recursive approximation (10), we get

$$y(t)=y_0(t)+\sum_{j=0}^{\infty}\left[y_{j+1}(t)-y_j(t)\right]. \tag{11}$$

We have that

$$y_{j+1}(t)-y_j(t)=\frac{1}{m^2-n^2}\int_0^t \left[-n\sin\left(n\left(t-s\right)\right)+m\sin\left(m\left(t-s\right)\right)\right] \tag{12}$$

$$\times\left[f(s,y_j(s)),y'_j(s))-f(s,y_{j-1}(s)),y'_{j-1}(s))\right]ds$$

$$+\frac{\alpha}{m^2-n^2}\int_0^t \left[\cos\left(n\left(t-s\right)\right)-\cos\left(m\left(t-s\right)\right)\right]$$

$$\times\left[f(s,y_j(s)),y'_j(s))-f(s,y_{j-1}(s)),y'_{j-1}(s))\right]ds$$

$$+\frac{a}{m^2-n^2}\int_0^t \left[-n\sin\left(n\left(t-s\right)\right)+m\sin\left(m\left(t-s\right)\right)\right]$$

$$\times\left[f(s,y_j(s)),y'_j(s))-f(s,y_{j-1}(s)),y'_{j-1}(s))\right]ds$$

$$-\frac{b}{m^2-n^2}\int_{-t}^0 \left[-\frac{1}{n}\sin\left(n\left(t+s\right)\right)+\frac{1}{m}\sin\left(m\left(t+s\right)\right)\right]$$

$$\times\left[f(s,y_j(s)),y'_j(s))-f(s,y_{j-1}(s)),y'_{j-1}(s))\right]ds,\ j\geq 1,$$

therefore, applying the triangle inequality, formula (12) and Lipschitz condition (8), we get

$$\left|y_{j+1}(t)-y_j(t)\right|,\left|y'_{j+1}(t)-y'_j(t)\right|$$

$$\leq M(a,b,\alpha)l\int_{-|t|}^{|t|}\left[\left|y_j(s)-y_{j-1}(s)\right|+\left|y'_j(s)-y'_{j-1}(s)\right|\right]ds \tag{13}$$

for any $t\in I$ and $j\geq 1$. Moreover, applying the triangle inequality, we get

$$\left|y_0(t)\right|,\left|y'_0(t)\right|\leq M_1(a,b,\alpha,\varphi,\psi),$$

$$|y_1(t)-y_0(t)|\,,|y_1'(t)-y_0'(t)|\le M_2(a,b,\alpha)\,|t|\,, \tag{14}$$

for any $t\in I$. Applying estimates (13) and (14), we can prove that

$$|y_{j+1}(t)-y_j(t)|\,,\left|y_{j+1}'(t)-y_j'(t)\right|\le\left[4M(a,b,\alpha)lM_2(a,b,\alpha)\right]^j\frac{|t|^{j+1}}{(j+1)!} \tag{15}$$

for any $t\in I$ and $j\ge 1$. Therefore, applying the triangle inequality, formula (11) and estimates (13) and (15), we get

$$|y(t)-y_n(t)|\,,|y'(t)-y_n'(t)|$$

$$\le\sum_{j=n+1}^{\infty}\left[4M(a,b,\alpha)lM_2(a,b,\alpha)\right]^j\frac{|t|^{j+1}}{(j+1)!}\to 0,\ n\to\infty,$$

$$|y(t)|\,,|y'(t)|\le M_1(a,b,\alpha,\varphi,\psi)+M_2(a,b,\alpha)\,|t|$$

$$+\sum_{j=1}^{\infty}\left[4M(a,b,\alpha)lM_2(a,b,\alpha)\right]^j\frac{|t|^{j+1}}{(j+1)!}$$

for any $t\in I$. Theorem 3 is proved.

Conclusion

In the present paper the initial value problem for the second order differential equation with damping term and involution is investigated. We obtained equivalent initial value problem for the fourth order ordinary differential equations to the initial value problem for second order differential equations with damping term and involution. Theorem on stability estimates for the solution of the initial value problem for the second order ordinary linear differential equation with damping term and involution is proved. Theorem on existence and uniqueness of bounded solution of initial value problem for the second order ordinary nonlinear differential equation with damping term and involution is established. Moreover, applying this result, the two-step stable difference schemes for the numerical solution of the initial value linear and nonlinear problems (2) and (7) for the second order linear and nonlinear differential equations with damping term and involution can be presented and studied.

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Инволюциясы мен жойылып бара жатқан мүшесі бар екінші ретті қарапайым дифференциалдық теңдеудің шектелген шешімі туралы

Мақалада демпингтік мүше пен инволюциясы бар қарапайым екінші ретті дифференциалдық теңдеудің бастапқы есебі зерттелді. Екінші ретті сызықтық дифференциалдық теңдеулер үшін қарапайым, төртінші ретті дифференциалдық теңдеулер үшін бастапқы есептерге эквивалентті есептер алынды. Демпингтік мүше мен инволюциясы бар қарапайым екінші ретті сызықтық дифференциалдық теңдеу үшін бастапқы есепті шешудің тұрақтылығын бағалау теоремасы дәлелденді. Инволюциясы мен жойылып бара жатқан мүшесі бар екінші ретті қарапайым сызықты емес дифференциалдық теңдеу үшін бастапқы есепті шектелген шешімнің бар болуы мен жалғыздығы туралы теорема анықталды.

Кілт сөздер: жойылатын мүшесі және инволюциясы бар дифференциалдық теңдеу, тұрақтылық, шектелген, бар болуы мен жалғыздығы.

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Об ограниченности решения обыкновенного дифференциального уравнения второго порядка с затухающим членом и инволюцией

В статье исследована начальная задача для обыкновенного дифференциального уравнения второго порядка с демпинговым членом и инволюцией. Получены задачи, эквивалентные начальной задаче для обыкновенных дифференциальных уравнений четвертого порядка, начальной задаче для линейных дифференциальных уравнений второго порядка с затухающим членом и инволюцией. Доказана теорема об оценках устойчивости решения начальной задачи для обыкновенного линейного дифференциального уравнения второго порядка с демпинговым членом и инволюцией. Установлена теорема о существовании и единственности ограниченного решения начальной задачи для обыкновенного нелинейного дифференциального уравнения второго порядка с затухающим членом и инволюцией.

Ключевые слова: дифференциальное уравнение с затухающим членом и инволюцией, устойчивость, ограниченность, существование и единственность.

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On stability of the third order partial delay differential equation with involution and Dirichlet condition

In this paper the stability of the initial value problem for the third order partial delay differential equation with involution is investigated. The first order of accuracy absolute stable difference scheme for the solution of the differential problem is presented. Stability estimates for the solution of this difference scheme are proved. Numerical results are provided.

Keywords: time delay, third order partial differential equations, stability, difference scheme.

Introduction

Local and nonlocal boundary value problems for third order partial differential equations have been studied widely in the literature (see, for instance, [1–8]).

The time delay is one of the most common phenomena occurring in many engineering applications. In control theory the process of sampled-data control is a typical example where time delay happens in the transmission from measurement to controller.

Theory and applications of delay linear and nonlinear third order ordinary differential and difference equations with the delay term were widely investigated (see, for instance, [9–14] and the references given therein).

Our goal in this paper is to investigate the initial value problem for third order partial delay differential and difference equations with convolution. The paper is organized as follows. Section 1 is the introduction. In section 2 the theorem on stability of the initial value problem for the third order partial delay differential equation with convolution is established. In section 3 the first order of accuracy difference scheme for the solution of this problem is studied. Stability estimates for the solution of this difference scheme are proved. In section 4 numerical results are provided. Finally, section 5 is a conclusion.

Stability of differential problem

In $[0, \infty) \times (-l, l)$ the initial boundary value problem for the third order partial differential equation with time delay and involution

$$\begin{cases} \frac{\partial^3 u(t,x)}{\partial t^3} - (a(x)u_{tx}(t,x))_x + \beta\left(a(-x)u_{t,-x}(t,-x)\right)_{-x} \\ = -b\left(-a(x)u_x(t-w,x)\right)_x + \beta\left(a(-x)u_{-x}(t,-x)\right)_{-x} \\ +f(t,x),\ 0 < t < \infty, (-l,l), \\ \\ u(t,x) = g(t,x), -w \le t \le 0, x \in [-l,l], \\ \\ u(t,-l) = u(t,l) = 0,\ \ 0 \le t < \infty \end{cases} \tag{1}$$

is considered. Throughout this paper we will assume that $w > 0$, $\overline{a} \ge a(x) = a(-x) \ge \underline{a} > 0$, $x \in (-l, l)$ and $\underline{a} - \overline{a}|\beta| \ge 0$.

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We consider the Hilbert space $L_2[-l,l]$ of the all square integrable functions defined on $[-l,l]$, equipped with the norm

$$\| f \|_{L_2[-l,l]}= \left(\int_{-l}^{l} |f(x)|^2 dx \right)^{\frac{1}{2}}.$$

Under compatibility conditions problem (1) has a unique solution $u(t,x)$ for the smooth functions $a(x)$, $x \in (-l,l)$, $g(t,x), -w \leq t \leq 0,\ x \in [-l,l], f(t,x),\ 0 < t < \infty,\ x \in (-l,l)$, and $b \in R^1$.

Let us give theorem on stability of problem (1).

Theorem 1. For the solutions of problem (1) we have following stability estimates

$$\max_{0\leq t\leq nw} \|v_{tt}(t,\cdot)\|_{W_2^1[-l,l]}, \max_{0\leq t\leq nw} \|v_t(t,\cdot)\|_{W_2^2[-l,l]}, \max_{0\leq t\leq nw} \|v(t,\cdot)\|_{W_2^3[-l,l]}$$

$$\leq M_2 \left[(2+|b|\, w)^n a_0 + \sum_{j=1}^{n} (2+|b|\, w)^{n-j} \int_{(j-1)\omega}^{jw} \|f(s,\cdot)\|_{W_2^1[-l,l]} ds \right],$$

$$a_0 = \max \left\{ \max_{-w\leq t\leq 0} \|g_{tt}(t,\cdot)\|_{W_2^1[-l,l]}, \max_{-w\leq t\leq 0} \|g(t,\cdot)\|_{W_2^3[-l,l]} \right\},$$

where M_2 does not depend on $g(t,x)$ and $f(t,x)$. Here, $W_2^1[-l,l], W_2^2[-l,l]$ and $W_2^3[-l,l]$ are Sobolev spaces of all square integrable functions $\psi(x)$ defined on $[-l,l]$ equipped with the norm

$$\|\psi\|_{W_2^k[-l,l]} = \left(\int_{-l}^{l} \sum_{j=0}^{k} \left(\psi_{\underbrace{x\cdots x}_{j \text{ time}}}(x) \right)^2 dx \right)^{\frac{1}{2}}.$$

Proof. This allows us to reduce the problem (1) to the initial value problem

$$\begin{cases} \frac{d^3v(t)}{dt^3} + A\frac{dv(t)}{dt} = bAv(t-w) + f(t),\ 0 < t < \infty, \\ v(t) = g(t), -w \leq t \leq 0 \end{cases} \tag{2}$$

in a Hilbert space $H = L_2[-l,l]$ with a self-adjoint positive definite operator A defined by formula

$$Au(x) = -(a(x)u_x(x))_x + \beta(a(-x)u_{-x}(-x))_{-x} \tag{3}$$

with domain

$$D(A) = \{u(x) : u(x),\ u_x(x),\ (a(x)u_x)_x \in L_2[-l,l], u(\pm l) = 0\ \}.$$

The proof of Theorem 1 is based on the self-adjointness and positive definiteness of the space operator A defined by formula (3), paper [15] and the following theorem on stability of the solution of the abstract problem (2).

Theorem 2. [16] For the solution of problem (2) the following estimate holds:

$$\max_{0\leq t\leq nw} \left\| A^{\frac{1}{2}} \frac{d^2v(t)}{dt^2} \right\|_H, \max_{0\leq t\leq nw} \left\| A\frac{dv(t)}{dt} \right\|_H, \frac{1}{2} \max_{0\leq t\leq nw} \left\| A^{\frac{3}{2}} v(t) \right\|_H$$

$$\leq (2+|b|\, w)^n a_0 + \int_0^{nw} \left\| A^{\frac{1}{2}} f(s) \right\|_H ds, n = 1,2,...,$$

where

$$a_0 = \max \left\{ \max_{-w\leq t\leq 0} \left\| A^{\frac{1}{2}} \frac{d^2g(t)}{dt^2} \right\|_H, \max_{-w\leq t\leq 0} \left\| A\frac{dg(t)}{dt} \right\|_H, \max_{-w\leq t\leq 0} \left\| A^{\frac{3}{2}} g(t) \right\|_H \right\}.$$

Stability of the difference scheme

Now, we study the stable difference scheme for the approximate solution of the problem (1). The discretization of the problem (1) is carried out in two steps.

In the first step, the spatial discretization is carried out. We define the grid space

$$[-l,l]_h = \left\{x = x_n \mid x_n = nh,\ -M \le n \le M,\ Mh = \ell\right\}.$$

We introduce the Hilbert space $L_{2h} = L_2([-l,l]_h)$ of the grid functions $\varphi^h(x) = \{\varphi^n\}_{-M}^{M}$ defined on $[-l,l]_h$, equipped with the norm

$$\left\|\varphi^h\right\|_{L_{2h}} = \left(\sum_{x\in[-l,l]_h} \left|\varphi^h(x)\right|^2 h\right)^{1/2}.$$

To the differential operator A defined by the formula (3), we assign the difference operator A_h^x by the formula

$$A_h^x\varphi^h(x) = \left\{-\left(a(x)\varphi_{\bar{x}}^n\right)_x - \beta\left(a(-x)\varphi_{\bar{x}}^{-n}\right)_x\right\}_{-M+1}^{M-1}, \tag{4}$$

acting in the space of grid functions $\varphi^h(x) = \{\varphi^n\}_{-M}^{M}$ and satisfying the conditions $\varphi^{-M} = \varphi^M = 0$. Here

$$\varphi_{\bar{x}}^n = \frac{\varphi^n - \varphi^{n-1}}{h},\ -M+1 \le n \le M, \quad \varphi_x^n = \frac{\varphi^{n+1} - \varphi^n}{h},\ -M \le n \le M-1.$$

It is well-known that A_h^x, defined by (4), is a self-adjoint positive definite operator in L_{2h}. With the help of A_h^x the first discretization step results in the following problem

$$\begin{cases} \dfrac{\partial^3 u^h(t,x)}{\partial t^3} + A_h^x u^h(t,x) = -bA_h^x u^h(t-w,x) \\ +f^h(t,x),\ x\in[-l,l]_h,\ 0<t<\infty, \\ u^h(t,x) = g^h(t,x), -w\le t\le 0,\ x\in[-l,l]_h,\ -w<t<0. \end{cases} \tag{5}$$

In the second step we replace the problem (5) with the following first order of accuracy difference scheme

$$\begin{cases} \frac{u_{k+2}^h(x) - 3u_{k+1}^h(x) + 3u_k^h(x) - u_{k-1}^h(x)}{\tau^3} + A_h^x \frac{u_{k+2}^h(x) - u_{k+1}^h(x)}{\tau} \\ = bA_h^x u_{k-N}^h(x) + f_k^h(x), f_k^h(x) = f^h(t_k,x),\ k\ge 1,\quad x\in[-l,l]_h, \\ u_k^h(x) = g^h(t_k,x),\ -N\le k\le 0, \\ (I_h + \tau^2 A_h^x)\frac{u_1^h(x) - u_0^h(x)}{\tau} = g_t^h(0,x), \\ (I_h + \tau^2 A_h^x)\frac{u_2^h(x) - 2u_1^h(x) + u_0^h(x)}{\tau^2} = g_{tt}^h(0,x),\ x\in[-l,l]_h, \\ (I_h + \tau^2 A_h^x)\frac{u_{mN+1}^h(x) - u_{mN}^h(x)}{\tau} = \frac{u_{mN}^h(x) - u_{mN-1}^h(x)}{\tau},\ x\in[-l,l]_h, \\ (I_h + \tau^2 A_h^x)\frac{u_{mN+2}^h(x) - 2u_{mN+1}^h(x) + u_{mN}^h(x)}{\tau^2} \\ = \frac{u_{mN}^h(x) - 2u_{mN-1}^h(x) + u_{mN-2}^h(x)}{\tau^2},\ x\in[-l,l]_h, m=1,2,..., \end{cases} \tag{6}$$

where $\tau = 1/N$ and $t_k = k\tau$, $-N\le k<\infty$.

Theorem 3. Let τ and h be sufficiently small numbers. For the solution of difference scheme (6) the following estimates

$$\max_{0\le k\le(m+1)N-2}\left\|\frac{u_{k+2}^h - 2u_{k+1}^h + u_k^h}{\tau^2}\right\|_{W_{2h}^1}, \max_{1\le k\le(m+1)N}\left\|\frac{u_k^h - u_{k-1}^h}{\tau}\right\|_{W_{2h}^2},$$

$$\max_{0\le k\le(m+1)N}\|u_k^h\|_{W_{2h}^3} \le C_1\left[(2+\tau|b|(N-2))^m b_0^h\right.$$

$$\left. + \sum_{j=1}^{m}(2+\tau|b|(N-2))^{m-j}\tau\sum_{s=(j-1)N+1}^{jN}\|f(t_s)\|_{W_{2h}^1}\right], m=0,1,...,$$

$$b_0^h = \max\left\{ \max_{-N\le k\le 0} \|g_{tt}^h(t_k)\|_{W_{2h}^1},\ \max_{-N\le k\le 0} \|g_t^h(t_k)\|_{W_{2h}^2},\ \max_{-N\le k\le 0} \|g^h(t_k)\|_{W_{2h}^3} \right\}$$

hold, where C_1 does not depend on $\tau, h,\ g^h(t_k)$, and $f_k^h(x)$. Here, W_{2h}^1, W_{2h}^2 and W_{2h}^3 are spaces of all mesh functions $\psi^h(x)$ defined on $[-l,l]_h$ equipped with the norm

$$\|\psi^h\|_{W_{2h}^k} = \left(\sum_{x\in[-l,l]} \sum_{j=0}^{k} \left(\psi^h_{\underbrace{x\cdots x}_{j \text{ time}}}(x) \right)^2 h^k \right)^{\frac{1}{2}}.$$

Proof. Difference scheme (6) can be written in abstract form

$$\begin{cases} \frac{u_{k+2}^h-3u_{k+1}^h+3u_k^h-u_{k-1}^h}{\tau^3} + A_h \frac{u_{k+2}^h-u_{k+1}^h}{\tau} = bA_h u_{k-N}^h + f_k^h,\ k\ge 1, \\ u_k^h = g_k^h,\ -N\le k\le 0, \\ (I_h+\tau^2 A_h)\frac{u_1^h-u_0^h}{\tau} = g_t^h(0), (I_h+\tau^2A_h)\frac{u_2^h-2u_1^h+u_0^h}{\tau^2} = g_{tt}^h(0), \\ (I_h+\tau^2A_h)\frac{u_{mN+2}^h-2u_{mN+1}^h+u_{mN}^h}{\tau^2} = \frac{u_{mN}^h-2u_{mN-1}^h+u_{mN-2}^h}{\tau^2}, \\ (I_h+\tau^2A_h)\frac{u_{mN+1}^h-u_{mN}^h}{\tau} = \frac{u_{mN}^h-u_{mN-1}^h}{\tau},\ m=1,2,... \end{cases} \tag{7}$$

in a Hilbert space L_{2h} with self-adjoint positive definite operator $A_h = A_h^x$, which is defined by formula (4). Here, $g_k^h = g_k^h(x)$, $f_k^h = f_k^h(x)$ and $u_k^h = u_k^h(x)$ are known and unknown abstract mesh functions defined on $[-l,l]_h$ with the values in $H = L_{2h}$. Therefore, the proof of Theorem 2 is based on the self-adjointness and positive definiteness of the space operator A_h (4) [17] and the following theorem on stability of the solution of the difference scheme (7).

Theorem 4. [18] For the solution of difference scheme (7) the following estimate holds:

$$\frac{1}{2}\max_{0\le k\le(m+1)N-2} \left\| A_h^{\frac{1}{2}} \frac{u_{k+2}^h-2u_{k+1}^h+u_k^h}{\tau^2} \right\|_H,\ \max_{1\le k\le(m+1)N} \left\| A_h \frac{u_k^h-u_{k-1}^h}{\tau} \right\|_H,$$

$$\max_{0\le k\le(m+1)N} \|A_h^{\frac{3}{2}} u_k^h\|_H \le C_1 \left[(2+\tau|b|(N-2))^m b_0^h \right.$$

$$\left. + \sum_{j=1}^{m} (2+\tau|b|(N-2))^{m-j} \tau \sum_{s=(j-1)N+1}^{jN} \|A_H^{\frac{1}{2}} f(t_s)\|_H \right], m=0,1,...,$$

where

$$b_0 = \max\left\{ \max_{-N\le k\le 0} \|A_h^{\frac{1}{2}} g''(t_k)\|_H,\ \max_{-N\le k\le 0} \|A_h g_t^h(t_k)\|_H,\ \max_{-N\le k\le 0} \|A_h^{\frac{3}{2}} g^h(t_k)\|_H \right\}.$$

Numerical results

The numerical methods for obtaining the approximate solutions of partial differential equations play an important role in applied mathematics when the analytical methods do not work properly. In this section we will use the first order of accuracy difference scheme to approximate the solution of a simple test problem

$$\begin{cases} \frac{\partial^3 u(t,x)}{\partial t^3} - \frac{\partial^3 u(t,x)}{\partial t\partial x^2} + 16\frac{\partial u(t,x)}{\partial t} - \frac{1}{8}\frac{\partial^3 u(t,-x)}{\partial t\partial x^2} + 2\frac{\partial u(t,-x)}{\partial t} \\ = -0.1(-\frac{\partial^2 u(t-1,x)}{\partial x^2} + 16u(t-1,x)) - 43e^{-2t}\sin 2x + 2e^{-2(t-1)}\sin 2x, \\ 0<t<\infty,\ -\pi<x<\pi, \\ u(t,x) = e^{-2t}\sin 2x,\ -1\le t\le 0,\ -\pi\le x\le\pi, \\ u(t,-\pi) = u(t,\pi) = 0,\quad 0\le t<\infty. \end{cases} \tag{8}$$

The exact solution of problem (8) is $u(t,x) = e^{-2t}\sin 2x, -\pi \le x \le \pi, -1 \le t < \infty$. For the approximate solutions of the problem (8), using the set of grid points

$$[-1,\infty)_\tau \times [-\pi,\pi]_h = \{(t_k,x_n) : t_k = k\tau,\ -N \le k,\ N\tau = 1,\ x_n = nh,\ -M \le n \le M,\ Mh = \pi\},$$

we get the first order of accuracy in t difference scheme

$$\begin{cases}
\frac{u_n^{k+2}-3u_n^{k+1}+3u_n^{k}-u_n^{k-1}}{\tau^3} - \frac{u_{n+1}^{k+2}-u_{n+1}^{k+1}-2(u_n^{k+2}-u_n^{k+1})+u_{n-1}^{k+2}-u_{n-1}^{k+1}}{\tau h^2} \\
+16\frac{u_n^{k+2}-u_n^{k+1}}{\tau} - \frac{1}{8}\frac{u_{-n+1}^{k+2}-u_{-n+1}^{k+1}-2(u_{-n}^{k+2}-u_{-n}^{k+1})+u_{-n-1}^{k+2}-u_{-n-1}^{k+1}}{\tau h^2} \\
+2\frac{u_{-n}^{k+2}-u_{-n}^{k+1}}{\tau} = -(0.1)\left(-\frac{u_{n+1}^{k-N}-2u_n^{k-N}+u_{n-1}^{k-N}}{h^2} + 16u_n^{k-N}\right) \\
-43e^{-2t_k}\sin 2x_n + 2e^{-2(t_{k-N})}\sin 2x_n,\ t_k = k\tau, \\
mN+1 \le k \le (m+1)N-2,\ m = 0,1,...,-M+1 \le n \le M-1, \\
N\tau = 1,\ x_n = nh,\ -M+1 \le n \le M-1,\ Mh = \pi,\ u_n^0 = \sin(2nh), \\
\frac{u_n^1-u_n^0}{\tau} + \tau(-\frac{u_{n+1}^1-2u_n^1+u_{n-1}^1}{h^2} + 16u_n^1) \\
+\tau(\frac{u_{n+1}^0-2u_n^0+u_{n-1}^0}{h^2} - 16u_n^0) = -2\sin(2nh), \\
\frac{u_n^2-2u_n^1+u_n^0}{\tau^2} + (-\frac{u_{n+1}^2-2u_n^2+u_{n-1}^2}{h^2} + 16u_n^2) \\
+2(\frac{u_{n+1}^1-2u_n^1+u_{n-1}^1}{h^2} - 16u_n^1) \\
+(-\frac{u_{n+1}^0-2u_n^0+u_{n-1}^0}{h^2} + 16u_n^0) = 4\sin(2nh),\ -M \le n \le M, \\
\frac{u_n^{mN+1}-u_n^{mN}}{\tau} + \tau(-\frac{u_{n+1}^{mN+1}-2u_n^{mN+1}+u_{n-1}^{mN+1}}{h^2} + 16u_n^{mN+1}) \\
+\tau(\frac{u_{n+1}^{mN}-2u_n^{mN}+u_{n-1}^{mN}}{h^2} - 16u_n^{mN}) = \frac{u_n^{mN}-u_n^{mN-1}}{\tau}, \\
\frac{u_n^{mN+2}-2u_n^{mN+1}+u_n^{mN}}{\tau^2} + (-\frac{u_{n+1}^{mN+2}-2u_n^{mN+2}+u_{n-1}^{mN+2}}{h^2} + 16u_n^{mN+2}) \\
+2(\frac{u_{n+1}^{mN+1}-2u_n^{mN+1}+u_{n-1}^{mN+1}}{h^2} - 16u_n^{mN+1}) \\
+(-\frac{u_{n+1}^{mN}-2u_n^{mN}+u_{n-1}^{mN}}{h^2} + 16u_n^{mN}) = \frac{u_n^{mN}-2u_n^{mN-1}+u_n^{mN-2}}{\tau^2}, \\
-M \le n \le M,\ m = 1,2,\ldots,\ u_{-M}^k = u_M^k = 0, \\
0 \le k < \infty,\ mN \le k \le (m+1)N,\ m = 1,2,\ldots.
\end{cases} \tag{9}$$

We can write (9) in the matrix form

$$\begin{cases}
BU^{k+2} + CU^{k+1} + DU^k + EU^{k-1} = \varphi(U^{k-N}),\ k = 1,2,3,\ldots \\
U^0 = \begin{bmatrix} 0 \\ \sin(2(-M+1)h) \\ \vdots \\ \sin(2(M-1)h) \\ 0 \end{bmatrix}, \\
U^1 = (1-2\tau)U^0, \\
U^2 = 2U^1 - (1-4\tau^2)U^0, \\
U^{mN+1} = F^{-1}HU^{mN} - F^{-1}U^{mN-1}, \\
U^{mN+2} = 2U^{mN+1} + F^{-1}PU^{mN} - 2F^{-1}U^{mN-1} + F^{-1}U^{mN-2}, \\
m = 1,2,\ldots,
\end{cases}$$

where B, C, D, E, F, H and P are $(2M+1)\times(2M+1)$ matrices, $\varphi(U^{k-N}),\ U^0,\ U^1$ and $U^r, r = k, k\pm 1, k+2$ are $(2M+1)\times 1$ column vectors defined by

$$B=\begin{bmatrix} 1 & 0 & 0 & \cdot & 0 & 0 & 0 & 0 & 0 & \cdot & 0 & 0 & 0 \\ a & b & a & \cdot & 0 & 0 & 0 & 0 & 0 & \cdot & a^* & b^* & a^* \\ 0 & a & b & \cdot & 0 & 0 & 0 & 0 & 0 & \cdot & b^* & a^* & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & b & a & 0 & a^* & b^* & \cdot & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdot & a & b & a+a^* & b^* & a^* & \cdot & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdot & 0 & a+a^* & b+b^* & a+a^* & 0 & \cdot & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdot & a^* & b^* & a^*+a & b & a & \cdot & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdot & b^* & a^* & 0 & a & b & \cdot & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & a^* & b^* & \cdot & 0 & 0 & 0 & 0 & 0 & \cdot & b & a & 0 \\ a^* & b^* & a^* & \cdot & 0 & 0 & 0 & 0 & 0 & \cdot & a & b & a \\ 0 & 0 & 0 & \cdot & 0 & 0 & 0 & 0 & 0 & \cdot & 0 & 0 & 1 \end{bmatrix},$$

$$C=\begin{bmatrix} 0 & 0 & 0 & \cdot & 0 & 0 & 0 & 0 & 0 & \cdot & 0 & 0 & 0 \\ z & c & z & \cdot & 0 & 0 & 0 & 0 & 0 & \cdot & z^* & c^* & z^* \\ 0 & z & c & \cdot & 0 & 0 & 0 & 0 & 0 & \cdot & c^* & z^* & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & c & z & 0 & z^* & c^* & \cdot & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdot & z & c & z+z^* & c^* & z^* & \cdot & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdot & 0 & z+z^* & c+c^* & z+z^* & 0 & \cdot & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdot & z^* & c^* & z^*+z & c & z & \cdot & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdot & c^* & z^* & 0 & z & c & \cdot & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & z^* & c^* & \cdot & 0 & 0 & 0 & 0 & 0 & \cdot & c & z & 0 \\ z^* & c^* & z^* & \cdot & 0 & 0 & 0 & 0 & 0 & \cdot & z & c & z \\ 0 & 0 & 0 & \cdot & 0 & 0 & 0 & 0 & 0 & \cdot & 0 & 0 & 0 \end{bmatrix},$$

$$D=\begin{bmatrix} 0 & 0 & 0 & \cdot & 0 & 0 & 0 \\ 0 & d & 0 & \cdot & 0 & 0 & 0 \\ 0 & 0 & d & \cdot & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & d & 0 & 0 \\ 0 & 0 & 0 & \cdot & 0 & d & 0 \\ 0 & 0 & 0 & \cdot & 0 & 0 & 0 \end{bmatrix}, E=\begin{bmatrix} 0 & 0 & 0 & \cdot & 0 & 0 & 0 \\ 0 & e & 0 & \cdot & 0 & 0 & 0 \\ 0 & 0 & e & \cdot & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & e & 0 & 0 \\ 0 & 0 & 0 & \cdot & 0 & e & 0 \\ 0 & 0 & 0 & \cdot & 0 & 0 & 0 \end{bmatrix},$$

$$F=\begin{bmatrix} 1 & 0 & 0 & \cdot & 0 & 0 & 0 \\ -s & q & -s & \cdot & 0 & 0 & 0 \\ 0 & -s & q & \cdot & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & q & -s & 0 \\ 0 & 0 & 0 & \cdot & -s & q & -s \\ 0 & 0 & 0 & \cdot & 0 & 0 & 1 \end{bmatrix}, P=\begin{bmatrix} 1 & 0 & 0 & \cdot & 0 & 0 & 0 \\ s & p & s & \cdot & 0 & 0 & 0 \\ 0 & s & p & \cdot & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & p & s & 0 \\ 0 & 0 & 0 & \cdot & s & p & s \\ 0 & 0 & 0 & \cdot & 0 & 0 & 1 \end{bmatrix},$$

$$H=\begin{bmatrix} 1 & 0 & 0 & \cdot & 0 & 0 & 0 \\ -s & t^* & -s & \cdot & 0 & 0 & 0 \\ 0 & -s & t^* & \cdot & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & t^* & -s & 0 \\ 0 & 0 & 0 & \cdot & -s & t^* & -s \\ 0 & 0 & 0 & \cdot & 0 & 0 & 1 \end{bmatrix}, \varphi(U^{k-N})=\begin{bmatrix} 0 \\ \varphi^k_{-M+1} \\ \vdots \\ \varphi^k_{M-1} \\ 0 \end{bmatrix},\ U^r=\begin{bmatrix} U^r_0 \\ U^r_{-M+1} \\ \vdots \\ U^r_{M-1} \\ U^r_M \end{bmatrix},$$

$r = k, k \pm 1, k + 2$, where

$$\varphi_n^k = -(0.1)\left(-\frac{u_{n+1}^{k-N} - 2u_n^{k-N} + u_{n-1}^{k-N}}{h^2} + 16u_n^{k-N}\right)$$
$$- 43e^{-2t_k}\sin 2x_n + 2e^{-2(t_{k-N})}\sin 2x_n,$$
$$t_k = k\tau,\ mN + 1 \le k \le (m+1)N - 2,\ m = 0, 1, ...,\ -M + 1 \le n \le M - 1.$$

Here, we denote $a = -\frac{1}{\tau h^2},\ a^* = -\frac{1}{8\tau h^2},\ b = \frac{1}{\tau^3} + \frac{2}{\tau h^2} + \frac{16}{\tau}$, $b^* = \frac{2}{8\tau h^2} + \frac{2}{\tau}, c^* = -b^*,\ z^* = -a^*,\ z = -a,\ \ c = -\frac{3}{\tau^3} - \frac{2}{\tau h^2} - \frac{16}{\tau}, d = \frac{3}{\tau^3},\ e = -\frac{1}{\tau^3}$, $t^* = 2 + \frac{2\tau^2}{h^2} + 16\tau^2,\ p = -\frac{2\tau^2}{h^2} - 16\tau^2, q = 1 + \frac{2\tau^2}{h^2} + 16\tau^2,\ \ s = \frac{\tau^2}{h^2}$.

The numerical solutions are recorded for different values of N and M, and u_n^k represents the numerical solution of this difference scheme at $u(t_k,\ x_n)$. Table 1 is constructed for $N = M = 40, 80, 160$ in $t \in [0, 1]$, $t \in [1, 2],\ t \in [2, 3]$ respectively and the errors are computed by

$$mE_M^N = \max_{mN+1 \le k \le (m+1)N,\ -M \le n \le M} |u(t_k, x_n) - u_n^k|.$$

Table 1

Errors of Difference Scheme (9)

$(\mathbf{N}, \mathbf{M})$	$\mathbf{N = M = 40}$	$\mathbf{N = M = 80}$	$\mathbf{N = M = 160}$
$t \in [0, 1]$	0.0784	0.0397	0.0198
$t \in [1, 2]$	0.0852	0.0423	0.0210
$t \in [2, 3]$	0.0679	0.0312	0.0139

If N and M are doubled, the values of the errors are decreased by a factor of approximately 1/2 for the first order difference scheme (9). The errors presented in this table indicates the accuracy of difference scheme.

Conclusion

In this paper the stability of the initial boundary value problem for the third order partial delay differential equation with involution is investigated. The first order of accuracy difference scheme for the solution of this problem is presented. Stability estimates for the solution of this difference scheme are proved. Numerical results are provided. Some statements of the present paper were published, without proof, in [16, 19].

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Инволюция және Дирихле шарты бар үшінші ретті дербес туындылы кешігуі бар дифференциалдық теңдеудің тұрақтылығы туралы

Мақалада үшінші ретті дербес туындылы кешігуі бар дифференциалдық теңдеудің бастапқы есебінің тұрақтылығы зерттелген. Дифференциалдық есепті шешу үшін бірінші ретті дәлдікті абсолютті тұрақты айырымдық схемасы ұсынылған. Осы айырымдық схема үшін шешімнің тұрақтылығының бағалаулары дәлелденді. Сандық нәтижелер келтірілген.

Кілт сөздер: кешігу, үшінші ретті дербес туындылы теңдеулер, тұрақтылық, айырымдық схема.

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Об устойчивости запаздывающего дифференциального уравнения в частных производных третьего порядка с инволюцией и условием Дирихле

В статье исследована устойчивость начальной задачи для запаздывающего дифференциального уравнения в частных производных третьего порядка. Представлена абсолютно устойчивая разностная схема первого порядка точности для решения дифференциальной задачи. Доказаны оценки устойчивости решения этой разностной схемы. Приведены численные результаты.

Ключевые слова: запаздывание, уравнения в частных производных третьего порядка, устойчивость, разностная схема.

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On the Crank-Nicolson difference scheme for the time-dependent source identification problem

In this study the source identification problem for the one-dimensional Schrödinger equation with non-local boundary conditions is considered. A second order of accuracy Crank-Nicolson difference scheme for the numerical solution of the differential problem is presented. Stability estimates are proved for the solution of this difference scheme. Numerical results are given.

Keywords: identification problem, Schrödinger equation, difference scheme, Crank-Nicolson, stability.

Introduction

Source identification problems (SIPs) have the significant role in natural science, applied sciences, engineering, quantum mechanics, diffusion equations, heat equations (see [1–4] and references therein). The theory and applications of SIPs for partial differential equations (PDEs) were studied in many works (see [5–32] and references therein). The time-dependent SIP

$$\left\{\begin{array}{l} i\frac{\partial u(t,x)}{\partial t}-\frac{\partial}{\partial x}\left(a\left(x\right)\frac{\partial u(t,x)}{\partial x}\right)+\delta u(t,x) \\ =p\left(t\right)q\left(x\right)+f\left(t,x\right),t\in\left(0,T\right),x\in\left(0,l\right), \\ u\left(0,x\right)=\varphi\left(x\right),x\in\left[0,l\right], \\ u\left(t,0\right)=u\left(t,l\right),u_{x}\left(t,0\right)=u_{x}\left(t,l\right), \\ \int_{0}^{l}u\left(t,x\right)dx=\zeta\left(t\right),t\in\left[0,T\right] \end{array}\right. \tag{1}$$

for the one-dimensional Schrödinger equation (SE) was investigated [33]. Here $0<a\leq a\left(x\right),\ f\left(t,x\right),\ \zeta\left(t\right),$ $\varphi\left(x\right),\ q\left(x\right)$ and $a\left(x\right)$ are given sufficiently smooth functions and $q\left(0\right)=q\left(l\right)$, $q^{'}\left(0\right)=q^{'}\left(l\right)$ and $\int_{0}^{l}q\left(x\right)dx\neq 0$. Stability estimates were established for the solution of source identification problem (1). A first order of accuracy difference scheme was investigated for the numerical solution of this problem.

In this paper a second order of accuracy Crank-Nicolson difference scheme for the numerical solution of problem (1) is presented. Stability estimates are proved for the solution of the difference scheme. Numerical results are provided.

Stability of difference problem

To formulate results on difference problem we introduce the normed space. Let $C_{\tau}\left(H\right)=C\left(\left[0,T\right]_{\tau},H\right)$ of all mesh functions $\phi^{\tau}=\left\{\phi_{k}\right\}_{k=0}^{N}$ defined on

$$\left[0,T\right]_{\tau}=\left\{t_{k}=k\tau,0\leq k\leq N,N\tau=T\right\}$$

with values in H equipped with the norm

$$\left\|\phi^{\tau}\right\|_{C_{\tau}(H)}=\max_{0\leq k\leq N}\left\|\phi_{k}\right\|_{H}.$$

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Moreover, $L_{2h} = L_2 \left[0, l\right]_h$ are normed spaces of all mesh functions $\gamma^h\left(x\right) = \left\{\gamma_n\right\}_{n=0}^{M}$ defined on

$$\left[0,l\right]_h = \left\{x_n = nh, 0 \leq n \leq M, Mh = l\right\}$$

equipped with the norm

$$\left\|\gamma^h\right\|_{L_{2h}} = \left\{\sum_{i=0}^{M} \left|\gamma_i\right|^2 h\right\}^{\frac{1}{2}},$$

and $W_{2h}^2 = W_2^2 \left[0, l\right]_h$ is the Sobolev space with norm

$$\left\|\gamma^h\right\|_{W_{2h}^2} = \left\{\sum_{i=0}^{M} \left|\gamma_i\right|^2 h + \sum_{i=1}^{M-1} \left|\frac{\gamma_{i+1} - 2\gamma_i + \gamma_{i-1}}{h^2}\right|^2 h\right\}^{\frac{1}{2}}.$$

To the differential operator A defined by (2) we introduce the difference operator A^h defined by the formula

$$A^h\psi^h\left(x\right) = \left\{-\frac{1}{h}\left(a_{n+1}\frac{\psi_{n+1} - \psi_n}{h} - a_n\frac{\psi_n - \psi_{n-1}}{h}\right) + \delta\psi_n\right\}_{n=1}^{M-1}, a_n = a\left(x_n\right) \tag{2}$$

acting in the space of grid functions $\psi^h\left(x\right) = \left\{\psi_n\right\}_{n=0}^{M}$ defined on $\left[0, l\right]_h$ satisfying the conditions $\psi_M^k = \psi_0^k$, $\psi_M^k - \psi_{M-1}^k = \psi_1^k - \psi_0^k$. For the numerical solution $\left\{u_n^\tau\right\}_{n=0}^{M}$ of problem (1) we consider the second order of accuracy Crank-Nicolson difference scheme

$$\begin{cases} i\frac{u_n^k - u_n^{k-1}}{\tau} - \frac{1}{2h}\left(a_{n+1}\frac{u_{n+1}^k - u_n^k}{h} - a_n\frac{u_n^k - u_{n-1}^k}{h}\right) \\ -\frac{1}{2h}\left(a_{n+1}\frac{u_{n+1}^{k-1} - u_n^{k-1}}{h} - a_n\frac{u_n^{k-1} - u_{n-1}^{k-1}}{h}\right) + \delta\frac{u_n^k + u_n^{k-1}}{2} \\ = \frac{p_k + p_{k-1}}{2}q_n + f_k\left(x_n\right), f_k\left(x_n\right) = f\left(t_k - \frac{\tau}{2}, x_n\right), \\ a_n = a\left(x_n\right), q_n = q\left(x_n\right), 1 \leq k \leq N, 1 \leq n \leq M-1, \\ u_n^0 = \varphi_n, \varphi_n = \varphi\left(x_n\right), 0 \leq n \leq M, \\ u_M^k = u_0^k, u_M^k - u_{M-1}^k = u_1^k - u_0^k, \sum_{i=1}^{M} u_i^k h = \zeta_k, \\ \zeta_k = \zeta\left(t_k\right), 0 \leq k \leq N. \end{cases} \tag{3}$$

Let us give the following result on the stability of DS (3).

Theorem 1. For the solution of DS (3) the stability estimates are satisfied:

$$\left\|\left\{\frac{1}{\tau}\left(u_k^h - u_{k-1}^h\right)\right\}_{k=1}^{N}\right\|_{C_\tau\left(L_{2h}\right)} + \left\|\left\{\frac{u_k^h + u_{k-1}^h}{2}\right\}_{k=1}^{N}\right\|_{C_\tau\left(W_{2h}^2\right)} + \left\|\left\{\frac{p_k + p_{k-1}}{2}\right\}_{k=1}^{N}\right\|_{C[0,T]_\tau}$$

$$\leq Q\left(q\right)\left[\left\|\varphi^h\right\|_{W_{2h}^2} + \left\|f_1^h\right\|_{L_{2h}} + \left|\zeta_0\right|\right.$$

$$\left. + \left\|\left\{\frac{1}{\tau}\left(f_k^h - f_{k-1}^h\right)\right\}_{k=2}^{N}\right\|_{C_\tau\left(L_{2h}\right)} + \left\|\left\{\frac{1}{\tau}\left(\zeta_k - \zeta_{k-1}\right)\right\}_{k=1}^{N}\right\|_{C[0,T]_\tau}\right].$$

Proof. Denote that

$$u_n^k = w_n^k - i\eta_k q_n, \tag{4}$$

where

$$\frac{p_k + p_{k-1}}{2} = \frac{\eta_k - \eta_{k-1}}{\tau}, 1 \leq k \leq N, \eta_0 = 0 \tag{5}$$

and w_n^k is the solution of the following Crank-Nicolson difference scheme

$$\begin{cases} i\frac{w_n^k-w_n^{k-1}}{\tau}-\frac{1}{2h}\left(a_{n+1}\frac{w_{n+1}^k-w_n^k}{h}-a_n\frac{w_n^k-w_{n-1}^k}{h}\right) \\ -\frac{1}{2h}\left(a_{n+1}\frac{w_{n+1}^{k-1}-w_n^{k-1}}{h}-a_n\frac{w_n^{k-1}-w_{n-1}^{k-1}}{h}\right)+\delta\frac{w_n^k+w_n^{k-1}}{2} \\ =f_k\left(x_n\right)-i\frac{\eta_k+\eta_{k-1}}{2}\left[-\frac{1}{2h}\left(a_{n+1}\frac{q_{n+1}-q_n}{h}-a_n\frac{q_n-q_{n-1}}{h}\right)+\delta q_n\right], \\ 1\le k\le N, 1\le n\le M-1, \\ w_n^0=\varphi_n, 0\le n\le M, \\ w_M^k=w_0^k, u_M^k-u_{M-1}^k=u_1^k-u_0^k, 0\le k\le N. \end{cases} \tag{6}$$

Now, we estimate $|\frac{p_k+p_{k-1}}{2}|$. Using the conditions $\sum_{m=1}^{M}u_m^k h=\zeta_k$ and (4), we obtain

$$\eta_k=\frac{i}{d_1}\left(\sum_{m=1}^{M}w_m^k h-\zeta_k\right), d_1=\sum_{m=1}^{M}q_m h, 1\le k\le N,$$

$$\frac{p_k+p_{k-1}}{2}=\frac{\sum_{m=1}^{M}(w_m^k-w_m^{k-1})h-(\zeta_k-\zeta_{k-1})}{i\tau d}, 1\le k\le N.$$

Using the Cauchy-Schwartz inequality and triangle inequality, we get

$$\left|\frac{p_k+p_{k-1}}{2}\right|$$

$$\le Q_1(q)\left(\left\|\left\{\frac{w_m^k-w_m^{k-1}}{\tau}\right\}_{m=1}^{M}\right\|_{L_{2h}}+\left|\frac{\zeta_k-\zeta_{k-1}}{\tau}\right|\right)$$

for all $1\le k\le N$ and

$$\left\|\left\{\frac{p_k+p_{k-1}}{2}\right\}_{k=1}^{N}\right\|_{C[0,T]_\tau}$$

$$\le Q_1(q)\left[\left\|\left\{\frac{1}{\tau}\left(w_k^h-w_{k-1}^h\right)\right\}_{k=1}^{N}\right\|_{C_\tau(L_{2h})}+\left\|\left\{\frac{\zeta_k-\zeta_{k-1}}{\tau}\right\}_{k=1}^{N}\right\|_{C[0,T]_\tau}\right].$$

Now, applying formulas (4) and (5), we obtain

$$\frac{u_n^k-u_n^{k-1}}{\tau}=\frac{w_n^k-w_n^{k-1}}{\tau}-i\frac{p_k+p_{k-1}}{2}q_n$$

and

$$\left\|\left\{\frac{1}{\tau}\left(u_k^h-u_{k-1}^h\right)\right\}_{k=1}^{N}\right\|_{C_\tau(L_{2h})}\le\left\|\left\{\frac{1}{\tau}\left(w_k^h-w_{k-1}^h\right)\right\}_{k=1}^{N}\right\|_{C_\tau(L_{2h})}$$

$$+\left\|\left\{\frac{p_k+p_{k-1}}{2}\right\}_{k=1}^{N}\right\|_{C[0,T]_\tau}\left\|\{q_n\}_{n=1}^{M}\right\|_{L_{2h}}.$$

Then, the proof of Theorem 1 is based on the following theorem.

Theorem 2. For the solution of DS (6) the stability estimate is satisfied:

$$\left\|\left\{\frac{1}{\tau}\left(w_k^h-w_{k-1}^h\right)\right\}_{k=1}^{N}\right\|_{C_\tau(L_{2h})}\le Q_2(a)\left[\left\|\varphi^h\right\|_{W_{2h}^2}+|\zeta_0|\right. \tag{7}$$

$$\left.+\left\|f_1^h\right\|_{L_{2h}}+\left\|\left\{\frac{1}{\tau}\left(f_k^h-f_{k-1}^h\right)\right\}_{k=2}^{N}\right\|_{C_\tau(L_{2h})}+\left\|\left\{\frac{\zeta_k-\zeta_{k-1}}{\tau}\right\}_{k=1}^{N}\right\|_{C[0,T]_\tau}\right].$$

Proof. We can write problem (6) as the abstract problem

$$\begin{cases} i\frac{w_k^h - w_{k-1}^h}{\tau} - \frac{A^h}{2}\left(w_k^h + w_{k-1}^h\right) = f_k^h + iA^h q^h \frac{\eta_k + \eta_{k-1}}{2}, \\ t_k = k\tau, 1 \le k \le N, N\tau = T, w_0^h = \varphi^h \end{cases}$$

in L_{2h}. Then,

$$w_k^h = R^k \varphi^h - i\sum_{j=1}^{k} R^{k-j} C\tau \left(f_j^h + iA^h q^h \frac{\eta_j + \eta_{j-1}}{2}\right),$$

where

$$R = \left(I - i\tau \frac{A^h}{2}\right) C, C = \left(I + i\tau \frac{A^h}{2}\right)^{-1}.$$

Taking the difference derivative and applying the Abel's formula, we get

$$\frac{w_k^h - w_{k-1}^h}{\tau} = -iR^{k-1} C A^h \varphi^h - iC\left(f_1^h + iA^h q^h \frac{\eta_1}{2}\right)$$

$$-iC\sum_{j=1}^{k} R^{k-j} (f_j^h - f_{j-1}^h) + C\sum_{j=1}^{k} R^{k-j} A^h q^h \frac{\eta_j - \eta_{j-1}}{2}. \tag{8}$$

Applying formula (8) and estimates

$$\|R\|_{H \longrightarrow H} \le 1, \|C\|_{H \longrightarrow H} \le 1,$$

we get

$$\left\|\frac{w_k^h - w_{k-1}^h}{\tau}\right\|_{L_{2h}} \le \left\|A_h^x \varphi^h\right\|_H + \left\|f_1^h\right\|_{L_{2h}} + \sum_{j=2}^{k} \left\|f_k^h - f_{k-1}^h\right\|_{L_{2h}}$$

$$+Q_4(q)\,\tau \sum_{j=2}^{k} \left[\left|\frac{\zeta_j - \zeta_{j-1}}{\tau}\right| + \left\|\frac{w_j^h - w_{j-1}^h}{\tau}\right\|_{L_{2h}}\right]$$

for any k. Then, applying the discrete analogy of the integral inequality, we get

$$\left\|\frac{w_k^h - w_{k-1}^h}{\tau}\right\|_{L_{2h}}$$

$$\le \left[\left\|A_h^x \varphi^h\right\|_H + \left\|f_1^h\right\|_{L_{2h}} + \sum_{j=2}^{k} \left\|f_k^h - f_{k-1}^h\right\|_{L_{2h}} + Q_4(q) \sum_{j=2}^{k} \left|\zeta_j - \zeta_{j-1}\right|\right] e^{\frac{Q_4(q) k\tau}{1 - Q_4(q)\tau}}$$

for any k. From that it follows (7).

Numerical results

We study the numerical solution of the identification problem

$$\begin{cases} i\frac{\partial u(t,x)}{\partial t} - \frac{\partial^2 u(t,x)}{\partial x^2} + u(t,x) = p(t)(1 + \sin 2x) \\ + (3\sin(2x) - 1)\,e^{it}, x \in (0,\pi), t \in (0,1), \\ u(0,x) = 1 + \sin 2x, x \in [0,\pi], \\ u(t,0) = u(t,\pi), u_x(t,0) = u_x(t,\pi), \\ \int_0^\pi u(t,x)\,dx = \pi e^{it}, t \in [0,1] \end{cases} \tag{9}$$

for a one dimensional Schrodinger differential equation. The exact solution of this problem is $(u(t,x), p(t)) =$ $= \left((1 + \sin 2x)e^{it}, e^{it}\right)$. Applying difference scheme (3) for problem (9), we get

$$\begin{cases} i\frac{u_n^k-u_n^{k-1}}{\tau}-\frac{u_{n+1}^k-2u_n^k+u_{n-1}^k}{2h^2}-\frac{u_{n+1}^{k-1}-2u_n^{k-1}+u_{n-1}^{k-1}}{2h^2}+\frac{u_n^k+u_n^{k-1}}{2} \\ =\frac{p_k+p_{k-1}}{2}\left(1+\sin 2x_n\right)+\left(3\sin 2x_n-1\right)e^{i\left(t_k-\frac{\tau}{2}\right)}, \\ t_k=k\tau, x_n=nh, 1\le k\le N, 1\le n\le M-1, \\ u_n^0=1+\sin 2x_n, 0\le n\le M, Mh=\pi, N\tau=1, \\ u_M^k=u_0^k, u_M^k-u_{M-1}^k=u_1^k-u_0^k, \\ \sum_{m=1}^M u_m^k h=\pi e^{it_k}, 0\le k\le N. \end{cases} \tag{10}$$

The algorithm for obtaining the solution $\left\{\left\{u_n^k\right\}_0^N\right\}_0^M$ and $\{p_k\}_1^N$ of DS (10) contains three steps. We introduce η_k by the formula

$$\eta_k=\frac{p_0+p_k}{2}\tau+\sum_{m=1}^{k-1}p_m\tau, k\in\overline{1,N}, \eta_0=0. \tag{11}$$

Then,

$$\frac{p_k+p_{k-1}}{2}=\frac{\eta_k-\eta_{k-1}}{\tau}, k\in\overline{1,N}, \tag{12}$$

$$u_n^k=w_n^k-i\eta_k(1+sin2x_n), k\in\overline{0,N}, n\in\overline{0,M}. \tag{13}$$

Here w_n^k is the solution of the DS

$$\begin{cases} i\frac{w_n^k-w_n^{k-1}}{\tau}-\frac{w_{n+1}^k-2w_n^k+w_{n-1}^k}{2h^2}-\frac{w_{n+1}^{k-1}-2w_n^{k-1}+w_{n-1}^{k-1}}{2h^2}+\frac{w_n^k+w_n^{k-1}}{2} \\ -z_nh\sum_{k=1}^M w_m^k-z_nh\sum_{k=1}^M w_m^{k-1}=z_n\pi(e^{it_k}+e^{it_{k-1}}) \\ +\left(3\sin 2x_n-1\right)e^{i\left(t_k-\frac{\tau}{2}\right)}, k\in\overline{1,N}, n\in\overline{1,M-1} \\ w_n^0=1+\sin 2x_n, n\in\overline{1,M-1}, \\ w_M^k=w_0^k, w_M^k-w_{M-1}^k=w_1^k-w_0^k, \end{cases} \tag{14}$$

where

$$z_n=\frac{1}{\pi+dh}\left[sin2x_n\left(\frac{1-cos2h}{h^2}-\frac{1}{2}\right)-\frac{1}{2}\right], n\in\overline{1,M-1}.$$

Using the discrete analogy of integral condition in (14), we get

$$\eta_k=\frac{\sum_{m=1}^M w_m^k h-\pi e^{it_k}}{i(\pi+dh)}, d=\sum_{m=1}^M sin2x_m, k\in\overline{1,N}. \tag{15}$$

Step 1: According to DS (10), we obtain $\left\{\left\{w_n^k\right\}_0^N\right\}_0^M$.

We can write (14) as difference equation with matrix coefficients

$$Aw^k+Bw^{k-1}=\varphi^k, 1\le k\le N$$

for any k. Here A and B are $(M+1)$x$(M+1)$ square matrices and φ is $(M+1)$x1 colomn matrix

$$A=\begin{bmatrix} 1 & 0 & 0 & \cdot & 0 & -1 \\ a & b-hz_1 & -hz_1 & \cdot & -hz_1 & -hz_1 \\ 0 & a-hz_2 & a-hz_2 & \cdot & -hz_2 & -hz_2 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & -hz_{M-1} & -hz_{M-1} & \cdot & b-hz_{M-1} & a-hz_{M-1} \\ 1 & -1 & 0 & \cdot & -1 & 1 \end{bmatrix}_{(M+1)\times(M+1)},$$

$$B=\begin{bmatrix} 0 & 0 & 0 & \cdot & 0 & 0 \\ a & a-hz_1 & c-hz_1 & \cdot & -hz_1 & 0 \\ 0 & a-hz_2 & c-hz_2 & \cdot & -hz_2 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & -hz_{M-1} & -hz_{M-1} & \cdot & c-hz_{M-1} & a \\ 0 & 0 & 0 & \cdot & 0 & 0 \end{bmatrix}_{(M+1)\times(M+1)},$$

$a = -\frac{1}{2h^2}$, $b = \frac{i}{\tau} + \frac{1}{h^2} + \frac{1}{2}$, $c = -\frac{i}{\tau} + \frac{1}{h^2} + \frac{1}{2}$,

$$\varphi_n^k = \begin{bmatrix} 0 \\ \varphi_1^k \\ \cdot \\ \varphi_{M-1}^k \\ 0 \end{bmatrix}_{(M+1)\times 1}, w^k = \begin{bmatrix} w_0^k \\ w_1^k \\ \cdot \\ w_{M-1}^k \\ w_M^k \end{bmatrix}_{(M+1)\times 1},$$

$$w^0 = \begin{bmatrix} 1 + sin2x_0 \\ 1 + sin2x_1 \\ \cdot \\ 1 + sin2x_{M-1} \\ 1 + sin2x_M \end{bmatrix}_{(M+1)\times 1},$$

$$\varphi_n^k = z_n \pi (e^{it_k} + e^{it_{k-1}}) + (3sin2x_n - 1)e^{i(t_k - \tau/2)}, 1 \leq k \leq N.$$

Therefore

$$w^k = inv(A)\left(\varphi^k - Bw^{k-1}\right).$$

Step 2: We will find $\{\eta_k\}_0^N$, $\left\{\frac{p_k + p_{k-1}}{2}\right\}_1^N$ by formulas (12) and (15).

Step 3: We will find $\left\{\{u_n^k\}_0^N\right\}_0^M$ by formulas (11) and (13). The errors are computed by

$$E_u = \max_{k \in \overline{0,N}} \left(\sum_{n=0}^{M} \left| u(t_k, x_n) - u_n^k \right|^2 h \right)^{\frac{1}{2}},$$

$$E_p = \max_{k \in \overline{1,N}} \left| p(t_k) - \frac{p_k + p_{k-1}}{2} \right|.$$

Numerical solutions of problem (9) $u(t,x)$ at (t_k, x_n) is u_n^k and of $p(t)$ at t_k is $\frac{p_k + p_{k-1}}{2}$. The result of numerical experience for problem (9) is provided in Table 1.

Table 1

Error Analysis

Error	$M=N=20$	$M=N=40$	$M=N=80$
E_p	0.0002	0.00005	0.00001
E_u	0.017	0.0043	0.0011

Conclusion

In this article the SIP for the one-dimensional SE with non-local boundary conditions is studied. A second order of accuracy Crank-Nicolson difference scheme for the numerical solution of the differential problem is presented. Theorem on stability of this difference scheme is established. The numerical results are given. Finally, this operator approach permits us to investigate one-dimensional SE with classical boundary conditions.

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Дереккөзді идентификациялау бейстационарлы есебі үшін Кранк-Николсонның айырымдық схемасы туралы

Мақалада бейлокалды шекаралық шарттары бар Шредингердің бір өлшемді теңдеуі үшін дереккөзді идентификациялау есебі қарастырылды. Дифференциалдық есепті сандық шешуге арналған екінші дәлдік ретті Кранк-Николсонның айырымдық схемасы ұсынылған. Осы айырымдық схеманың шешімінің тұрақтылығын бағалаулары дәлелденді және сандық нәтижелер келтірілген.

Кілт сөздер: идентификациялау мәселесі, Шредингер теңдеуі, айырымдық схемасы, Кранка-Николсон, тұрақтылық.

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О разностной схеме Кранка-Николсона для нестационарной задачи идентификации источника

В статье рассмотрена задача идентификации источника для одномерного уравнения Шредингера с нелокальными граничными условиями. Представлена разностная схема Кранка-Николсона второго порядка точности для численного решения дифференциальной задачи. Доказаны оценки устойчивости решения этой разностной схемы, и приведены численные результаты.

Ключевые слова: проблема идентификации, уравнение Шредингера, разностная схема Кранка-Николсона, устойчивость.

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On fourth order accuracy stable difference scheme for a multi-point overdetermined elliptic problem

In this paper fourth order of accuracy difference scheme for approximate solution of a multi-point elliptic overdetermined problem in a Hilbert space is proposed. The existence and uniqueness of the solution of the difference scheme are obtained by using the functional operator approach. Stability, almost coercive stability, and coercive stability estimates for the solution of difference scheme are established. These theoretical results can be applied to construct a stable highly accurate difference scheme for approximate solution of multi-point overdetermined boundary value problem for multidimensional elliptic partial differential equations.

Keywords: overdetermined elliptic problem, multi-point condition, high order difference scheme, difference scheme, inverse, source identification problem, well-posedness, stability, coercive stability, almost coercive stability.

Introduction

Methods of solutions of nonlocal and source identification boundary value problems for partial differential equations have been widely investigated by several researchers (see [1–21] and references therein). Construction of highly accurate difference schemes (DSs) for problems of this type is important, especially for their specific theoretical and practical aspects and also usefulness in wide applications [5, 8] and bibliography herein).

Let H be a Hilbert space, A be a self-adjoint positive definite operator (SAPDO) and I be identity operator.

In paper [10] to find an element $p \in H$ and function $v \in C^2([0,T],H) \cap C([0,T],D(A))$ the following multi-point elliptic overdetermined problem

$$\begin{cases} -v_{tt}(t) + Av(t) = g(t) + p, 0 < t < T, \\ v(0) = \phi, v(T) = \sum\limits_{i=1}^{q} \beta_i v(\lambda_i) + \eta, v(\lambda_0) = \zeta \end{cases} \tag{1}$$

was investigated. Here $q \in N$, $\lambda_0, \lambda_i \in (0,T)$, $\beta_i \in R, \beta_i \geq 0, i = 1, ..., q$ are known numbers, $\zeta, \phi, \eta \in D(A)$, $g \in C^2([0,T],H) \cap C([0,T],D(A))$ are given elements and function, respectively. Moreover,

$$\lambda_1 < \lambda_2 < ... < \lambda_q, \beta = \sum_{i=1}^{q} \beta_i \leq 1.$$

In paper [10] the first and second order of accuracy stable DSs were proposed. The objective of this work is to study the fourth order of ADS for multi-point elliptic overdetermined problem (1) in an arbitrary Hilbert space H with a SAPDO A.

Let $[\cdot]$ be the greatest integer function and

$$l_i = \left[\tfrac{\lambda_i}{\tau}\right], \mu_i = \tfrac{\lambda_i}{\tau} - l_i,$$

$$\mu_{i,1} = \tfrac{1}{12}\mu_i - \tfrac{1}{12}\mu_i^3, \mu_{i,2} = -\tfrac{2}{3}\mu_i + \tfrac{1}{2}\mu_i^2 + \tfrac{1}{6}\mu_i^3,$$

$$\mu_{i,3} = 1 - \mu_i^2, \mu_{i,4} = \tfrac{2}{3}\mu_i + \tfrac{1}{2}\mu_i^2 - \tfrac{1}{6}\mu_i^3,$$

$$\mu_{i,5} = -\mu_{i,1}, i = 0, 1, 2, 3, 4, 5.$$

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Introduce the following notations

$$C = A + \frac{\tau^2}{12}A, D = \frac{1}{2}\left(\tau C + \left(4C + \tau^2 C^2\right)^{\frac{1}{2}}\right),$$

$$P = (I + \tau D)(2I + \tau D)^{-1} D^{-1}, R = (I + \tau D)^{-1}.$$

Stability estimates

Lemma 1. The following estimates hold [5]:

$$\begin{aligned} &\left\|\exp\left(k\tau A^{\frac{1}{2}}\right)R^k\right\|_{H\to H} \leq M(\delta)\tau^{-k}, \left\|R^k\right\|_{H\to H} \leq M(1+\delta\tau)^{-k}, k \geq 1, \\ &\left\|\left(I - R^{2N}\right)^{-1}\right\|_{H\to H} \leq M(\delta),\ k\tau\left\|DR^k\right\|_{H\to H} \leq M(\delta), \\ &\left\|D^\beta\left(R^{k+r} + R^k\right)\right\|_{H\to H} \leq M(\delta)\frac{(r\tau)^\alpha}{(k\tau)^{\alpha+\beta}}, \\ &1 \leq k \leq k+r \leq N, 0 \leq \alpha, \beta \leq 1, \delta > 0. \end{aligned} \tag{2}$$

Lemma 2. For $1 \leq l_i \leq N-1$, $1 \leq l_0 \leq N-1$, the operator ([10; 861]

$$\Delta = \left(I - R^{2N}\right)\left(I - R^{l_0}\right)\left(I - \sum_{i=1}^{q} k_i R^{N-l_i}\right)\left(I - \sum_{i=1}^{q} k_i R^{N-l_0+l_i}\right) \tag{3}$$

has a bounded inverse Δ^{-1} such that

$$\left\|\Delta^{-1}\right\|_{H\to H} \leq M(\delta).$$

Let us take $1 \leq l_i \leq N-1, 0 \leq i \leq q$. Denote by

$$\begin{aligned} J_1 = &-\left(I - R^{2N}\right)\left\{\mu_{0,1}\left(R^{l_0-2} - R^{2N-l_0+2}\right) - \mu_{0,2}\left(R^{l_0-1} - R^{2N-l_0+1}\right)\right. \\ &+\mu_0^2\left(R^{l_0} - R^{2N-l_0}\right) - \mu_{0,4}\left(R^{l_0+1} - R^{2N-l_0-1}\right) \\ &-\mu_{0,5}\left(R^{l_0+2} - R^{2N-l_0-2}\right) + \sum_{i=1}^{q} k_i\left[-\mu_{i,1}\left(R^{N-l_i+2} - R^{N+l_i-2}\right)\right. \\ &-\mu_{i,2}\left(R^{N-l_i+1} - R^{N+l_i-1}\right) + \mu_i^2\left(R^{N-l_i} - R^{N+l_i}\right) \\ &\left.\left.-\mu_{i,2}\left(R^{N-l_i-1} - R^{N+l_i+1}\right) - \mu_{i,5}\left(R^{N-l_i-2} - R^{N+l_i+2}\right)\right]\right\} \end{aligned} \tag{4}$$

$$\begin{aligned} J_2 = &-\left(I - R^{2N}\right)\left\{\sum_{i=1}^{q} k_i\left\{\mu_{0,1}\mu_{i,5}\left(R^{N-l_0+l_i+4} - R^{N+l_0-l_i-4}\right)\right.\right. \\ &+\left(\mu_{0,1}\mu_{i,4} + \mu_{0,2}\mu_{i,5}\right)\left(R^{N-l_0+l_i+3} - R^{N+l_0-l_i-3}\right) \\ &+\left(\mu_{0,1}\mu_{i,3} + \mu_{0,2}\mu_{i,4} + \mu_{0,3}\mu_{i,5}\right)\left(R^{N-l_0+l_i+2} - R^{N+l_0-l_i-2}\right) \\ &\left.+\left(\mu_{0,1}\mu_{i,2} + \mu_{0,2}\mu_{i,3} + \mu_{0,3}\mu_{i,4} + \mu_{0,4}\mu_{i,5}\right)\left(R^{N-l_0+l_i+1} - R^{N+l_0-l_i-1}\right)\right\} \end{aligned} \tag{5}$$

$$\begin{aligned} J_3 = &-\left(I - R^{2N}\right)\sum_{i=1}^{q} k_i\left\{\left(\mu_{0,1}\mu_{i,1} + \mu_{0,2}\mu_{i,2} + \mu_0^2\mu_i^2 - \mu_0^2 - \mu_i^2\right.\right. \\ &\left.\left.+\mu_{0,4}\mu_{i,4} + \mu_{0,5}\mu_{i,5}\right)\right\}\left(R^{N-l_0+l_i} - R^{N+l_0-l_i}\right) \end{aligned} \tag{6}$$

$$J_4 = -\left(I - R^{2N}\right)\left\{\sum_{i=1}^{q} k_i \left\{(\mu_{0,2}\mu_{i,1} + \mu_{0,3}\mu_{i,2} + \mu_{0,4}\mu_{i,3} + \mu_{0,5}\mu_{i,4})\right.\right.$$
$$\times \left(R^{N-l_0+l_i-1} - R^{N+l_0-l_i+1}\right) + (\mu_{0,3}\mu_{i,1} + \mu_{0,4}\mu_{i,2} + \mu_{0,5}\mu_{i,3}) \tag{7}$$
$$\times \left(R^{N-l_0+l_i-2} - R^{N+l_0-l_i+2}\right) + (\mu_{0,4}\mu_{i,1} + \mu_{0,5}\mu_{i,2})$$
$$\left.\left.\times \left(R^{N-l_0+l_i-3} - R^{N+l_0-l_i+3}\right) + \mu_{0,5}\mu_{i,1}\left(R^{N-l_0+l_i-4} - R^{N+l_0-l_i+4}\right)\right\}\right\}.$$

Lemma 3. Let the operators $\Delta, J_1, J_2, J_3, J_4$ be defined by (3), (4), (5), (6), (7), correspondingly. Then, the operator

$$G = \Delta + J_1 + J_2 + J_3 + J_4 \tag{8}$$

has a bounded inverse G^{-1} such that

$$\left\|G^{-1}\right\|_{H\to H} \le M(\delta). \tag{9}$$

Proof. We have

$$G^{-1} - \Delta^{-1} = G^{-1}\Delta^{-1}K, \tag{10}$$

where

$$K = J_1 + J_2 + J_3 + J_4.$$

Applying (2), it can be showed that the estimates

$$\|J_i\|_{H\to H} \le M\tau, i = 1, 2, 3, 4$$

hold for constant M which does not depend on τ.

Consequently,

$$\|K\|_{H\to H} \le M\tau. \tag{11}$$

By using (10), (11) and triangle inequality, we can get

$$\left\|G^{-1}\right\|_{H\to H} \le \left\|\Delta^{-1}\right\|_{H\to H} + \left\|\Delta^{-1}\right\|_{H\to H}\left\|G^{-1}\right\|_{H\to H}\|K\|_{H\to H}$$
$$\le M(\delta) + \left\|G^{-1}\right\|_{H\to H} M(\delta)\, M\tau$$

for any small positive number τ. Therefore, estimate (9) is valid.

Let $[0,T]_\tau = \{t_k = k\tau, 0 \le k \le N, N\tau = T\}$ be space of grid points and $v_k = v(t_k), 0 \le k \le N$.

Denote by $C(H)$ and $C_{0T}^{\alpha,\alpha}(H)$ the corresponding Banach spaces of H-valued grid functions $\{w_k\}_0^N$ with norms

$$\left\|\{w_k\}_1^{N-1}\right\|_{C(H)} = \max_{0\le k\le N-1}\|w_k\|_H,$$

$$\left\|\{w_k\}_1^{N-1}\right\|_{C_{0T}^{\alpha,\alpha}(H)} = \left\|\{w_k\}_1^{N-1}\right\|_{C(H)} + \sup_{0\le k<k+\tau\le N-1}(k\tau + n\tau)^\alpha n\tau^{-\alpha}(T - k\tau)^\alpha\|w_{k+n} - w_k\|_H,$$

respectively.

Applying the fourth order of approximation for function v at point λ_i, $i = 0, 1, ..., q$

$$v(\lambda_i) = \mu_{i,1}v_{l_i-2} + \mu_{i,2}v_{l_i-1} + \mu_{i,3}\, v_{l_i} + \mu_{i,4}v_{l_i+1} + \mu_{i,5}v_{l_i+2}$$

and fourth order of accuracy approximation of differential equation, one can get the next DS

$$\begin{cases} -\tau^{-2}\left(v_{k+1} - 2v_k + v_{k-1}\right) + Cv_k = \theta_k + p, \\ \psi_k = g(t_k) + \frac{\tau^2}{12}\left(\frac{g(t_{k+1})-2g(t_k)+g(t_{k-1})}{\tau^2} + Ag(t_k)\right), \\ t_k = k\tau, 1 \le k \le N-1, N\tau = T, v_0 = \phi, \\ \mu_{0,1}v_{l_0-2} + \mu_{0,2}v_{l_0-1} + \mu_{0,3}v_{l_0} + \mu_{0,4}v_{l_0+1} + \mu_{0,5}v_{l_0+2} = \zeta, \\ v_N = \sum_{i=1}^{q} k_i\left\{\mu_{i,1}v_{l_i-2} + \mu_{i,2}v_{l_i-1} + \mu_{i,3}v_{l_i} + \mu_{i,4}v_{l_i+1} + \mu_{i,5}v_{l_i+2}\right\} + \eta \end{cases} \tag{12}$$

for approximately solution of problem (1).

We will find solution of DS (1) by formula

$$v_k = u_k + A^{-1}p, \tag{13}$$

where grid function $\{u_k\}_0^N$ is a solution of the following difference problem:

$$\begin{cases} -\tau^{-2}\left(u_{k+1} - 2u_k + u_{k-1}\right) + Au_k + \frac{\tau^2}{12}A^2u_k = \psi_k, \\ t_k = k\tau, 1 \le k \le N-1, N\tau = T, \\ u_0 - \mu_{0,1}u_{l_0-2} - \mu_{0,2}u_{l_0-1} - \mu_{0,3}u_{l_0} - \mu_{0,4}u_{l_0+1} - \mu_{0,5}u_{l_0+2} = \phi - \zeta, \\ u_N - \sum\limits_{i=1}^{q} k_i\left\{\mu_{i,1}u_{l_i-2} + \mu_{i,2}u_{l_i-1} + \mu_{i,3}u_{l_i} + \mu_{i,4}u_{l_i+1} + +\mu_{i,5}u_{l_i+2}\right\} = \eta. \end{cases} \tag{14}$$

After soving DS (14), unknown element p is defined by

$$p = A\phi - Au_0. \tag{15}$$

Theorem 1. Let $\phi, \zeta, \eta \in D(A)$ and $\{\psi_k\}_1^{N-1} \in C(H)$ be given. Then, the difference problem (12) has a solution $(\{v_k\}_1^{N-1}, p)$ which satisfies the stability estimates in below:

$$\left\|\{v_k\}_1^{N-1}\right\|_{C(H)} \le M(\delta)\left[\|\phi\|_H + \|\zeta\|_H + \|\eta\|_H + \left\|\{\psi_k\}_1^{N-1}\right\|_{C(H)}\right], \tag{16}$$

$$\left\|A^{-1}p\right\|_H \le M(\delta)\left[\|\phi\|_H + \|\zeta\|_H + \|\eta\|_H + \left\|\{\psi_k\}_1^{N-1}\right\|_{C(H)}\right],$$

where $M(\delta)$ is independent from ϕ, ζ, η and $\{\psi_k\}_1^{N-1}$.

Proof. For given u_0 and u_N the solution of difference problem

$$-\tau^2\left(u_{k+1} - 2u_k + u_{k-1}\right) + Au_k = \psi_k, 1 \le k \le N-1 \tag{17}$$

is defined by [5]

$$\begin{aligned} u_k = {} & \left(I - R^{2N}\right)^{-1}\Big[\left(R^k - R^{2N-k}\right)u_0 + \left(R^{N-k} - R^{N+k}\right)u_N - \left(R^{N-k} - R^{N+k}\right) \\ & \times (I + \tau D)(2I + \tau D)^{-1}D^{-1}\sum_{j=1}^{N-1}\left(R^{N-j} - R^{N+j}\right)\psi_j\tau\Big] + (I + \tau D) \\ & \times (2I + \tau D)^{-1}D^{-1}\textstyle\sum_{j=1}^{N-1}\left(R^{|k-j|} - R^{k+j}\right)\psi_j\tau. \end{aligned} \tag{18}$$

Applying formula (18) to nonlocal conditions of the difference problem (14), we get a system equation for u_0 and u_N:

$$s_{11}u_0 + s_{12}u_N = S_1, s_{21}u_0 + s_{22}u_N = S_2. \tag{19}$$

Here operators $s_{11}, s_{12}, s_{21}, s_{22}, S_1$ and S_2 are defined by

$$s_{11} = \left(I - R^{2N}\right) - \mu_{0,1}\left(R^{l_0-2} - R^{2N-l_0+2}\right) - \mu_{0,2}\left(R^{l_0-1} - R^{2N-l_0+1}\right)$$

$$-\mu_{0,3}\left(R^{l_0} - R^{2N-l_0}\right) - \mu_{0,4}\left(R^{l_0+1} - R^{2N-l_0-1}\right) - \mu_{0,5}\left(R^{l_0+2} - R^{2N-l_0-2}\right),$$

$$s_{12} = -\left\{\mu_{0,1}\left(R^{N-l_0+2} - R^{N+l_0-2}\right) + \mu_{0,2}\left(R^{N-l_0+1} - R^{N+l_0-1}\right)\right.$$

$$\left.+\mu_{0,3}\left(R^{N-l_0} - R^{N+l_0}\right) + \mu_{0,4}\left(R^{N-l_0-1} - R^{N+l_0+1}\right) + \mu_{0,5}\left(R^{N-l_0-2} - R^{N+l_0+2}\right)\right\},$$

$$s_{21}=\sum_{i=1}^{q}k_i\left\{-\mu_{i,1}\left(R^{l_i-2}-R^{2N-l_i+2}\right)-\mu_{i,2}\left(R^{l_i-1}-R^{2N-l_i+1}\right)\right.$$
$$\left.-\mu_{i,3}\left(R^{l_i}-R^{2N-l_i}\right)-\mu_{i,4}\left(R^{l_i+1}-R^{2N-l_i-1}\right)-\mu_{i,5}\left(R^{l_i+2}-R^{2N-l_i-2}\right)\right\},$$
$$s_{22}=\sum_{i=1}^{q}k_i\left\{\left(I-R^{2N}\right)-\mu_{i,1}\left(R^{N-l_i+2}-R^{N+l_i-2}\right)-\mu_{i,2}\left(R^{l_i-1}-R^{2N-l_i+1}\right)\right.$$
$$\left.-\mu_{i,3}\left(R^{l_i}-R^{2N-l_i}\right)-\mu_{i,4}\left(R^{N-l_i-1}-R^{N+l_i+1}\right)-\mu_{i,5}\left(R^{N-l_i-2}-R^{N+l_i+2}\right)\right\}, \tag{20}$$

$$S_1=\left(I-R^{2N}\right)(\phi-\zeta)+\left[\tfrac{1}{12}\mu_0-\tfrac{1}{12}\mu_0^3\right]\left\{-\left(R^{N-l_0+2}-R^{N+l_0-2}\right)\right.$$
$$\left.\times P\sum_{j=1}^{N-1}\left(R^{N-j}-R^{N+j}\right)\theta_j\tau+\left(I-R^{2N}\right)P\sum_{j=1}^{N-1}\left(R^{|l_0-2-j|}-R^{l_0-2+j}\right)\theta_j\tau\right\}$$
$$+\left[-\tfrac{8}{12}\mu_0+\tfrac{1}{2}\mu_0^2+\tfrac{1}{6}\mu_0^3\right]\left\{-\left(R^{N-l_0+1}-R^{N+l_0-1}\right)\right.$$
$$\left.\times P\sum_{j=1}^{N-1}\left(R^{N-j}-R^{N+j}\right)\theta_j\tau+\left(I-R^{2N}\right)P\sum_{j=1}^{N-1}\left(R^{|l_0-1-j|}-R^{l_0-1+j}\right)\theta_j\tau\right\}$$
$$+\left[1-\mu_0^2\right]\left\{-\left(R^{N-l_0}-R^{N+l_0}\right)P\sum_{j=1}^{N-1}\left(R^{N-j}-R^{N+j}\right)\theta_j\tau+\left(I-R^{2N}\right)\right.$$
$$\left.\times P\sum_{j=1}^{N-1}\left(R^{|l_0-j|}-R^{l_0+j}\right)\theta_j\tau\right\}$$
$$+\left[\tfrac{8}{12}\mu_0+\tfrac{1}{2}\mu_0^2-\tfrac{1}{6}\mu_0^3\right]\left\{-\left(R^{N-l_0-1}-R^{N+l_0+1}\right)\right.$$
$$\left.\times P\sum_{j=1}^{N-1}\left(R^{N-j}-R^{N+j}\right)\theta_j\tau+\left(I-R^{2N}\right)P\sum_{j=1}^{N-1}\left(R^{|l_0+1-j|}-R^{l_0+1+j}\right)\theta_j\tau\right\}$$
$$+\left[-\tfrac{1}{12}\mu_0+\tfrac{1}{12}\mu_0^3\right]\left\{-\left(R^{N-l_0-2}-R^{N+l_0+2}\right)\right.$$
$$\left.\times P\sum_{j=1}^{N-1}\left(R^{N-j}-R^{N+j}\right)\theta_j\tau+\left(I-R^{2N}\right)P\sum_{j=1}^{N-1}\left(R^{|l_0+2-j|}-R^{l_0+2+j}\right)\theta_j\tau\right\}, \tag{21}$$

$$S_2=\left(I-R^{2N}\right)\eta+\sum_{i=1}^{q}k_i\left\{\mu_{i,1}\left[-\left(R^{N-l_i+2}-R^{N+l_i-2}\right)\times\right.\right.$$
$$\left.\times P\sum_{j=1}^{N-1}\left(R^{N-j}-R^{N+j}\right)\psi_j\tau+\left(I-R^{2N}\right)P\sum_{j=1}^{N-1}\left(R^{|l_i-2-j|}-R^{l_i-2+j}\right)\psi_j\tau\right]+$$
$$+\mu_{i,2}\left[-\left(R^{N-l_i+1}-R^{N+l_i-1}\right)P\sum_{j=1}^{N-1}\left(R^{N-j}-R^{N+j}\right)\psi_j\tau+\right.$$
$$\left.+\left(I-R^{2N}\right)P\sum_{j=1}^{N-1}\left(R^{|l_i-1-j|}-R^{l_i-1+j}\right)\psi_j\tau\right]+$$

$$+\mu_{i,3}\left[-\left(R^{N-l_i}-R^{N+l_i}\right)P\sum_{j=1}^{N-1}\left(R^{N-j}-R^{N+j}\right)\psi_j\tau+\left(I-R^{2N}\right)\times\right.$$

$$\left.\times P\sum_{j=1}^{N-1}\left(R^{|l_i-j|}-R^{l_i+j}\right)\psi_j\tau\right]+\mu_{i,4}\left[-\left(R^{N-l_i-1}-R^{N+l_i+1}\right)\right.$$

$$\left.\times P\sum_{j=1}^{N-1}\left(R^{N-j}-R^{N+j}\right)\psi_j\tau+\left(I-R^{2N}\right)P\sum_{j=1}^{N-1}\left(R^{|l_i+1-j|}-R^{l_i+1+j}\right)\psi_j\tau\right]+$$

$$+\mu_{i,5}\left[-\left(R^{N-l_i-2}-R^{N+l_i+2}\right)P\sum_{j=1}^{N-1}\left(R^{N-j}-R^{N+j}\right)\psi_j\tau+\right. \tag{22}$$

$$\left.\left.+\left(I-R^{2N}\right)P\sum_{j=1}^{N-1}\left(R^{|l_i+2-j|}-R^{l_i+2+j}\right)\psi_j\tau\right]\right\}.$$

Determinant operator $G=s_{11}s_{22}-s_{12}s_{21}$ of the system equations (19) can be rewritten as (8). Consequently, according to Lemma 3, the operator G has bounded inverse G^{-1}. So, the system of equations (19) has a unique solution:

$$u_0=G^{-1}\left(S_1s_{22}-S_2s_{21}\right),u_N=G^{-1}s_{11}S_2-s_{12}S_1. \tag{23}$$

Thus, difference problem (14) has a unique solution $\{u_k\}_0^N$ which is defined by formula (18) with corresponding $s_{11},s_{12},s_{21},s_{22},S_1,S_2,u_0,\ u_N$ by (20)–(23).

For the solution of problem (17) the following inequality [5]

$$\left\|\{u_k\}_0^{N-1}\right\|_{C(H)}\le M\left[\left\|\{\psi_k\}_1^{N-1}\right\|_{C(H)}+\|Ru_0\|_H+\|Ru_N\|_H\right] \tag{24}$$

is valid. By virtue triangle, Cauchy-Schwarz inequalities and (2) one can obtain

$$\max\left\{\|S_1\|_{C(H)},\|S_2\|_{C(H)}\right\}\le M(\delta)\left(\|\phi\|_H+\|\zeta\|_H+\left\|\{\psi_k\}_1^{N-1}\right\|_{C(H)}\right).$$

Applying Cauchy-Schwarz and triangle inequalities to (19) and by using (2), (9), we have

$$\|Ru_0\|_H\le M(\delta)\left[\|\phi\|_H+\|\zeta\|_H+\|\eta\|_H+\left\|\{\psi_k\}_1^{N-1}\right\|_{C(H)}\right],$$

$$\|Ru_N\|_H\le M(\delta)\left[\|\phi\|_H+\|\zeta\|_H+\|\eta\|_H+\left\|\{\psi_k\}_1^{N-1}\right\|_{C(H)}\right].$$

So, by using (24) we get

$$\left\|\{u_k\}_1^{N-1}\right\|_{C(H)}\le M(\delta)\left[\|\phi\|_H+\|\zeta\|_H+\|\eta\|_H+\left\|\{\psi_k\}_1^{N-1}\right\|_{C(H)}\right]. \tag{25}$$

Finally, by virtue (15) and (23) we can establish

$$\left\|A^{-1}p\right\|_H\le M\left(\delta\right)\left[\|\phi\|_H+\|\zeta\|_H+\|\eta\|_H+\left\|\{\psi_k\}_1^{N-1}\right\|_{C(H)}\right].$$

Now, from (25), (13) and triangle inequality one can get inequality (16).

Then, DS (12) has a solution $(\{v_k\}_1^{N-1},p)$, which satisfies stability estimates given in the below theorems.

Theorem 2. Let $\phi,\zeta,\eta\in D(A)\cap D(C)$ and $\{\psi_k\}_1^{N-1}\in C(H)$ be given. Then, for solution $(\{v_k\}_1^{N-1},p)$ of DS (12) almost coercive stability estimate hold:

$$\left\|\left\{\tfrac{(v_{k+1}-2v_k+v_{k-1})}{\tau^2}\right\}_1^{N-1}\right\|_{C(H)}+\left\|\{Cv_k\}_1^{N-1}\right\|_{C(H)}+\|p\|_H$$
$$\le M(\delta)\left\{\min\left[\ln\left(\tfrac{1}{\tau}\right),1+\left|\ln\|D\|_{H\to H}\right|\right]\left\|\{\psi_k\}_1^{N-1}\right\|_{C(H)}+\|C\phi\|_H+\|C\zeta\|_H+\|C\eta\|_H\right\},$$

where $M(\delta)$ does not depend on ϕ, ζ, η and $\{\psi_k\}_1^{N-1}$.

Theorem 3. Let $\phi, \zeta, \eta \in D(A) \cap D(C)$ and $\{\psi_k\}_1^{N-1} \in C_{0T}^{\alpha,\alpha}(H)(0<\alpha<1)$ be given. Then, for solution $(\{v_k\}_1^{N-1}, p)$ of difference problem (12) coercive stability estimate

$$\left\|\left\{\tau^{-2}\left(v_{k+1}-2v_k+v_{k-1}\right)\right\}_1^{N-1}\right\|_{C_{0T}^{\alpha,\alpha}(H)} + \left\|\left\{Cv_k\right\}_1^{N-1}\right\|_{C_{0T}^{\alpha,\alpha}(H)} + \|p\|_H$$
$$\leq M(\delta)\left[\frac{1}{(1-\alpha)\alpha}\left\|\{\psi_k\}_1^{N-1}\right\|_{C_{0T}^{\alpha,\alpha}(H)} + \|C\phi\|_H + \|C\zeta\|_H + \|C\eta\|_H\right]$$

is true. Here $M(\delta)$ is independent from $\alpha, \phi, \zeta, \eta$ and $\{\psi_k\}_1^{N-1}$.

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Көпнүктелі қайтаанықталған эллипстік есеп үшін төртінші ретті дәлдікті тұрақтылық айырымдық схемасы туралы

Мақалада гильберттік кеңістікте көпнүктелі эллипстік қайтаанықталған есептің жуық шешімін табу үшін төртінші ретті дәлдікті айырымдық схема ұсынылды. Айырымдық схеманың шешімінің бар және жалғыз болуы функционалды-операторлық тәсілді қолдану арқылы алынады. Айырымдық схеманың шешімінің тұрақтылық, дерлік тұрақтылық және коэрцитивті тұрақтылық бағалаулары анықталды. Бұл теориялық нәтижелерді дербес туындылы көп өлшемді эллипстік теңдеулер үшін көпнүктелі қайтаанықталған шеттік есептің жуық шешімін табу үшін тұрақты жоғары дәлдіктегі айырымдық схеманы құру үшін қолдануға болады.

Кілт сөздер: қайтаанықталған эллипстік есеп, көпнүктелі шарт, жоғары ретті айырымдық схема, айырымдық схема, кері, дереккөзді идентификациялау есебі, корректілік, тұрақтылық, коэрцитивті тұрақтылық, дерлік коэрцитивті тұрақтылық.

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Об устойчивой разностной схеме четвертого порядка точности для многоточечной переопределенной эллиптической задачи

В статье предложена разностная схема четвертого порядка точности для приближенного решения многоточечной эллиптической переопределенной задачи в гильбертовом пространстве. Существование и единственность решения разностной схемы получены с использованием функционально-операторного подхода. Установлены оценки устойчивости, почти коэрцитивной устойчивости и коэрцитивной устойчивости решения разностной схемы. Эти теоретические результаты могут быть применены для построения устойчивой высокоточной разностной схемы для приближенного решения многоточечной переопределенной краевой задачи для многомерных эллиптических уравнений в частных производных.

Ключевые слова: переопределенная эллиптическая задача, многоточечное условие, разностная схема высокого порядка, разностная схема, обратная, задача идентификации источника, корректность, устойчивость, коэрцитивная устойчивость, почти коэрцитивная устойчивость.

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A new finite difference method for computing approximate solutions of boundary value problems including transition conditions

This article is aimed at computing numerical solutions of new type of boundary value problems (BVPs) for two-linked ordinary differential equations. The problem studied here differs from the classical BVPs such that it contains additional conditions at the point of interaction, so-called transition conditions. Naturally, such type of problems is much more complicated to solve than classical problems. It is not clear how to apply the classical numerical methods to such type of boundary value transition problems (BVTPs). Based on the finite difference method (FDM) we have developed a new numerical algorithm for computing numerical solution of BVTPs for two-linked ordinary differential equations. To demonstrate the reliability and efficiency of the presented algorithm we obtained numerical solution of one BVTP and the results are compared with the corresponding exact solution. The maximum absolute errors (MAEs) are presented in a table.

Keywords: finite difference method, transition condition, boundary value problems, second order differential equation.

Introduction

Sturm-Liouville BVPs arise as mathematical models of many problems in physics and engineering, such as Newton's law of cooling, the population growth of decay, Kirchoff's law in electrical circuits, the steady-state temperature in heated rod, thermodynamics, resistor, and inductor circuits, etc (see, for example, [1–7] and references cited therein). It is obvious that not all BVPs can be solved analytically. Even if a BVP can be solved analytically, the closed-form of the analytical solution may take some complicated form that is unhelpful to use. Therefore we have to apply various numerical methods for determining the approximate solution. There have been developed different numerical methods to solve various type of BVPs. One of them, the so-called FDM, can be applied to a wide class of BVPs, provided that the problem considered has a complete set of continuty and boundary conditions. In this study we will consider a BVP of a new type. The main feature of this problem is the nature of the imposed boundary conditions, which include not only the ends of the interval under consideration but also one inner point of the singularity. Naturally, such type of singular problems is much more difficult to solve than regular problems. We will develop a new modification of classical FDM to solve BVPs involving additional transition conditions at the point of singularity. Such type of singular problems arises in heat and mass transfer problems, in vibrating string problems, and in a varied assortment of physical transfer problems (see, for example, [8–12] and references cited therein).

A new modification of finite difference method

Let us consider a linear BVP for the second order differential equation given by

$$u^{''}(x) + p(x)u^{'}(x) + q(x)u(x) = f(x), \quad x \in [a, c) \cup (c, b], \tag{1}$$

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subject to the boundary conditions (BCs) at end-points $x = a$ and $x = b$ given by

$$u(a) = \alpha, \quad u(b) = \beta \tag{2}$$

together with additional transition conditions at the interior point of singularity $x = c$ given by

$$u(c-0) = \xi u(c+0), \quad u'(c-0) = \psi u'(c+0), \tag{3}$$

where $p(x)$, $q(x)$ and $f(x)$ are continuous on [a,c) and (c,b] with the finite limits $p(c \mp 0)$, $q(c \mp 0)$, $f(c \mp 0)$, α, β, ξ ,ψ are real constants. For convenience, we will use the notations $[a_1, b_1] = [a, c]$, $[a_2, b_2] = [c, b]$. To discretize BVTP (1)-(3) the interval $[a_k, b_k]$, $k = 1, 2$ are divided into finite number f intervals $[x_{k,0}, x_{k,1}]$, $[x_{k,1}, x_{k,2}]$, ..., $[x_{k,N-1}, x_{k,N}]$ with

$$a_k = x_{k,0} < x_{k,1} < ... < x_{k,N} = b_k,$$

where

$$x_{k,i} = a_k + ih_k, \quad h_k = \frac{b_k - a_k}{N}, \quad k = 1, 2 \quad , i = 1, 2, ..., N.$$

Below we will use the central finite difference discretization. Namely, we will express the first and second derivatives of the unknown function

$$u(x) = \begin{cases} u_1(x), & for \quad x \in [a_1, b_1) , \\ u_2(x), & for \quad x \in (a_2, b_2] \end{cases}$$

as

$$u'_k(x) \approx \frac{u_k(x + h_k) - u_k(x - h_k)}{2h_k}$$

and

$$u''_k(x) \approx \frac{u_k(x + h_k) - 2u_k(x) + u_k(x - h_k)}{{h_k}^2},$$

respectively. Let us denote the value of the unknown function $u(x)$ at the nodal point $x_{k,i}$ by $u_{k,i}$ and substitute in equation (1). We have the following linear system of equations for each $k = 1, 2$

$$\left(1 - \frac{1}{2}h_k p_{k,i}\right) u_{k,i-1} + \left(-2 + {h_k}^2 q_{k,i}\right) u_{k,i} + \left(1 + \frac{1}{2}h_k u_{k,i+1}\right) u_{i+1} = {h_k}^2 f\left(x_{k,i}\right), \tag{4}$$

$$i = 1, 2, 3, ..., N - 1,$$

where

$$u(a) = u_{1,0} = \alpha, \quad u(b) = u_{2,N} = \beta.$$

Let us introduce to two new parameters $\beta_1 := u_{1,N}$ and $\alpha_2 := u_{2,0}$ that will be calculated later. For convenience, we will use the notations $\alpha_1 := \alpha$ and $\beta_2 := \beta$.

Note that each of finite difference equation (4) involves solutions $u_{k,i-1}$, $u_{k,i}$, and $u_{k,i+1}$ at the nodal points $x_{k,i-1}$, $x_{k,i}$, and $x_{k,i+1}$, respectively.

This system of linear equations can be written in matrix form

$$A_k U_k = B_k, \quad k = 1, 2, \tag{5}$$

where A_k is the $(N-1)\times(N-1)$ matrix given by

$$A_k = \begin{pmatrix} -2+{h_k}^2q_{k,1} & 1+\frac{1}{2}h_kp_{k,1} & 0 & \cdots & 0 & 0 \\ 1-\frac{1}{2}h_kp_{k,2} & -2+h^2{}_kq_{k,2} & 1+\frac{1}{2}h_kp_{k,2} & \cdots & 0 & 0 \\ 0 & 1-\frac{1}{2}h_kp_{k,3} & -2+h^2{}_kq_{k,3} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -2+h^2{}_kq_{k,N-3} & 1+\frac{1}{2}h_kp_{k,N-3} & 0 \\ 0 & 0 & \cdots & 1-\frac{1}{2}h_kp_{k,N-2} & -2+h^2{}_kq_{k,N-2} & 1+\frac{1}{2}h_kp_{k,N-2} \\ 0 & 0 & \cdots & 0 & 1-\frac{1}{2}h_kp_{k,N-1} & -2+h^2{}_kq_{k,N-1} \end{pmatrix}$$

and U_k and B_k are column vectors given by

$$U_k = \begin{pmatrix} u_{k,1} \\ u_{k,2} \\ u_{k,3} \\ \vdots \\ u_{k,N-3} \\ u_{k,N-2} \\ u_{k,N-1} \end{pmatrix} \quad and \quad B_k = \begin{pmatrix} b_{k,1} \\ b_{k,2} \\ b_{k,3} \\ \vdots \\ b_{k,N-3} \\ b_{k,N-2} \\ b_{k,N-1} \end{pmatrix},$$

where

$$b_{k,i} = \begin{cases} h^2{}_kf_{k,1} - \left(1-\frac{1}{2}hp_{k,1}\right), & i=1, \\ h^2{}_kf_{k,i}, & i=2,3,...,N-2, \\ h^2{}_kf_{k,N-1} - \left(1+\frac{1}{2}hp_{k,N-1}\right)\beta_k, & i=N-1 \end{cases}$$

and $f_{k,i} = f(x_{k,i})$.

Since the linear system of algebraic equations (5) is tridiagonal, it can be solved by the Crout or Cholesky algorithm (see [2]). To satisfy transition conditions (3) we have the following equations

$$u_{1,N} = \xi u_{2,0},$$

$$\frac{u_{1,N}-u_{1,N-1}}{h_1} = \psi\frac{u_{2,1}-u_{2,0}}{h_2},$$

From which we can easily find the numerical solutions $u_{1,N}$ and $u_{2,0}$. Thus we find all the numerical solutions $u_{k,0}, u_{k,1}, ..., u_{k,N}, k=1,2$.

Numerical illustration

Let us consider the following BVP on the disjoint intervals $[-1,0)$ and $(0,1]$ consisting of linear differential equation

$$u'' = (1+2tan(x)^2)u, \quad x\in[-1,0)\cup(0,1] \tag{6}$$

together with boundary conditions at the end-points $x = -1, 1$ given by

$$u\,(-1) = 2, \quad u\,(1) = -1 \tag{7}$$

and with additional transition conditions at the interior point of singularity $x = 0$ given by

$$u(-0) = 5u(+0), \quad 3u'(-0) = u'(+0). \tag{8}$$

At first we will investigate this problem without transition conditions.We can show that the exact solution of the BVP (6) and BVP (7) is

$$u(x) = \frac{cos(1)}{2} sec(x) + \frac{3cos(1)}{2 + 2sin(x)} \left(sin(x) + xsec(x)\right). \tag{9}$$

Consider the uniform cartesian grid $x_i = -1 + ih$, $i = 1, \quad \ldots, 49$ for $N = 50$, i.e, $h = \frac{x_0 - x_{20}}{N} = \frac{1-(-1)}{50} = 0,04$ where in particular $x_0 = -1, x_{20} = 1$, $u_0 = 2, u_{50} = -1$. By using the central FDM at a typical grid point x_i, we obtain

$$u_{i-1} + (-2 - h^2(1 + 2tan^2(x_i)))u_i + u_{i+1} = 0 \tag{10}$$

for $i = 1, 2, \ldots, 49$. Consequently, the finite difference solution $u_i \approx u(x_i)$ is defined as the solution of the linear algebraic system of equations (10). In a tridiagonal matrix-vector form, this linear algebraic system of equations can be written as

$$Au = B, \tag{11}$$

where

$$A = \begin{pmatrix} -2 - h^2(1 + 2tan^2(x_1)) & 1 & 0 & \cdots & 0 \\ 1 & -2 - h^2(1 + 2tan^2(x_2)) & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & -2 - h^2(1 + 2tan^2(x_{49})) \end{pmatrix},$$

$$u = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_{48} \\ u_{49} \end{pmatrix}, \quad B = \begin{pmatrix} -2 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}.$$

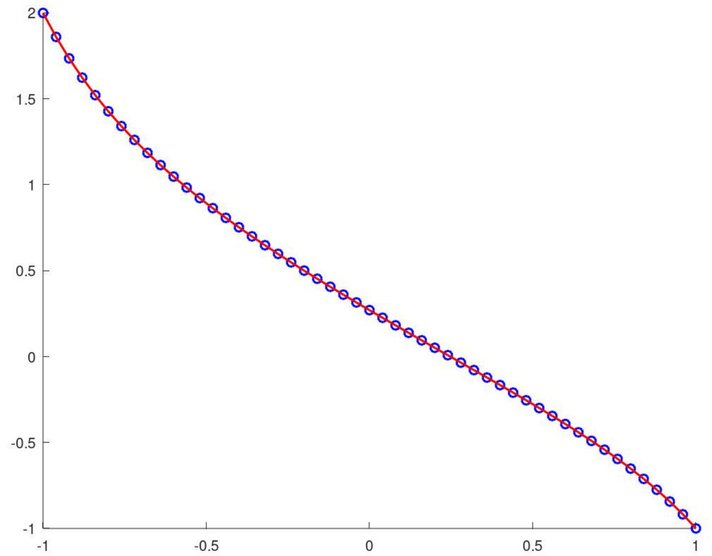

Figure 1. Comparison of FDM-solution and exact solution for the equation (6) and equation (7) for N=50.

Now we shall investigate of BVP (6) and BVP (7) under additional transition conditions (8). We can find the exact solution of this problem in the following form:

$$u = \begin{cases} \frac{25cos(1)}{16} sec(x) - \frac{7cos(1)}{8(2+sin(1))}(sin(x) + xsec(x)) \ , \quad x \in [-1, 0), \\ \frac{5cos(1)}{16} sec(x) - \frac{21cos(1)}{8(2+sin(2))}(sin(x) + xsec(x)) \ , \quad x \in [0, 1). \end{cases}$$

Letting $N = 49$ and applying the transition conditions (8) we have two additional algebraic equations

$$u_{24} - 5u_{25} = 0, \quad 3u_{23} - 3u_{24} - u_{25} + u_{26} = 0. \tag{12}$$

The solution of the algebraic system of equations (11) and (12) is obtained by MATLAB/Octave.

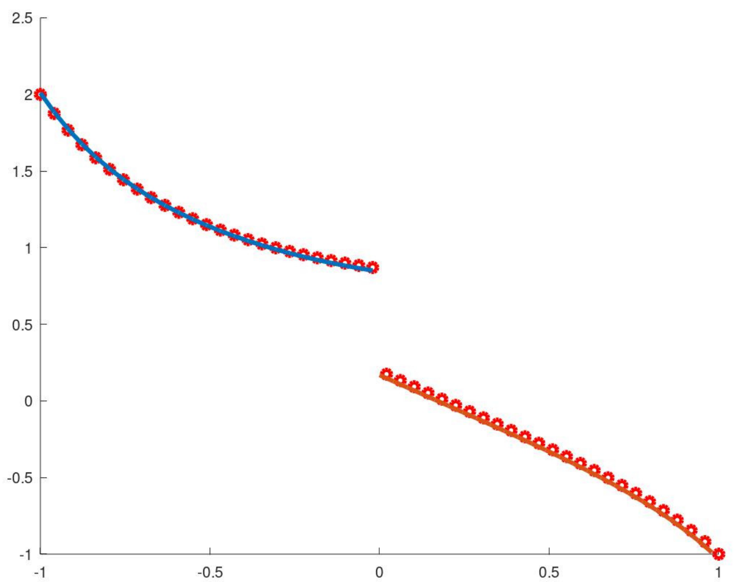

Figure 2. Comparison of FDM-solution and exact solution of BVTP (8)-(10) for N=49.

Conclusion

We have considered BVTP (6)–(8) to test the computational efficiency of the proposed modification of the classical FDM. When solving BVTP (6)–(8) numerically for different values of $N = 20, 50, 100, 1000$ presented in Table 1, we observed that if N increases, h decreases, then maximum absolute error in computed solution decreases.

Table 1

Maximum absolute error

h=2/N	N	MAE
1/10	20	0.0039503
1/25	50	0.00064601
1/50	100	0.00016199
1/500	1000	0.0000016217

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Ауысу шарттарын қамтитын шеттік есептердің жуық шешімдерін есептеу үшін ақырлы айырымдардың жаңа әдісі

Мақала қос байланысымды қарапайым дифференциалдық теңдеулер үшін шеттік есептердің жаңа түрінің сандық шешімдерін есептеуге бағытталған. Мұнда зерттелетін есеп классикалық шеттік есептерден ерекше, онда өзара әрекеттесу нүктесінде ауысу шарттары деп аталатын қосымша шарттар бар. Мұндай есептерді классикалық есептерге қарағанда шешу әлдеқайда қиын. Классикалық сандық әдістерді шеттік ауысу есептердің осы түріне қалай қолдану керектігі түсініксіз. Ақырлы айырымды әдісіне сүйене отырып, қос байланысымды қарапайым дифференциалдық теңдеулер үшін шеттік ауысу есептерді шешудің жаңа сандық алгоритмі құрылды. Ұсынылған алгоритмнің сенімділігі мен тиімділігін көрсету үшін бір шеттік ауысу есептің сандық шешімі табылды және нәтижелер тиісті дәл шешіммен салыстырылды. Максималды абсолютті қателер кестеде келтірілген.

Кілт сөздер: ақырлы айырымдық әдісі, ауысу шарты, классикалық сандық әдістер, шеттік ауысу есептерін шешудің алгоритмі.

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Новый метод конечных разностей для вычисления приближенных решений краевых задач, включающих условия перехода

Статья направлена на вычисление численных решений нового типа краевых задач для двусвязных обыкновенных дифференциальных уравнений. Изучаемая здесь задача отличается от классических краевых задач тем, что она содержит дополнительные условия в точке взаимодействия, так называемые переходные условия. Естественно, что такие задачи гораздо сложнее решать, чем классические. Непонятно, как применить классические численные методы к такому типу краевых переходных задач. На основе метода конечных разностей разработан новый численный алгоритм решения краевых переходных задач для двусвязных обыкновенных дифференциальных уравнений. Для демонстрации надежности и эффективности представленного алгоритма проведено численное решение одной краевой переходной задачи, и результаты сравнивались с соответствующим точным решением. Максимальные абсолютные погрешности представлены в таблице.

Ключевые слова: метод конечных разностей, условие перехода, классические численные методы, алгоритм решения краевых переходных задач.

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Paracompact-type mappings

Recently a new direction of uniform topology called the uniform topology of uniformly continuous mappings has begun to develop intensively. This direction is devoted, first of all, to the extension to uniformly continuous mappings of the basic concepts and statements concerning uniform spaces. In this case a uniform space is understood as the simplest uniformly continuous mapping of this uniform space into a one-point space. The investigations carried out have revealed large uniform analogs of continuous mappings and made it possible to transfer to uniformly continuous mappings many of the main statements of the uniform topology of spaces. The method of transferring results from spaces to mappings makes it possible to generalize many results. Therefore, the problem of extending some concepts and statements concerning uniform spaces to uniformly continuous mappings is urgent. In this article, we introduce and study uniformly R-paracompact, strongly uniformly R-paracompact, and uniformly R-superparacompact mappings. In particular, we solve the problem of preserving R-paracompact (respectively, strongly uniformly R-paracompact, uniformly R-superparacompact) spaces towards the preimage under uniformly R-paracompact (respectively, strongly uniformly R-paracompact, uniformly R-superparacompact) mappings.

Keywords: uniformly continuous mapping, uniformly locally finite open cover, uniformly star finite open cover, uniformly finite-component open cover.

Introduction

The most important properties of the paracompact-types are paracompact, strongly paracompact, and superparacompact spaces in General Topology. One of the interesting problems of Uniform Topology is extending the basic properties of uniform spaces to mappings.

For coverings α and β of the set X, the symbol $\alpha \succ \beta$ means that the covering α is a refinement of the covering β, i.e., for any $A \in \alpha$ there exists $B \in \beta$ such that $A \subset B$. For coverings α, β of a set X and $x \in X$, $M \subset X$ we have: $\alpha \wedge \beta = \{A \bigcap B : A \in \alpha, B \in \beta\}$, $\alpha(x) = \bigcup St(\alpha, x)$, $St(\alpha, x) = \{A \in \alpha : A \ni x\}$, $\alpha(M) = \bigcup St(\alpha, M)$, $St(\alpha, M) = \{A \in \alpha : A \bigcap M \neq \varnothing\}$.

A topological space X is called paracompact if every open cover α has a locally finite open refinement [1]. A topological space X is called strongly paracompact if every open cover α has a star finite open refinement [1]. A topological space X is called superparacompact if every open cover α has a finite-component open refinement [2]. A uniform space (X, U) called uniformly R-paracompact if every open covering has a uniformly locally finite open refinement [3]. A uniform space (X, U) called strongly uniformly R-paracompact if every open covering has a uniformly star finite open refinement [2]. A uniform space (X, U) called uniformly R-superparacompact if every open covering has a uniformly finite-component open refinement [2].

Let $f : (X, \tau) \to (Y, \eta)$ be a continuous mapping of topological space (X, τ) to a topological space (Y, η). A mapping $f : (X, \tau) \to (Y, \eta)$ is called a paracompact (strongly paracompact, superparacompact) mappings if for each open covering α of (X, τ) there exist a open covering β of (Y, η) and locally finite (star finite, finite component) open covering γ of (Y, η) such that $f^{-1}\beta \wedge \gamma \succ \alpha$ [4].

Throughout this article by a uniformity we understand a uniformity defined with help of covers, for the uniformity U by τ_U we understand the topology generated by this uniformity, for the Tychonoff space X by U_X we understand a universal uniformity, all uniform spaces are assumed to be Hausdorff and mappings are uniformly continuous.

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Uniformly R-paracompact, strongly uniformly R-paracompact and uniformly R-superparacompact mappings

Let $f:(X,U)\to(Y,V)$ be a uniformly continuous mapping of a uniform space (X,U) to a uniform space (Y,V).

Definition 1. A uniformly continuous mapping $f:(X,U)\to(Y,V)$ of a uniform space (X,U) to a uniform space (Y,V) is called

(P_1) uniformly R-paracompact,

(P_2) strongly uniformly R-paracompact,

(P_3) uniformly R-superparacompact

mapping if for any open covering α of the uniform space (X,U) there exist such open covering β of the uniform space (Y,V) and

(p_1) uniformly locally finite

(p_2) uniformly star finite

(p_3) uniformly finite-component

an open covering γ of a space (X,U) such that $f^{-1}\beta\wedge\gamma\succ\alpha$.

Proposition 1. Let $f:(X,U)\to(Y,V)$ be a uniformly continuous mapping. If (X,U) is a uniformly R-paracompact (strongly uniformly R-paracompact, uniformly R-superparacompact) space, then the mapping f is a uniformly R-paracompact (strongly uniformly R-paracompact, uniformly R-superparacompact) mapping.

Proof. Let $f:(X,U)\to(Y,V)$ be a uniformly continuous mapping of a uniformly R-paracompact (strongly uniformly R-paracompact, uniformly R-superparacompact) uniform space (X,U) to a uniform space (Y,V) and α be an arbitrary open covering of the space (X,U). Then there exists a uniformly locally finite (uniformly star finite, uniformly finite-component) open covering λ of the space (X,U) such that $\lambda\succ\alpha$. Let β be an arbitrary open covering of the space (Y,V). Then, $f^{-1}\beta$ is an open covering of the space (X,U). It is clear that $f^{-1}\beta\wedge\lambda\succ\alpha$. Therefore, the mapping f is a uniformly R-paracompact (strongly uniformly R-paracompact, uniformly R -superparacompact) mapping.

Proposition 2. If $f:(X,U)\to(Y,V)$ is a uniformly R-paracompact (strongly uniformly R-paracompact, uniformly R-superparacompact) mapping and $Y=\{y\}$, then the uniform space (X,U) is a uniformly R-paracompact (strongly uniformly R-paracompact, uniformly R-superparacompact) space.

Proof. Let α be an arbitrary open covering of the space (X,U). Then there exist an open covering β of a space (Y,V) and a uniformly locally finite open covering γ of a space (X,U) such that $f^{-1}\beta\wedge\gamma\succ\alpha$. It is clear that $f^{-1}\beta\wedge\gamma=\gamma$. So, (X,U) is a uniformly R-paracompact (strongly uniformly R-paracompact, uniformly R-superparacompact) space.

Lemma 1. If α and β are the uniformly locally finite (uniformly star finite, uniformly finite-component) coverings of the space (X,U), then $\alpha\wedge\beta$ is also a uniformly locally finite (uniformly star finite, uniformly finite-component) covering of the space (X,U).

Proof. We carry out the proof for a uniformly locally finite case, and the rest of the cases proceed similarly. Let α, β be the uniformly locally finite coverings of a space (X,U). Then there exist such uniform coverings $\lambda,\eta\in U$ that for any $L\in\lambda$, $E\in\eta$. We have $L\subset\bigcup\limits_{i=1}^{n}A_i$ and $E\subset\bigcup\limits_{j=1}^{k}B_j$, where $A_i\in\alpha$, $B_j\in\beta$, $i=1,2,...,n$, $j=1,2,...,k$. Hence, $L\bigcap E\subset(\bigcup\limits_{i=1}^{n}A_i)\bigcap(\bigcup\limits_{j=1}^{k}B_k)\subset\bigcup\limits_{i=1}^{n}\bigcup\limits_{j=1}^{k}(A_i\bigcap B_j)$. It is clear that $\lambda\wedge\eta$ is a uniform covering and $L\bigcap E\in\lambda\wedge\eta$. Note that each $L\bigcap M\in\lambda\wedge\eta$ meets with a finite number of elements of the covering $\alpha\wedge\beta$. Hence, $\alpha\wedge\beta$ is a uniformly locally finite covering of the space (X,U).

Lemma 2. Let $f:(X,U)\to(Y,V)$ be a uniformly continuous mapping. If β is a uniformly locally finite (uniformly star finite, uniformly finite-component) covering of a uniform space (Y,V), then $f^{-1}\beta$ is a uniformly locally finite (uniformly star finite, uniformly finite-component) covering of a uniform space (X,U).

Proof. We also carry out the proof for a uniformly locally finite case, and the rest of the cases can be proceeded similarly. Let β be a uniformly locally finite covering of the space (Y,V). Let us show that the covering $f^{-1}\beta$ is a uniformly locally finite covering of the space (X,U). Since β is a uniformly locally finite, there exists a uniform covering $\lambda\in V$ such that each element of which meets only with a finite number of elements of the covering β. For each $L\in\lambda$ there exist $B_1,B_2,...,B_n$ from β such that $L\subset\bigcup\limits_{i=1}^{n}B_i$. Therefore, $f^{-1}L\subset f^{-1}(\bigcup\limits_{i=1}^{n}B_i)=\bigcup\limits_{i=1}^{n}f^{-1}(B_i)$, where $f^{-1}(B_i)\in\beta$, $i=1,2,...,n$. It's clear that $f^{-1}\lambda\in U$. Then $f^{-1}\lambda$ is the required uniform covering. So, the covering $f^{-1}\beta$ is a uniformly locally finite covering of the space (X,U).

Proposition 3. A composition of two uniformly R-paracompact (strongly uniformly R-paracompact, uniformly R-superparacompact) mappings is again a uniformly R-paracompact (strongly uniformly paracompact, uniformly R-superparacompact) mapping.

Proof. Let $f : (X, U) \to (Y, V)$ and $g : (Y, V) \to (Z, W)$ be the uniformly R-paracompact (strongly uniformly R-paracompact, uniformly R-superparacompact) mappings. Let α be an arbitrary open covering of the space (X, U). Then, there exist an open covering β of a space (Y, V) and a uniformly locally finite (uniformly star finite, uniformly finite-component) open covering γ of a space (X, U) such that $f^{-1}\beta \wedge \gamma \succ \alpha$. Due to the uniformly R-paracompactness (strongly uniformly R-paracompactness, uniformly R-superparacompactness) of the mapping g, for an open covering β there exist such an open covering λ of the space (Z, W) and a uniformly locally finite (uniformly star finite, uniformly finite-component) open covering η of the space (Y, V) that $g^{-1}\lambda \wedge \eta \succ \beta$.
Then, $(g \circ f)^{-1}\lambda \wedge (f^{-1}\eta \wedge \gamma) \succ f^{-1}\beta \wedge \gamma \succ \alpha$. Put $f^{-1}\eta \wedge \gamma = \mu$. According to Lemmas 1 and 2, a covering μ is a uniformly locally finite (uniformly star finite, uniformly finite-component) open covering of a space (X, U). Therefore, $g \circ f : (X, U) \to (Z, W)$ is a uniformly R-paracompact (strongly uniformly R-paracompact, uniformly R-superparacompact) mapping.

Proposition 4. Let $f : (X, U) \to (Y, V)$ be a uniformly continuous mapping of a uniform space (X, U) to a uniform space (Y, V) and (M, U_M) be a closed subspace of a space (X, U). If f is a uniformly R-paracompact (strongly uniformly R-paracompact, uniformly R-superparacompact) mapping, then its restriction $f|_M : (M, U_M) \to (Y, V)$ is also a uniformly R-paracompact (strongly uniformly R-paracompact, uniformly R-superparacompact) mapping.

Proof. Let $f : (X, U) \to (Y, V)$ be a uniformly R-paracompact (strongly uniformly R-paracompact, uniformly R-superparacompact) mapping and let (M, U_M) be a closed subspace. Let α_M be an arbitrary open covering of the space (M, U_M). Then there is an open covering α of a space (X, U) such that $\alpha_M = \alpha \wedge \{M\}$. Let β be an open covering of a space (Y, V) and γ is a uniformly locally finite (uniformly star finite, uniformly finite-component) open space (X, U) such that $f^{-1}\beta \wedge \gamma \succ \alpha$. Then, $f|_M^{-1}\beta \wedge \gamma_M \succ \alpha_M$. Hence, the mapping $f|_M : (M, U_M) \to (Y, V)$ is a uniformly R-paracompact (strongly uniformly R-paracompact, uniformly R-superparacompact) mapping.

Theorem 1. If the mapping f and the space (Y, V) are uniformly R-paracompact (strongly uniformly R-paracompact, uniformly R-superparacompact), then (X, U) is R-paracompact (strongly uniformly R-paracompact, uniformly R-superparacompact).

Proof. Let f and (Y, V) be a uniformly R-paracompact (strongly uniformly R-paracompact, uniformly R-superparacompact) and α be an arbitrary open covering of the space (X, U). Then, there are an open covering β of a space (Y, V) and a uniformly locally finite (uniformly star finite, uniformly finite-component) open covering γ of a space (X, U) such that $f^{-1}\beta \wedge \gamma \succ \alpha$. In the covering β a refinement a uniformly locally finite (uniformly star finite, uniformly finite-component) open covering η of a space (Y, V). Then $f^{-1}\eta \wedge \gamma \succ \alpha$ and, according to Lemmas 1 and 2, the covering $f^{-1}\eta \wedge \gamma$ is a uniformly locally finite (uniformly star finite, uniformly finite-component) covering. So, (X, U) is a uniformly R-paracompact (strongly uniformly R-paracompact, uniformly R-superparacompact) space.

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Паракомпактілі бейнелеудің түрлері

Кейінгі кездері бірқалыпты үздіксіз бейнелеулердің бірқалыпты топологиясы деп аталатын жаңа бағыт қарқынды дами бастады. Бұл бағыт, ең алдымен, бірқалыпты кеңістіктерге қатысты негізгі ұғымдар мен мәлімдемелерді бірқалыпты үзіліссіз бейнелеулерді таратуға арналған. Сонымен қатар, бірқалыпты кеңістік осы бірқалыпты кеңістікті бір нүктелі кеңістікке қарапайым бірқалыпты үзіліссіз бейнелеу ретінде түсініледі. Жүргізілген зерттеулер үзіліссіз бейнелеулердің үлкен, бірқалыпты аналогтарын анықтады және бірқалыпты кеңістік топологиясының көптеген негізгі тұжырымдарын бірқалыпты үзіліссіз бейнелеулерге өткізуге мүмкіндік берді. Нәтижелерді кеңістіктен бейнелеуге ауыстыру көптеген нәтижелерді қорытындылауға мүмкіндік береді. Сондықтан бірқалыпты кеңістіктерге қатысты кейбір ұғымдар мен мәлімдемелерді бірқалыпты үзіліссіз бейнелеулерге тарату есебі өзекті болып табылады. Осы жұмыста R-паракомпакт, қатты бірқалыпты R-паракомпакт және бірқалыпты R-суперпаракпакт бейнелері енгізіліп, зерттелді. Дербес жағдайда, R-паракомпакт (тиісінше, өте бірқалыпты R-паракомпакт, бірқалыпты R-суперпаракпакт) кеңістіктерін бейнеге қарай бірқалыпты R-паракомпакт (тиісінше, өте бірқалыпты R-паракомпакт, бірқалыпты R-суперпаракпакт) бейнелеулермен сақтау есебі шешіледі.

Кілт сөздер: бірқалыпты үзіліссіз бейнелеу, бірқалыпты локалды ақырлы ашық жабын, бірқалыпты жұлдызды ақырлы ашық жабын, бірқалыпты ақырлы компонентті ашық жабын.

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Паракомпактные типы отображений

В последнее время интенсивно развивается новое направление равномерной топологии — равномерная топология равномерно непрерывных отображений. Это направление посвящено, в первую очередь, распространению на равномерно непрерывные отображения основных понятий и утверждений, касающихся равномерных пространств. При этом равномерное пространство понимается как простейшее равномерно непрерывное отображение этого равномерного пространства в одноточечное пространство. Проведенные исследования выявили большие равномерные аналоги непрерывных отображений и позволили перенести на равномерно непрерывные отображения многие основные утверждения равномерной топологии пространств. Метод перенесения результатов с пространств на отображения позволяет обобщить многие результаты. Поэтому задача распространения некоторых понятий и утверждений, касающихся равномерных пространств, на равномерно непрерывные отображения является актуальной. В настоящей работе введены и исследованы равномерно R-паракомпактные, сильно равномерно R-паракомпактные и равномерно R-суперпаракомпактные отображения. В частности, решена задача сохранения R-паракомпактных (соответственно, сильно равномерно R-паракомпактных, равномерно R-суперпаракомпактных) пространств в сторону прообраза при равномерно R-паракомпактных (соответственно, сильно равномерно R-паракомпактных, равномерно R-суперпаракомпактных) отображениях.

Ключевые слова: равномерно непрерывное отображение, равномерно локально-конечное открытое покрытие, равномерно звездное конечное открытое покрытие, равномерно конечно-компонентное открытое покрытие.

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On the solvability of the tracking problem in the optimization of the thermal process by moving point controls

In the present article we investigate problems of tracking in the moving point control of thermal processes described by Fredholm integro-differential equations in partial derivatives with the Fredholm integral operator, in the case when the functions of point sources are nonlinear with respect to the control function. It is found that optimal controls are defined as solutions to a system of linear integral equations, and an algorithm for constructing its solution is developed. Sufficient conditions for the unique solvability of the tracking problem are found and an algorithm for constructing a complete solution to the nonlinear optimization problem was indicated.

Keywords: generalized solution, Dirac function, functional, tracking problem, optimal control, integral equation, complete solution.

Introduction

Problems of tracking are an interesting branch of optimal control theory. Control problems, where the process should be controlled so that the deviation of the state of the controlled process differs little from the specified trajectory during the entire control time, are called tracking problems. Such problems are encountered in various branches of science and are of great practical importance. Few works are devoted to the research of tracking problems for optimal control of processes described by integro-differential equations in partial derivatives, in particular with the Fredholm integral operator [1: 55–60; 2]. In the article solvability of the tracking problem is investigated for moving point controls of thermal processes described by integro-differential equations in partial derivatives with the Fredholm integral operator in the case when the functions of point sources are nonlinear in control.

In this article we will use the concept of a generalized solution of a boundary value problem for controlled process, as such approach allows us to adequately describe the actually occurring process. The quality criteria of control is the minimization of the generalized quadratic functional. It is established that optimal controls are defined since solutions of a system of nonlinear integral equations containing unknown functions, both under the integral and outside the integral. An algorithm for constructing a solution to this system was developed and sufficient conditions for its unique solvability were found. A complete solution of the tracking problem is constructed.

Statement of the tracking problem and optimality conditions

We consider a case when mathematical formalization of the tracking problem for optimal control of the thermal process is reduced to the problem of minimizing the integral generalized quadratic functional

$$J[u_1(t), ..., u_m(t)] = \int_0^T \int_0^1 [V(t,x) - \xi(t,x)]^2 dxdt + \beta \int_0^T \sum_{k=1}^m p_k^2[u_k(t)]dt, \quad \beta > 0 \tag{1}$$

on the set of solutions of the boundary value problem

$$V_t = V_{xx} + \lambda \int_0^T K(t,\tau)V(\tau,x)d\tau + \sum_{k=1}^m g_k(x)\delta(x - x_k(t))f_k[u_k(t)], \quad 0 < x < 1, \ 0 < t \leq T, \tag{2}$$

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$$V(0,x)=\varphi(x), \quad x\in(0,1), \tag{3}$$

$$V_x(t,0)=0, \quad V_x(t,1)+\alpha V(t,1)=0, \quad 0<t\le T, \quad \alpha>0, \tag{4}$$

where $\xi(t,x)\in H(Q_T)$ function describing a given trajectory, $Q_T=(0,1)\cdot(0,T)$, $g_k(x)\in H(0,1)$, $\varphi(x)\in H_1(0,1)$, $K(t,\tau)\in H(D)$ are given functions, $D=\{0\le t,\tau\le T\}$; $\delta(x)$ is Dirac Delta function, $x_k(t)$ are given functions that describe the laws of motion of the points of application of external forces and take values from $[0,1]$; functions $f_k[u_k(t)]\in H(0,T), p_k[u_k(t)]\in H(0,T)$ for any controls $u_k(t)\in H(0,T)$ and have the property of monotony, i.e.

$$f_{ku_k}[u_k(t)]\ne 0, \; p_{ku_k}[u_k(t)]\ne 0, \quad \forall t\in[0,T]; \tag{5}$$

λ is a parameter, T is fixed point in time; $H(Y)$ is Hilbert space of quadratically summable functions defined on the set Y; $H_1(Y)$ is Sobolev space of the first order.

Note that according to condition (5) one-to-one correspondences are established between the elements $\{u_1(t),...,u_m(t)\}\in H^m(0,T)=H(0,T)\times...\times H(0,T)$ of the control space and the elements $V(t,x)$ of the state controlling process $\{V(t,x)\}$ space.

Given this commitment, we calculate the increments of the functional (1). By direct calculation we have the equality

$$\begin{gathered}\Delta I[u_1(t),...,u_m(t)]=I[u_1(t)+\Delta u_1(t),...,u_m(t)+\Delta u_m(t)]-I[u_1(t),...,u_m(t)]=\\ =-\int_0^T \Delta\Pi[t,V(t,x),\omega(t,x),u_1(t),...,u_m(t)]dt+\int_0^T\int_0^1 \Delta V^2(t,x)dxdt,\end{gathered} \tag{6}$$

where

$$\begin{gathered}\Delta\Pi[t,V(t,x),\omega(t,x),u_1(t),...,u_m(t)]=\\ =\Pi[t,V(t,x),\omega(t,x),u_1(t)+\Delta u_1(t),...,u_m(t)+\Delta u_m(t)]-\Pi[t,V(t,x),\omega(t,x),u_1(t),...,u_m(t)],\end{gathered} \tag{7}$$

$$\Pi[t,V(t,x),\omega(t,x),u_1(t),...,u_m(t)]=\sum_{k=1}^m\{g_k[x_k(t)]\omega[t,x_k(t)]f_k[u_k(t)]-\beta p_k^2[u_k(t)]\},$$

function $\omega(t,x)$ is a generalized solution to a boundary value problem

$$\begin{gathered}\omega_t+\omega_{xx}=-\lambda\int_0^T K(\tau,t)\omega(\tau,x)d\tau+2[V(t,x)-\xi(t,x)], \quad 0<x<1, \; 0\le t<T,\\ \omega(T,x)=0, \quad 0<x<1,\\ \omega_x(t,0)=0, \quad \omega_x(t,1)+\alpha\omega(t,1)=0, \quad 0\le t<T,\end{gathered} \tag{8}$$

$V(t,x)$ is a generalized solution to the main boundary value problem (2–5). The problem (8) is called a conjugate boundary value problem.

From (6) and (7) it follows that $\Delta I[u_1(t),...,u_m(t)]\ge 0$ on the controls satisfying the condition $\Delta\Pi[t,V(t,x),\omega(t,x),u_1(t),...,u_m(t)]\le 0$. These relations are at the basis of maximum principle for the considered problem of optimal control, i.e., on controls, where the function $\Pi(\cdot)$ reaches its maximum, and $I[u_1(t),...,u_m(t)]$ reaches its minimum.

We investigate the function $\Pi[t,V(t,x),\omega(t,x),u_1(t),...,u_m(t)]$ for the maximum. For each fixed $t\in[0,T]$ and $x\in(0,1)$ it turns into a function of m variables $\{u_1,...,u_m\}\in R^m$ - m dimensional Euclidean space.

Consider the case when the set of admissible values of variables $u_1,...,u_m$ are open sets. Then, by applying the classical method of research for the extremum, we obtain the following relations

$$\Pi_{u_k}[\cdot,u_1(t),...,u_m(t)]=g_k[x_k(t)]\omega[t,x_k(t)]f_{ku_k}[u_k(t)]-2\beta p_k[u_k(t)]p_{ku_k}[u_k(t)]=0,$$

$k=1,2,...,m$, a necessary condition of the extremum of the first order [3: 379–380].

From this we obtain the necessary first-order optimality condition

$$2\beta\frac{p_k[u_k(t)]p_{ku_k}[u_k(t)]}{f_{ku_k}[u_k(t)]}=g_k[x_k(t)]\omega[t,x_k(t)], \quad k=1,2,...,m, \tag{9}$$

that is valid for almost all $t\in[0,T]$.

The Hess matrix for the function $\Pi(\cdot, u_1, ..., u_m)$ has the form

$$\Gamma[\Pi(\cdot, u_1, ..., u_m)] = diag\{g_k[x_k(t)]\omega[t, x_k(t)]f_{ku_ku_k}[u_k(t)] - 2\beta(p_k[u_k(t)]p_{ku_k}[u_k(t)])_{u_k}\}.$$

Then the necessary second-order optimality condition for the maximum, according to Sylvester's criterion, has the form of the inequality [3: 379–380]

$$(-1)^j \prod_{k=1}^{j} f_{ku_k}[u_k(t)]\left(\frac{p_k[u_k(t)]p_{ku_k}[u_k(t)]}{f_{ku_k}[u_k(t)]}\right)_{u_k} > 0, \quad j = 1, 2, ..., m \tag{10}$$

which was obtained taking into account of the condition (9) and it is for almost all $t \in [0, T]$.

Relations (9) and (12), which are realized almost at all $t \in [0, T]$ are called optimality conditions.

Since the remainder $\int_0^T \int_0^1 \Delta V^2(t, x)dxdt$ in relation (6) takes on a sufficiently small value, the condition $\Delta\Pi[\cdot, u] \leq 0$ is both necessary and sufficient for the optimality controls $u_1, ..., u_m$.

To determine the optimal controls $u_1^0(t), ..., u_m^0(t)$ it is necessary to use the first order optimality condition of equality (9). For this purpose we use generalized solutions of both the main and adjoint boundary value problems.

Generalized solutions of the main and conjugated boundary value problems of a controlled process

The generalized solution to the boundary value problem (2)–(5) has the form [4]

$$\begin{aligned} V(t, x) &= \sum_{n=1}^{\infty} V_n(t)z_n(x) \\ &= \sum_{n=1}^{\infty}\left(\lambda \int_0^T R_n(t, s, \lambda)a_n(s)ds + a_n(t)\right)z_n(x), \end{aligned} \tag{11}$$

where $R_n(t, s, \lambda)$ is the resolvent [5: 98-101] of the kernel $K_n(t, s) = \int_0^t e^{-\lambda_n^2(t-\tau)}K(\tau, s)d\tau$,

$$a_n(t) = e^{-\lambda_n^2 t}\varphi_n + \int_0^t e^{-\lambda_n^2(t-\tau)} \sum_{k=1}^{m} g_k[x_k(\tau)]z_n[x_k(\tau)]f_k[u_k(\tau)]d\tau.$$

The solution of the conjugate boundary value problem has the form

$$\omega(t, x) = \sum_{n=1}^{\infty}\left(\lambda \int_0^T B_n(\tau, t, \lambda)q_n(\tau)d\tau + q_n(t)\right)z_n(x), \tag{12}$$

where $B_n(\tau, t, \lambda)$ is the resolvent [5; 98-101] of the kernel $G_n(\tau, t) = \int_t^T e^{-\lambda_n^2(s-t)}K(\tau, s)ds$,

$$q_n(t) = -2\int_t^T e^{-\lambda_n^2(s-t)}[V_n(s) - \xi_n(s)]ds, \tag{13}$$

where $V_n(t)$ and $\xi_n(t)$ are the Fourier coefficients of the functions $V(t, x)$ and $\xi(t, x)$, respectively.

System of nonlinear integral equations of optimal controls

The desired controls $\{u_1^0(t), ..., u_m^0(t)\}$ we find are according to the optimality conditions (9) and (10). Note that the optimality condition (10) restrict the functions class $\{f_k[u_k(t)], p_k[u_k(t)]\}$. We assume that the functions $\{f_k[u_k(t)], p_k[u_k(t)]\}$ satisfy the condition (10). Then the controls $\{u_1^0(t), ..., u_m^0(t)\}$ defined by the condition (11) will be the desired optimal controls. In the formula (9) we replace the function $\omega(t, x)$ according to the formulas (12), (13) and obtain a system of equalities

$$2\beta\frac{p_k[u_k(t)]p_{ku_k}[u_k(t)]}{f_{ku_k}[u_k(t)]} = g_k[x_k(t)]\sum_{n=1}^{\infty}\left(\lambda \int_0^T B_n(s, t, \lambda)q_n(s)ds + q_n(t)\right)z_n[x_k(t)], \tag{14}$$

$k = 1, 2, ..., m.$

Now we transform this system of integral equations (14) to the form

$$\beta\frac{p_k[u_k(t)]p_{ku_k}[u_k(t)]}{f_{ku_k}[u_k(t)]}$$
$$= g_k[x_k(t)]\sum_{n=1}^{\infty} z_n[x_k(t)]h_n(t,\lambda) - g_k[x_k(t)]\sum_{n=1}^{\infty} z_n[x_k(t)]\int_0^T W_n(t,\eta,\lambda)\sum_{j=1}^{m} q_j[x_j(\eta)]d\eta, \tag{15}$$

$k = 1, 2, ..., m$, where

$$W_n(t,\eta,\lambda) = \int_0^T \epsilon_n(y,t,\lambda)Y_n(y,\eta,\lambda)dy,$$

$$\epsilon_n(t,\tau,\lambda) = \begin{cases} \lambda\int_0^y B_n(s,t,\lambda)e^{-\lambda_n^2(y-s)}ds, & 0\le y\le t, \\ e^{-\lambda_n^2(y-s)} + \lambda\int_0^y B_n(s,t,\lambda)e^{-\lambda_n^2(y-s)}ds, & t\le y\le T; \end{cases}$$

$$Y_n(y,\eta,\lambda) = \begin{cases} e^{-\lambda_n^2(y-\eta)} + \lambda\int_\eta^T R_n(y,\tau,\lambda)e^{-\lambda_n^2(\tau-\eta)}d\tau, & 0\le\eta\le y, \\ \lambda\int_\eta^T R_n(y,\tau,\lambda)e^{-\lambda_n^2(\tau-\eta)}d\tau, & y\le\eta\le T. \end{cases}$$

Next, we investigate the unique solvability of the nonlinear integral equations system (15). This system nonlinearly contains unknown functions $u_1(t), ..., u_m(t)$ under the integral and outside the integral.

Suppose

$$\beta\frac{p_k[u_k(t)]p_{ku_k}[u_k(t)]}{f_{ku_k}[u_k(t)]} = \sigma_k(t), \quad k = 1, 2, ..., m. \tag{16}$$

Then, according to the condition (10) there is a function $\psi[\cdot]$ such that

$$u_k(t) = \psi_k[t, \sigma_k(t), \beta], \quad k = 1, 2, ..., m. \tag{17}$$

Taking into account (16) and (17), we rewrite the system (15) in the form

$$\sigma_k(t) = \sum_{n=1}^{\infty}\theta_{kn}[x_k(t)]\Big(h_n(t,\lambda)$$
$$-\int_0^T W_n(t,\eta,\lambda)\sum_{j=1}^{m}\theta_{jn}[x_j(\eta)]f_j\big(\psi_j[\eta,\sigma_j(\eta),\beta]\big)d\eta\Big), \quad k=\overline{1,m}, \tag{18}$$

where

$$\theta_{kn}[x_k(t)] = g_k[x_k(t)]z_n[x_k(t)], \quad k = 1, 2, ..., m.$$

Now, we compose the vector-functions

$$\sigma(t) = \{\sigma_1(t), ..., \sigma_m(t)\}, \quad \overline{\theta_n}[t] = \{\theta_{1n}[x_1(t)], ..., \theta_{mn}[x_m(t)]\},$$
$$\psi[t,\sigma(t),\beta] = \{\psi_1[t,\sigma_1(t),\beta], ..., \psi_m[t,\sigma_m(t),\beta]\},$$
$$f\big(\psi[t,\sigma(t),\beta]\big) = \{f_1\big(\psi_1[t,\sigma_1(t),\beta]\big), ..., f_m\big(\psi_m[t,\sigma_m(t),\beta]\big)\}$$

and rewrite the system of equalities (18) in the vector form

$$\sigma(t) = \sum_{n=1}^{\infty}\tilde{\theta_n}(t)\Big(h_n(t,\lambda) - \int_0^T W_n(t,\eta,\lambda)\tilde{\theta_n}^*(\eta)f\big(\psi[\eta,\sigma(\eta),\beta]\big)d\eta\Big), \tag{19}$$

where $\sigma(t)$, $\tilde{\theta_n}(t)$, $f\big(\psi[\eta,\sigma(\eta),\beta]\big)$ are the column vectors, symbol $*$ is a transposition sign.

Further, the following lemmas are proved by direct calculations.

Lemma 1. Vector function

$$h(t,\lambda)=\{h^{(1)}(t,\lambda),...,h^{(m)}(t,\lambda)\}=\sum_{n=1}^{\infty}\tilde{\theta_n}(t)h_n(t,\lambda) \tag{20}$$

is an element of space $H^m(0,T)$, i.e., $h^{(k)}(t,\lambda)\in H(0,T)$ for each $k=1,2,...,m$.

Lemma 2. Vector function

$$W[\sigma(t)]=\{W_1[\sigma(t)],...,W_m[\sigma(t)]\}, \tag{21}$$

where

$$W_k[\sigma(t)]=\sum_{n=1}^{\infty}\theta_{nk}(t)\int_0^T W_n(t,\eta,\lambda)\tilde{\theta_n}^*(\eta)f\big(\psi[\eta,\sigma(\eta),\beta]\big)d\eta$$

$$=\sum_{n=1}^{\infty}g_k[x_k(t)]z_n[x_k(t)]\int_0^T W_n(t,\eta,\lambda)\sum_{j=1}^{m}g_j[x_j(t)]z_n[x_j(t)]f_j\big(\psi_j[\eta,\sigma_j(\eta),\beta]\big)d\eta \tag{22}$$

is an element of space $H^m(0,T)$, i.e., $W_k[\sigma(t)]\in H(0,T)$, $\quad k=1,2,...,m$.

According to Lemmas 1 and 2, equation (19) is considered in space $H^m(0,T)$. Taking into account formulas (20) and (21) system of integral equations (19) is rewritten in operator form as

$$\sigma=W[\sigma]+h. \tag{23}$$

Lemma 3. Let the functions $f_k[u_k(t)]$ and $\psi_k[t,\sigma_k(t),\beta]$ satisfy the Lipschitz condition with respect to the functional variable, i.e.

$$\|f_k[u_k(t)]-f_k[\tilde{u_k}(t)]\|_{H(0,T)}\leq f_k^0\|u_k(t)-\tilde{u_k}(t)\|_{H(0,T)},\quad f_k^0>0,\quad k=1,2,...,m, \tag{24}$$

$$\|\psi_k[t,\sigma_k(t),\beta]-\psi_k[t,\tilde{\sigma_k}(t),\beta]\|_{H(0,T)}\leq \psi_k^0(\beta)\|\sigma_k(t)-\tilde{\sigma_k}(t)\|_{H(0,T)},\quad \psi_k^0(\beta)>0,\quad k=1,2,...,m.$$

Then, under the condition

$$\gamma=T(g_k^0)^2\big(\frac{1}{\lambda_1^0}+\frac{1}{6}\big)\left(1+\frac{\lambda^2k_0T}{\big(\sqrt{2}\lambda_1-|\lambda|\sqrt{k_0T}\big)^2}\right)\sqrt{m}f^0\psi^0(\beta)<1,$$

$$f^0=max\{f_1^0,...,f_m^0\},\quad \psi^0=max\{\psi_1^0,...,\psi_m^0\},$$

where the operator $W[\sigma]:H^m(0,T)\to H^m(0,T)$ is contracting.

Theorem 1. Let conditions (22), (23), and (24) be satisfied. Then the operator equation (19) in space $H^m(0,T)$ has a unique solution.

Proof. Since the Hilbert space $H^m(0,T)$ is complete [6: 44–45], the operator $W[\cdot]$ transforms the space $H^m(0,T)$ into itself and becomes contracting, then according to the principle of contracting operators, the operator $W[\cdot]$ has a unique fixed point $\sigma^0(t)$.

This solution is defined as the limit of a sequence $\sigma^{(n)}(t)$, i.e., is determined by the successive approximation method

$$\sigma^{(n)}(t)=W[\sigma^{(n-1)}]+h(t,\lambda),\quad n=1,2,3,...,$$

where the initial guess $\sigma^{(0)}(t)$ is chosen arbitrarily, in particular $\sigma^{(0)}(t)=h(t,\lambda)$. Then, as is known, the estimate

$$\|\tilde{\sigma}(t)-\sigma^{(n)}(t)\|_{H^m(0,T)}\leq\frac{(2f^0\psi^0(\beta))^n}{1-2f^0\psi^0(\beta)}\|W[\sigma^{(0)}(t)]\|_{H^m(0,T)}$$

is valid.

Thus, substituting the obtained solution $\tilde{\sigma}(t)=\{\tilde{\sigma_1}(t),...,\tilde{\sigma_m}(t)\}$ into (16), we find the desired optimal controls

$$u_k^0(t)=\psi_k[t,\tilde{\sigma_k}(t),\beta],\quad k=1,2,...,m.$$

The solution of boundary value problem (2)–(5) corresponding to these controls, according to (11) is determined by the formula

$$V^0(t,x)=\sum_{n=1}^{\infty}\left(\lambda\int_0^T R_n(t,s,\lambda)a_n^0(s)ds+a_n^0(t)\right)z_n(x),$$

where

$$a_n^0(t) = e^{-\lambda_n^2 t}\psi_n + \int_0^t e^{-\lambda_n^2(t-\tau)} \sum_{k=1}^m g_k[x_k(\tau)] z_n[x_k(\tau)] f_k[u_k^0(\tau)].$$

Using the found $u_k^0(t), \quad k = 1, ..., m$ and $V^0(t,x)$ we calculate the minimum value of the functional (1)

$$J[u_1^0(t), ..., u_m^0(t)] = \int_0^T \int_0^1 [V^0(t,x) - \xi(t,x)]^2 dxdt + \beta \int_0^T \sum_{k=1}^m p_k^2[u_k^0(t)]dt.$$

Thus the triple found

$$\{\left(u_1^0(t), ..., u_m^0(t)\right), \quad V^0(t,x), \quad J[u_1^0(t), ..., u_m^0(t)]\}$$

is a complete solution to the tracking problem with nonlinear optimal control of the heat propagation process under the action of moving point sources.

Conclusion

In conclusion, we note some features of the investigated tracking problem for optimal point control of thermal processes described by integro-differential equations in partial derivatives.

The presence of the integral operator has led to the need in study of the Neumann series that appear when determining the Fourier coefficients of the boundary value problem. It was found that the convergence radius of the Neumann series with respect to the parameter λ expands with increasing number of the Fourier coefficient. The optimization problem can be solved only with a radius of convergence corresponding to the first Fourier coefficient.

A method for solving a system of nonlinear integral equations of a non-standard form has been developed. Such the system of equations appears in the case when the functions of external influences are nonlinear with respect to the control.

Using the property of the Dirac δ function, an algorithm has been developed for constructing a complete solution to the tracking problem using the example of controlling thermal processes, which can be used in solving and qualitative research of the problems of programmed control of various technological processes, described by functional equations of a more complex nature.

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Жылу процестерін қозғалмалы нүктелік басқару кезіндегі бақылау есебінің шешілімділігі жөнінде

Мақалада нүктелік көздер функциялары басқару функцияларына қатысты сызықтыемес болған жағдайда Фредгольм интегралдық операторлы дербес туындылы фредгольмдік интегралдық-дифференциалдық теңдеулермен сипатталатын жылу процестерін қозғалмалы нүктелік басқару кезіндегі қадағалау есебінің шешілімділігі зерттелген. Оптималды басқару сызықты интегралдық теңдеулер жүйесінің шешімі ретінде анықталды және шешімді тұрғызу алгоритмі құрылды. Қадағалау есебінің бірмәнді шешілуінің жеткілікті шарты табылды және сызықтыемес оңтайландыру есептерінің шешімін тұрғызудың толық алгоритмі көрсетілді.

Кілт сөздер: жалпыланған шешім, Дирак функциясы, функционал, қадағалау есебі, оптималды басқару, интегралдық теңдеу, толық шешім.

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О разрешимости задачи слежения при подвижном точечном управлении тепловыми процессами

В статье исследована разрешимость задачи слежения при подвижных точечных управлениях тепловыми процессами, описываемыми фредгольмовыми интегро-дифференциальными уравнениями в частных производных с интегральным оператором Фредгольма, в случае когда функции точечных источников нелинейны относительно функции управления. Установлено, что оптимальные управления определены как решения системы линейных интегральных уравнений, и разработан алгоритм построения ее решения. Найдены достаточные условия однозначной разрешимости задачи слежения, и указан алгоритм построения полного решения задачи нелинейной оптимизации.

Ключевые слова: обобщенное решение, функция Дирака, функционал, задача слежения, оптимальное управление, интегральное уравнение, полное решение.

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Existence and uniqueness results for the first-order non-linear impulsive integro-differential equations with two-point boundary conditions

The article discusses the existence and uniqueness of solutions for a system of nonlinear integro-differential equations of the first order with two-point boundary conditions. The Green function is constructed, and the problem under consideration is reduced to equivalent integral equation. Existence and uniqueness of a solution to this problem is analyzed using the Banach contraction mapping principle. Schaefer's fixed point theorem is used to prove the existence of solutions.

Keywords: two-point boundary conditions, impulsive systems, existence and uniqueness solutions, fixed point theorems, first order differential equation.

Introduction

A lot of problems of physics, engineering, biology and economy are described by differential and integro-differential equations. Such differential equations were studied rather well in [1–8]. In the above mentioned papers mainly the differential equations with local conditions are studied. However, in the last years there is a great interest to differential and integro-differential equations with nonlocal boundary conditions, by which a number of practical processes are described. Today, there exist a great number of works devoted to ordinary differential and integro-differential equations with nonlocal boundary conditions in which the theorem on the existence of solutions are proved for different types of nonlocal conditions [9–20].

Note that numerical methods for multipoint and integral boundary problems for first-order ordinary differential equations were developed in [21, 22].

It should be noted that the authors know about the study of boundary value problem as the form of

$$\begin{cases} \dot{x}(t) = f(t, x(t)), \ t \in [0, T], \\ Ax(0) + Bx(T) = C \end{cases} \tag{1}$$

from [23, 24], where $A, B \in R^{n\times n}$ are given matrices, $f \in R^n$ is a given function and it is assumed that the condition $detB \neq 0$ is satisfied. An approximate solution of the problem (1) was constructed using a numerical-analytical method developed by Samoilenko with the given initial conditions. In [11] the boundary value problem was investigated as follows:

$$\begin{cases} \dot{x}(t) = f(t, x(t)), \ t \in [0, T], \ t \neq t_i, \ i = 1, 2, ..., p, \\ Ax(0) + Bx(T) = C, \\ \Delta x(t_i) = I_i(x(t_i)), \ i = 1, 2, ..., p. \end{cases} \tag{2}$$

Here $A, B \in R^{n\times n}$ are given matrices, $f, \ I_i \in R^n, \ i = 1, 2, ..., p$ are given functions and $det(A + B) \neq 0$. Theorems on the existence and uniqueness of the solution of the boundary value problem (2) under suitable conditions have been proved. In this article the generalization of the boundary value problem (2) for the Volterra-Fredholm type system of integro-differential equations with two-point and impulse effect is studied.

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Problem statement

In this paper we study the existence and uniqueness of solutions of nonlinear integro-differential equations of the type

$$\dot{x} = f(t, x, \phi x(t), \varphi x(t)), \quad t \in [0, T], \ t \neq t_i, \quad i = 1, 2, ...p \tag{3}$$

with two-point boundary conditions

$$Ax(0) + Bx(T) = \alpha \tag{4}$$

and impulsive conditions

$$\Delta x(t_i) = I_i(x(t_i)), \quad i = 1, 2, ..., p, \ 0 = t_0 < t_1 < ... < t_p < T_{p+1} = T, \tag{5}$$

where A and B are constant square matrices of order n such that $\det N \neq 0$, $N = A + B$; $f : [0, T] \times R^n \times R^n \times \times R^n \to R^n$ and $I_i : R^n \to R^n$, $i = 1, 2, ..., p$ are a given function and
$\phi x(t) = \int\limits_0^t \mu(t, s) x(s) ds$, $\varphi x(t) = \int\limits_0^T \gamma(t, s) x(s) ds$, where $\mu, \gamma : R \times R \to R^{n \times n}$
with $\phi_0 = \max\limits_{t,s \in [0T]} \|\phi(t, s)\| < \infty$, $\gamma_0 = \max\limits_{t,s \in [0,T]} \|\gamma(t, s)\| < \infty$,

$$\Delta x(t_i) = x\left(t_i^+\right) - x(t_i), \quad i = 1, 2, ..., p,$$

where

$$x\left(t_i^+\right) = \lim_{h \to 0^+} x(t_i + h), \ x\left(t_i^-\right) = \lim_{h \to 0^-} x(t_i + h)$$

are the right-hand and left-hand limits of $x(t))$ at $t = t_i$, respectively.

In this work for the Green function is constructed for the two-point boundary value problem and the considered problem is reduced to the equivalent integral equations. Then the existence and uniqueness of the solutions is studied using the Banach contraction mapping principle. The existence of the solution is also proved by applying Schaefer's fixed point theorem.

This paper is organized as follows. In Section 2, we introduce definition and lemmas, which are the key tools for our main result. Section 3 focuses the theorems on the existence and uniqueness of the solution of problem (3)–(5) established under some sufficient conditions on the nonlinear terms. An example is included.

Preliminaries

In this section we present some basic definitions and preliminary facts which are used throughout the paper. We denote by $C([0, T]; R^n)$ the Banach space of all continuous functions from $[0, T]$ into R^n with the norm

$$\|x\| = \max\{|x(t)| : t \in [0, T]\},$$

where $|\cdot|$ is the norm in the space R^n.

We define the linear space

$$\begin{aligned} PC([0, T]; R^n) = \{&x : [0, T] \to R^n; \ x(t) \in C((t_i, t_{i+1}]; R^n), i = 1, 2, ..., p, \\ &x\left(t_i^-\right) \text{ and } x\left(t_i^+\right) \text{ exist, } i = 0, 1, ..., p \text{ and } x\left(t_i^-\right) = x(t_i)\}. \end{aligned}$$

Obviously, $PC([0, T]; R^n)$ is a Banach space with norm

$$\|x\|_{PC} = \max\left\{\|x\|_{(t_i, t_{i+1}]}, i = 0, 1, ..., p\right\}.$$

For the sake of simplicity, we can consider the following problem:

$$\dot{x}(t) = y(t), \qquad t \in [0, T], \tag{6}$$

$$Ax(0) + Bx(T) = \alpha, \tag{7}$$

$$\Delta x(t_i) = a_i, \quad i = 1, 2, ..., p. \tag{8}$$

Lemma 1. Let $y \in C([0, T]; R^n)$ and $a_i \in R^n$. The unique solution $x(t) \in PC([0, T]; R^n)$ of the boundary value problem for differential equation (6) with boundary conditions (7) and impulsive conditions (8) is given by

$$x(t) = N^{-1}\alpha + \int_0^T G(t,\tau)y(\tau)d\tau + \sum_{0<t_k<T} G(t_i,t_k)\, a_k \tag{9}$$

for $t \in (t_i, t_{i+1}]$, $i = 0, 1, ..., p$, where

$$G(t,\tau) = \begin{cases} N^{-1}A, & 0 \le \tau \le t, \\ -N^{-1}B, & t < \tau \le T. \end{cases}$$

Proof. If function $x = x(\cdot)$ is a solution of the differential equation (6), then for $t \in (0,T)$

$$\int_0^t y(s)ds = \int_0^t \dot{x}(s)ds = \left[x(t_1) - x(0^+)\right] + \left[x(t_2) - x({t_1}^+)\right] + ... + \left[x(t) - x({t_i}^+)\right]$$

$$= -x(0) - \left[x({t_1}^+) - x(t_1)\right] - \left[x({t_2}^+) - x(t_2)\right] - ... - \left[x({t_i}^+) - x(t_i)\right] + x(t),$$

where x_0 is an arbitrary constant vector. Using this formula and condition (8), we can write

$$x(t) = x(0) + \int_0^t y(s)ds + \sum_{0<t_i<t} a_i. \tag{10}$$

Now we define x_0 so that the function in equality (10) satisfies condition (7). Then we have

$$(A+B)\,x(0) = \alpha - B\int_0^T y(t)\,dt - B\sum_{0<t_k<T} a_k.$$

This obviously implies

$$x_0 = N^{-1}\alpha - N^{-1}B\int_0^T y(\tau)\,d\tau - N^{-1}B\sum_{0<t_k<T} a_k. \tag{11}$$

Now in (8) we take into account the value x_0 determined from the equality (11) and yield

$$x(t) = N^{-1}\alpha - N^{-1}B\int_0^T y(\tau)\,d\tau - N^{-1}B\sum_{0<t_k<T} a_k + \int_0^t y(s)\,ds + \sum_{0<t_i<t} a_i. \tag{12}$$

Since equality

$$\left(E - N^{-1}B\right) = N^{-1}A$$

is true, then we can introduce the following function:

$$G(t,\tau) = \begin{cases} N^{-1}A, & 0 \le \tau \le t, \\ -N^{-1}B, & t < \tau \le T. \end{cases}$$

Using this function, equality (12) can be written as an impulsive integral equation (9).

Lemma 2. Assume that $f \in C([0,T] \times R^n; R^n)$. Then the function $x(t)$ is a solution of boundary-value problem (3)–(5) if and only if $x(t)$ is a solution of the impulsive integral equation

$$x(t) = N^{-1}\alpha + \int_0^T G(t,\tau)f(\tau, x(\tau), \phi x(\tau), \varphi x(\tau))d\tau + \sum_{0<t_i<T} G(t_i,t_k)\, I_k(x(t_k)). \tag{13}$$

Proof. Let $x(t)$ be a solution of the boundary value problem (3)–(5). This lemma can be derived by a similar argument to Lemma 1. By checking directly, we make sure that the solution of integral equation (13) satisfies the boundary value problem (3)–(5). Lemma 2 is proved.

Main results

We introduce the following conditions:
(H1) The functions $f : [0, T] \times R^n \times R^n \times R^n \to R^n$ and $I_i : R^n \to R^n$ $i = 1, 2, ..., p$ are continuous;
(H2) There exist a constants $M \geq 0$ and $l_i \geq 0,\ i = 1, 2, ..., p$ such that

$$|f(t, x_1, x_2, x_3) - f(t, y_1, y_2, y_3)| \leq M(|x_1 - y_1| + |x_2 - y_2| + |x_3 - y_3|),$$

$$|I_i(x_1) - I_i(y_1)| \leq l_i |x_1 - y_1|,\ i = 1, 2, ..., p$$

for each $t \in [0, T]$ and all $x_1, x_2, x_3, y_1, y_2, y_3 \in R^n$;
(H3) There exists a constants K, k such that $|f(t, x)| \leq K,\ |J_i(x)| \leq k,\ i = 1, 2, ..., p$ for each $t \in [0, T]$ and all $x \in R^n$.

Theorem 1. Assume conditions (H1) and (H2) hold, and

$$L = S\left(TM(1 + T(\mu_0 + \gamma_0)) + \sum_{i=1}^{p} l_i\right) < 1, \tag{14}$$

where $S = \max\limits_{[0,T]\times[0,T]} \|G(t, \tau)\|$.
Then boundary-value problem (3)–(5) has a unique solution on $[0, T]$.

Proof. Transform the boundary value problem (3)–(5) into a fixed point problem. Consider the operator $F : PC([0, T]; R^n) \to PC([0, T]; R^n)$ defined by

$$(Fx)(t) = N^{-1}\alpha + \int_0^T G(t, \tau) f(\tau, x(\tau), \phi x(\tau), \varphi x(\tau)) d\tau + \sum_{0<t_i<T} G(t_i, t_k) I_k(x(t_k)). \tag{15}$$

Clearly, the fixed points of the operator F are solutions of the boundary problem (3)–(5).
Setting $\max\limits_{[0,T]} |f(t, 0, 0, 0)| = M_f$, $\max\limits_{k\in\{1,2,...,p\}} |I_k(0)| = m$ and let us select $r \geq \frac{\|N^{-1}\alpha\| + M_f TS + mp}{1-L}$. We show that $FB_r \subset B_r$, where

$$B_r = \{x \in PC([0, T]R^n) : \|x\| \leq r\}.$$

For $x \in B_r$, using (H1) and (H2), we get

$$\|Fx(t)\| \leq \left\|N^{-1}\alpha\right\| + \int_0^T |G(t, \tau)| \left(|f(\tau, x(\tau), \phi x(\tau), \gamma x(\tau)) - f(\tau, 0, 0, 0)| + |f(\tau, 0, 0.0)|\right) d\tau$$

$$+ \sum_{k=1}^{p} |G(t_i, t_k)| \left(|I_k(x(t_k)) - I_k(0)| + |I_k(0)|\right)$$

$$\leq \left\|N^{-1}d\right\| + S\int_0^T (M(|x| + |\phi x| + |\gamma x|) + M_f) dt + S\sum_{k=1}^{p} l_k |x(t_k)| + mp$$

$$\leq \left\|N^{-1}d\right\| + SMT(|x| + T(\phi_0 |x|) + \gamma_0 |x|) + M_f TS + S\sum_{k=1}^{p} l_k |x| + mp$$

$$\leq \left\|N^{-1}\alpha\right\| + S\left(MT(1 + T(\phi_0 + \gamma_0)) + \sum_{k=1}^{p} l_k\right)\|x\| + M_f TS + mp \leq r.$$

In order to show that the operator F is a contraction, let for any $x, y \in B_r$ we have

$$|Fx - Fy| \leq \int_0^T |G(t, \tau)\left(f(\tau, x(\tau), \phi x(t), \gamma x(t)) - f(\tau, y(\tau), \phi y(t), \gamma x(t))\right| d\tau$$

$$+ \left|\sum_{k=1}^{p} G(t_i, t_k)\left(I_k(x(t_k)) - I_k(y(t_k))\right)\right|$$

$$\leq MS\int_0^T \left(|x(t)-y(t)|+|\phi x\left(t\right)-\phi y\left(t\right)+|\gamma x\left(t\right)-\gamma y\left(t\right)||\right)dt+S\sum_{k=1}^{p}l_k\left\|x\left(t_k\right)-y\left(t_k\right)\right\|$$

$$\leq S\left(MT\left(1+T\left(\phi_0+\gamma_0\right)\right)+\sum_{k=1}^{p}l_k\right)\max_{[0,T]}|x(t)-y(t)|$$

or

$$\|Fx-Fy\|\leq L\left\|x-y\right\|.$$

It is seen that F is contraction by condition (14). So the boundary-value problem (3)–(5) has a unique solution.

Theorem 2. Assume conditions (H1)–(H3) hold. Then boundary-value problem (3)–(5) has at least one solution on $[0,T]$.

Proof. Let F be the operator defined in (15). We shall use Schaefer's fixed point theorem to prove that F has a fixed point. The proof will be given in several steps.

Step 1: F is continuous. Let $\{x_n\}$ be a sequence such that $x_n\to x$ in $PC\left([0,T];R^n\right)$. Then, for each $t\in\left(t_i,t_{i+1}\right],\ i=0,1,...,p$

$$|\left(Fx\right)(t)-\left(Fx_n\right)(t)|=\left|\int_0^T G\left(t,\tau\right)\left(f(\tau,x(\tau),\phi x\left(t\right),\gamma x\left(t\right))-f(\tau,x_n(\tau),\phi x_n\left(t\right),\gamma x_n\left(t\right))\right)d\tau\right|$$

$$+\left|\sum_{k=1}^{p}G\left(t_i,t_k\right)\left(I_k\left(x\left(t_k\right)\right)-I_k\left(x_n\left(t_k\right)\right)\right)\right|$$

$$\leq S\left(TM\left(1+T\left(\phi_0+\gamma_0\right)\right)+\sum_{k=1}^{p}l_k\right)|x\left(t\right)-x_n\left(t\right)|\leq L\left\|x-x_n\right\|.$$

From here we get $\|\left(Fx\right)(t)-\left(Fx_n\right)(t)\|\to 0$ as $n\to\infty$, which implies that the operator F is continuous.

Step 2: F maps bounded sets into bounded sets in $PC\left([0,T];R^n\right)$. Indeed, it is enough to show that for any $\eta>0$ there exists a positive constant ω such that for each $x\in B_\eta=\{x\in PC\left([0,T];R^n\right):\|x\|\leq\eta\}$ we have $\|F\left(x\right)\|\leq\omega$. We have for each $t\in[0,T]$

$$|\left(Fx\right)(t)|\leq\left\|N^{-1}\alpha\right\|+S\left(TK+pk\right).$$

This implies that

$$\|F\left(x\right)\|\leq\left\|N^{-1}\alpha\right\|+S\left(TK+pk\right)=\omega.$$

Step 3: F maps bounded sets into equicontinuous sets of $PC\left([0,T];R^n\right)$. Let $\xi_1,\xi_2\in[0,T]$, $\xi_1<\xi_2$,B_η be a bounded set of $PC\left([0,T];R^n\right)$ as in Step 2, and let $x\in B_\eta$.

Then, we have

$$F\left(x\left(\xi_2\right)\right)-F\left(x\left(\xi_1\right)\right)$$

$$=\left(A+B\right)^{-1}A\int_0^{\xi_2}f(\tau,x(\tau),\phi x\left(\tau\right),\gamma x\left(\tau\right))d\tau-\left(A+B\right)^{-1}B\int_{\xi_2}^{T}f(\tau,x(\tau),\phi x\left(\tau\right),\gamma x\left(\tau\right))d\tau$$

$$-\left(A+B\right)^{-1}A\int_0^{\xi_1}f(\tau,x(\tau),\phi x\left(\tau\right),\gamma x\left(\tau\right))d\tau-\left(A+B\right)^{-1}B\int_{\xi_1}^{T}f(\tau,x(\tau),\phi x\left(\tau\right),\gamma x\left(\tau\right))d\tau$$

$$=\left(A+B\right)^{-1}A\int_{\xi_1}^{\xi_2}f(\tau,x(\tau),\phi x\left(\tau\right),\gamma x\left(\tau\right))d\tau+\left(A+B\right)^{-1}B\int_{\xi_1}^{\xi_2}f(\tau,x(\tau),\phi x\left(\tau\right),\gamma x\left(\tau\right))d\tau$$

$$=\int_{\xi_1}^{\xi_2}f\left(\tau,x\left(\tau\right)\right)d\tau.$$

As $t_2 \to t_1$, the right-hand side of the above equalities tends to zero. As a consequence of Steps 1 to 3 together with the Ascoli–Arzela theorem, we can conclude that $F : PC([0,T];R^n) \to PC([0,T];R^n)$ is completely continuous.

Step 4: A priori bounds. Now it remains to show that the set $\Delta = \{x \in PC([0,T];R^n) : x = \lambda F(x)$ for some $0 < \lambda < 1\}$ is bounded. Let $x \in \Delta$. Then, $x = \lambda F(x)$ for some $0 < \lambda < 1$. Thus, for each $t \in (t_i, t_{i+1}]$ $i = 0, 1, ..., p$, we have

$$x(t) = \lambda N^{-1}\alpha + \lambda \int_0^T G(t,\tau) f(\tau, x(\tau), \phi x(\tau), \gamma x(\tau)) d\tau + \lambda \sum_{k=1}^{p} G(t_i, t_k) I_k(x(t_k)).$$

From here

$$\|x\| \le \left\|N^{-1}\alpha\right\| + S(TK + pk).$$

Therefore, the set Δ is bounded. The conclusion of Schaefer's fixed point theorem applies and the operator F has at least one fixed point. So, there exists at least one solution for the problems (3)–(5) on $[0,T]$.

Example

Consider the following system of integro-differential equations:

$$\begin{cases} \dot{x}_1 = 0.5x_2 + \sin\left(0.2\int_0^t tsx_1(s)\,ds\right), \\ \dot{x}_2 = 0.5x_1 + \cos\left(0.2\int_0^1 tsx_2(s)\,ds\right) \end{cases}$$

with two-point boundary conditions

$$x_1(0) = 0, x_2(1) = 1$$

and impulsive condition

$$\Delta x_1(0.5) = \frac{|x_2(0.5)|}{10(1 + |x_2(0.5)|)},$$

where $t \in [0,1]$, $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ and $\mu(t,s) = \begin{pmatrix} 0.2ts & 0 \\ 0 & 0 \end{pmatrix}$, $\gamma(t,s) = \begin{pmatrix} 0 & 0 \\ 0 & 0.2ts \end{pmatrix}$, and $S = 1$, $M = 0.5$, $\mu_0 = \gamma_0 = 0.2$, $l_1 = 0.1$.

Then $L = TSM(1 + T(\mu_0 + \gamma_0)) = 0.5(1 + 0.2 + 0.2) + 0.1 = 0.8 < 1$.

Thus, by Theorem 1, the boundary value problem has a unique solution on $[0,1]$.

Conclusion

The boundary conditions considered in this paper are general enough and can be used extensively in a wide class of problems. In this article the existence and uniqueness of the solutions for the first-order nonlinear impulsive differential equations with two-point conditions are established under sufficient conditions. Note that the methods given here can be used in similar multi-point problems for the ordinary differential equations as follows:

$$\dot{x}(t) = f(t, x(t), \mu x(t), \gamma x(t)), \quad t \in [0,T]$$

with multi-point and integral boundary conditions

$$\sum_{i=0}^{m} l_i x(t_i) + \int_0^T n(t)\, x(t)\, dt = \alpha.$$

Here $0 = t_0 < t_1 < ... < t_{m-1} < t_m = T$; $n(t) \in R^{n\times n}$ is a given function; $l_i \in R^{n\times n}$, $i = 1, 2, ..., m$ are given matrices; $\alpha \in R^n$is a given vector and

$$\det N \ne 0, \; N = \sum_{i=0}^{m} l_i + \int_0^T n(t)\, dt.$$

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Екі нүктелі шекаралық шарттары бар бірінші ретті сызықты емес импульсті интегро-дифференциалдық теңдеулер үшін шешімнің бар және жалғыз болуы

Мақалада импульстің әсерінен екі нүктелі шекаралық шарттары бар бірінші ретті сызықтыемес интегро-дифференциалдық теңдеулер жүйесінің шешімдерінің бар және жалғыз болуы талқыланды. Грин функциясы құрылды және қарастырылып отырған есепке эквивалентті интегралдық теңдеу келтірілді. Бұл есептің шешімінің бар және жалғыз болуы банахтың сығымдалған бейнелеуінің принципін қолдана отырып талданды. Қозғалмайтын нүкте туралы Шефер теоремасы шешімдердің бар екендігін дәлелдеу үшін қолданылды.

Кілт сөздер: екі нүктелі шекаралық шарттар, импульстік жүйелер, шешімнің бар және жалғыз болуы, қозғалмайтын нүкте теоремалары, бірінші ретті дифференциалдық теңдеу.

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Существование и единственность результатов для нелинейных импульсных интегро-дифференциальных уравнений первого порядка с двухточечными граничными условиями

В статье обсуждены существование и единственность решений системы нелинейных интегро-дифференциальных уравнений первого порядка с двухточечными граничными условиями при импульсных воздействиях. Построена функция Грина, и рассматриваемая задача сведена к эквивалентному интегральному уравнению. Существование и единственность решения этой задачи проанализированы с помощью банахова принципа сжимающего отображения. Теорема Шефера о неподвижной точке использовалась для доказательства существования решений.

Ключевые слова: двухточечные граничные условия, импульсные системы, существование и единственность решения, теорема о неподвижной точке, дифференциальное уравнение первого порядка.

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Ternary semigroups of topological transformations

A ternary semigroup is a nonempty set with a ternary operation which is associative. The purpose of the present paper is to give a characterization of open sets of finite-dimensional Euclidean spaces by ternary semigroups of pairs of homeomorphic transformations and extend to ternary semigroups certain results of L.M. Gluskin concerned with semigroups of homeomorphic transformations of finite-dimensional Euclidean spaces.

Keywords: Euclidean n-space, ternary semigroup, homeomorphic transformations.

Introduction

Lehmer [1] investigated certain triple systems called triplexes, Santiago and Sri Bala [2] developed regular and completely regular ternary semigroups. Dutta, Kar and Maity studied intra-regular ternary semigroups [3]. Wagner studied generalized heaps and generalized groups [4]. Gluskin showed that semigroups of topological transformations of bounded closed sets on Euclidean n-spaces define those sets exactly up to homeomorphism [5]. Mustafaev studied semiheaps of homeomorphic maps of open and closed sets of Euclidean n-spaces [6]. In this paper we study some properties of ternary semigroups of topological maps between open sets of Euclidean n-spaces.

A ternary semigroup is a nonempty set T together with a ternary operation $[abc]$ satisfying the associative law $[[abc]\,de] = [a\,[bcd]\,e] = [ab\,[cde]]$ for every $a, b, c, d, e \in T$. Any semigroup can be made into a ternary semigroup by defining the ternary product to be $[abc] = abc$. A nonempty subset L of a ternary semigroup T is called a left (right, lateral) *ideal* of T, if $[TTL] \subseteq L$ ($[LTT] \subseteq L, [TLT] \subseteq L$). A nonempty subset A of a ternary semigroup T is called a *two sided ideal* of T if it is a left and right ideal of T. A nonempty subset A of a ternary semigroup T is called an *ideal* of T if it is a left, right and lateral ideal of T. If the intersection K of all the ideals of a ternary semigroup T is not empty, we shall call K the *kernel* of T. A ternary semigroup is called (left, right) simple if it does not contain any proper (left, right) ideals [7]. A ternary semigroup is simple if it does not have nontrivial homomorphisms, that is, if each of its homomorphisms is either an isomorphism or a mapping onto a ternary semigroup consisting of one element. A zero "0"of a ternary semigroup T is an element such that for all $a, b \in T$, $[0ab] = [a0b] = [ab0] = 0$. An equivalence relation ρ on a ternary semigroup T is said to be a left congruence if $(a, b) \in \rho \Longrightarrow ([sta], [stb]) \in \rho$ for all $a, b, s, t \in T$. Similarly, ρ is a right congruence if $(a, b) \in \rho \Longrightarrow ([ast], [bst]) \in \rho$ for all $a, b, s, t \in T$ and a lateral congruence if $(a, b) \in \rho \Longrightarrow ([sat], [sbt]) \in \rho$ for all $a, b, s, t \in T$. An equivalence relation ρ on a ternary semigroup T is said to be a congruence if $(a, a') \in \rho, (b, b') \in \rho, (c, c') \in \rho \Longrightarrow ([abc], [a'b'c']) \in \rho$ for all $a, a', b, b', c, c' \in T$. An equivalence relation ρ on a ternary semigroup T is a congruence if and only if it is a left, a right and a lateral congruence on T [8].

Let X and Y be two nonempty sets and let $F(X, Y)$ be the set of all pairs of functions (γ, η), where $\gamma : X \to Y$ and $\eta : Y \to X$. The set $F(X, Y)$ is a ternary semigroup with respect to the ternary operation

$$[(\gamma_1, \eta_1)(\gamma_2, \eta_2)(\gamma_3, \eta_3)] = (\gamma_1\eta_2\gamma_3, \eta_1\gamma_2\eta_3),$$

where $(\gamma_1\eta_2\gamma_3)\,x = \gamma_1(\eta_2(\gamma_3(x)))$ and $(\eta_1\gamma_2\eta_3)\,y = \eta_1(\gamma_2(\eta_3(y)))$.

Let S be a ternary semigroup and a be any element of S. The set $SSa \cup a$ is a left ideal of S and is called the principal left ideal of S generated by a. Consider the following symmetric and reflexive relation on the set S defined by

$$\sigma_t : x\sigma_t y \leftrightarrow x,\ y \in SSa \cup a,\ (x, y \in S).$$

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$\overline{\sigma_t}$ denotes the transitive closure of σ_t. Each class L_ν of $\overline{\sigma_t}$ is the union of some principal left ideals of S and therefore is a left ideal of S. The partition

$$S = \cup L_\nu \tag{1}$$

of the ternary semigroup S into classes of $\overline{\sigma_t}$ is the representation of S as the union of the pairwise disjoint left ideals. We say that (1) is the most fractional partition of S into pairwise disjoint left ideals.

Characterization of open sets of Euclidean n-spaces by ternary semigroups

Let Ω_1 and Ω_2 be two open sets of a finite-dimensional Euclidean space. $H_i(\Omega_i, \Omega_j)$ denotes the set of all homeomorphic maps from Ω_i to Ω_j, where $i, j = 1, 2, (i \neq j)$. Let $K_i(\Omega_i, \Omega_j)$ denote the set of all $a \in H_i(\Omega_i, \Omega_j)$ for which there is an n-sized element $E_a \subset \Omega_j$ (a set homeomorphic to some closed n-ball) and a closed set $F_a \subset \Omega_j$ such that $a\Omega_i \subset F_a \subset IntE_a$, where $i, j = 1, 2$ $(i \neq j)$. Let $OH = OH(\Omega_1, \Omega_2) = H_1(\Omega_1, \Omega_2) \times H_2(\Omega_2, \Omega_1)$ be the set of all pairs of homeomorphic maps (a, b), where $a \in H_1(\Omega_1, \Omega_2)$, $b \in H_2(\Omega_2, \Omega_1)$. The set OH is a ternary semigroup with respect to the ternary operation

$$[(a_1, b_1)(a_2, b_2)(a_3, b_3)] = (a_1 b_2 a_3, b_1 a_2 b_3).$$

Clearly, the set $K = K(\Omega_1, \Omega_2) = K_1(\Omega_1, \Omega_2) \times K_2(\Omega_2, \Omega_1)$ is a ternary subsemigroup and even an ideal of the ternary semigroup OH.

Theorem 1. Let R and R' be finite-dimensional Euclidean spaces. Let Ω_1 and Ω_2 be open sets of a finite-dimensional Euclidean space R and Ω_1' and Ω_2' be open sets of a finite-dimensional Euclidean space R'. The ternary semigroups $K(\Omega_1, \Omega_2)$ and $K(\Omega_1', \Omega_2')$ are isomorphic if and only if the spaces Ω_i and Ω_i' are homeomorphic $(i = 1, 2)$.

Proof. Let Ω_1 and Ω_2 be open subsets of a finite-dimensional Euclidean space R and let Ω_1' and Ω_2' be open subsets of a finite-dimensional Euclidean space R'. Suppose that $\xi_1 : \Omega_1 \to \Omega_1'$ is a homeomorphism of Ω_1 onto Ω_1' and $\xi_2 : \Omega_2 \to \Omega_2'$ is a homeomorphism of Ω_2 onto Ω_2'. Then, the mapping $\varphi_{\xi_1,\xi_2} : K(\Omega_1, \Omega_2) \to K(\Omega_1', \Omega_2')$ defined by

$$\varphi_{\xi_1,\xi_2}(a, b) = \left(\xi_2 a \xi_1^{-1}, \xi_1 b \xi_2^{-1}\right)$$

is an isomorphism from $K(\Omega_1, \Omega_2)$ onto $K(\Omega_1', \Omega_2')$. The proof of the necessary condition follows from Lemmas 1–5.

Throughout this paper the symbol φ denotes an isomorphism $\varphi : K(\Omega_1, \Omega_2) \to K(\Omega_1', \Omega_2')$ unless otherwise stated.

Lemma 1. Let $(a_1, b_1), (a_2, b_2) \in K(\Omega_1, \Omega_2)$ such that

$$(a_2, b_2)(\Omega_1, \Omega_2) \subseteq (a_1, b_1)(\Omega_1, \Omega_2).$$

Then,

$$\varphi(a_2, b_2)(\Omega_1', \Omega_2') \subset \overline{\varphi(a_1, b_1)(\Omega_1', \Omega_2')}.$$

Proof. Let $(a_1, a_2), (b_1, b_2)$ be any two elements in $K(\Omega_1, \Omega_2)$ such that

$$(a_2, b_2)(\Omega_1, \Omega_2) \subseteq (a_1, b_1)(\Omega_1, \Omega_2).$$

If

$$[\varphi(a_1, b_1)(x_1', y_1')\varphi(a_1, b_1)] = [\varphi(a_1, b_1)(x_2', y_2')\varphi(a_1, b_1)] \tag{2}$$

is valid for some $(x_1', y_1'), (x_2', y_2') \in K(\Omega_1', \Omega_2')$, then there exist elements $(x_1, y_1), (x_2, y_2) \in K(\Omega_1, \Omega_2)$ such that $\varphi(x_1, y_1) = (x_1', y_1')$ and $\varphi(x_2, y_2) = (x_2', y_2')$. Therefore

$$[\varphi(a_1, b_1)\varphi(x_1, y_1)\varphi(a_1, b_1)] = [\varphi(a_1, b_1)\varphi(x_2, y_2)\varphi(a_1, b_1)].$$

From this it follows

$$\varphi[(a_1, b_1)(x_1, y_1)(a_1, b_1)] = \varphi[(a_1, b_1)(x_2, y_2)(a_1, b_1)]$$

and since φ is an isomorphism of $K(\Omega_1, \Omega_2)$ onto $K(\Omega_1', \Omega_2')$ we have

$$[(a_1, b_1)(x_1, y_1)(a_1, b_1)] = [(a_1, b_1)(x_2, y_2)(a_1, b_1)].$$

Then,
$$[(a_2,b_2)(x_1,y_1)(a_2,b_2)] = [(a_2,b_2)(x_2,y_2)(a_2,b_2)]$$
or
$$[\varphi(a_2,b_2)\varphi(x_1,y_1)\varphi(a_2,b_2)] = [\varphi(a_2,b_2)\varphi(x_2,y_2)\varphi(a_2,b_2)]$$
or
$$[\varphi(a_2,b_2)(x_1',y_1')\varphi(a_2,b_2)] = [\varphi(a_2,b_2)(x_2',y_2')\varphi(a_2,b_2)].$$
Since the last equality is valid for every $(x_1',y_1'),(x_2',y_2') \in K(\Omega_1',\Omega_2')$ satisfying (2), we have $\varphi(a_2,b_2)(\Omega_1',\Omega_2') \subset \overline{\varphi(a_1,b_1)(\Omega_1',\Omega_2')}$.

The following two lemmas are immediate consequences of Lemma 1.

Lemma 2. Let $(a_1,b_1),(a_2,b_2) \in K(\Omega_1,\Omega_2)$. If
$$(a_1,b_1)(\Omega_1,\Omega_2) \cap (a_2,b_2)(\Omega_1,\Omega_2) \neq \varnothing,$$
then
$$\overline{\varphi(a_1,b_1)(\Omega_1',\Omega_2')} \cap \overline{\varphi(a_2,b_2)(\Omega_1',\Omega_2')} \neq \varnothing.$$

Lemma 3. Let $(a_1,b_1),(a_2,b_2),(a_3,b_3) \in K(\Omega_1,\Omega_2)$. The equation
$$[(a_2,b_2)(a_1,b_1)(x_2,y_2)] = (a_3,b_3)$$
has a solution for $(x_2,y_2) \in K(\Omega_1,\Omega_2)$ if and only if there exist n-sized elements T_1 and T_2 such that
$$T_1 \subset b_2a_1\Omega_1, T_2 \subset a_2b_1\Omega_2, a_3\Omega_1 \subset IntT_2, b_3\Omega_2 \subset IntT_1.$$

Let $(\alpha,\beta) \in \Omega_1 \times \Omega_2$. We say that an infinite sequence $\{(a_i,b_i)\}_{i=1}^{\infty}$ of elements $(a_i,b_i) \in K(\Omega_1,\Omega_2)$ has a limit (α,β), if the following conditions are satisfied:

a) $(\cap_{i=2}^{\infty} b_i\Omega_2, \cap_{i=2}^{\infty} a_i\Omega_1) = (\alpha,\beta)$,

b) for every i there exists an element (x_{i+1},y_{i+1}) such that
$$[(a_{i+1},b_{i+1})(a_i,b_i)(x_{i+1},y_{i+1})] = (a_{i+2},b_{i+2}).$$
A sequence $\{(a_i,b_i)\}_{i=1}^{\infty}$ of elements $(a_i,b_i) \in K(\Omega_1,\Omega_2)$ converging to the point $(\alpha,\beta) \in \Omega_1 \times \Omega_2$ can be built, for example, as follows. Suppose that $E_1 \subset \Omega_1$ is a closed n-ball centered at α and $T_1 \subset \Omega_2$ is a closed n-ball centered at β. There exists $(a_1,b_1) \in K(\Omega_1,\Omega_2)$ such that
$$\alpha \in b_1\Omega_2 \subset IntE_1,\ \ \beta \in a_1\Omega_1 \subset IntT_1.$$
Suppose now that E_2 is a closed n-ball in $b_1\Omega_2$ and centered at α and T_2 is a closed n-ball in $a_1\Omega_1$ and centered at β. Then, there exists an element $(a_2,b_2) \in K(\Omega_1,\Omega_2)$ such that
$$\alpha \in b_2\Omega_2 \subset IntE_2,\ \ \beta \in a_2\Omega_1 \subset IntT_2.$$
Let $\alpha_1 = (b_2a_1)^{-1}(\alpha)$ and $\beta_1 = (a_2b_1)^{-1}(\beta)$. Let $A_1 \subset \Omega_1$ be a closed n-ball centered at α_1 and $B_1 \subset \Omega_2$ be a closed n-ball centered at β_1. Clearly,
$$\alpha \in b_2a_1A_1 \cap E_2, \beta \in a_2b_1B_1 \cap T_2.$$
Let $E_3 \subset Int(b_2a_1A_1 \cap E_2)$ be a closed n-ball centered at α, and let $T_3 \subset Int(a_2b_1B_1 \cap T_2)$ be a closed n-ball centered at β. Then, there exists an element $(a_3,b_3) \in K(\Omega_1,\Omega_2)$ such that
$$\alpha \in b_3\Omega_2,\ \ \ \beta \in a_3\Omega_1$$
and
$$b_3\Omega_2 \subset IntE_3,\ \ \ a_3\Omega_1 \subset IntT_3.$$
By Lemma 3, the point $x_2 = (a_2b_1)^{-1}a_3, y_2 = (b_2a_1)^{-1}b_3$ is the solution of the equation
$$[(a_2,b_2)(a_1,b_1)(x_2,y_2)] = (a_3,b_3).$$

Assume now that the first n terms of the sequence are already found. Denote $\alpha_{n-1} = (b_n a_{n-1})^{-1}(\alpha)$ and $\beta_{n-1} = (a_n b_{n-1})^{-1}(\beta)$. Suppose that $A_{n-1} \subset \Omega_1$ is a closed n-ball centered at α_{n-1} and $B_{n-1} \subset \Omega_2$ is a closed n-ball centered at β_{n-1}. Clearly,

$$\alpha \in b_n a_{n-1} A_{n-1} \cap E_n, \beta \in a_n b_{n-1} B_{n-1} \cap T_n.$$

Let $E_{n+1} \subset Int\,(b_n a_{n-1} A_{n-1} \cap E_n)$ be a closed n-ball centered at α, and let $T_{n+1} \subset Int\,(a_n b_{n-1} B_{n-1} \cap T_n)$ be a closed n-ball centered at β. Then, there exists an element $(a_{n+1}, b_{n+1}) \in K(\Omega_1, \Omega_2)$ such that

$$\alpha \in b_{n+1}\Omega_2, \beta \in a_{n+1}\Omega_1$$

and

$$b_{n+1}\Omega_2 \subset IntE_{n+1}, a_{n+1}\Omega_1 \subset IntT_{n+1}.$$

By Lemma 3, the point $x_n = (a_n b_{n-1})^{-1} a_{n+1}, y_n = (b_n a_{n-1})^{-1} b_{n+1}$ is the solution of the equation

$$[(a_n, b_n)(a_{n-1}, b_{n-1})(x_n, y_n)] = (a_{n+1}, b_{n+1}).$$

This sequence satisfies condition (b) and condition (a), if the sequences of radii of E_n and T_n converge to zero.

Lemma 4. If the ternary semigroups $K(\Omega_1, \Omega_2)$ and $K(\Omega_1', \Omega_2')$ are isomorphic, then there exist a bijective map f from $\Omega_1 \times \Omega_2$ onto $\Omega_1' \times \Omega_2'$ and bijective maps ξ_i from Ω_i onto Ω_i' $(i = 1, 2)$ such that $f(\alpha, \beta) = (\xi_1\alpha, \xi_2\beta)$ for every $(\alpha, \beta) \in \Omega_1 \times \Omega_2$.

Proof. Let (α, β) be any point in $\Omega_1 \times \Omega_2$ and let $\{(a_i, b_i)\}_{i=1}^{\infty}$ be a sequence of elements $(a_i, b_i) \in B(\Omega_1, \Omega_2)$ converging to the point (α, β). Denote $\varphi(a_i, b_i)$ by (a_i', b_i'). The sequence $\{(a_i', b_i')\}_{i=1}^{\infty}$ converges to the point (α', β'). Define a map $f : (\Omega_1, \Omega_2) \to (\Omega_1', \Omega_2')$ by $f(\alpha, \beta) = (\alpha', \beta')$. The point (α', β') does not depend on the choice of the sequence $\{(a_i, b_i)\}_{i=1}^{\infty}$ of elements $(a_i, b_i) \in B(\Omega_1, \Omega_2)$ converging to the point (α, β). The map f is one-to-one and there are one-to-one maps ξ_i from Ω_i onto Ω_i' for $(i = 1, 2)$ such that $\forall (\alpha, \beta) \in \Omega_1 \times \Omega_2, f(\alpha, \beta) =$ $= (\xi_1\alpha, \xi_2\beta)$.

Lemma 5. 1) The following implication holds

$$(\alpha, \beta) \in (a, b)(\Omega_1, \Omega_2) \to f(\alpha, \beta) = (\xi_1\alpha, \xi_2\beta) \in \overline{\varphi(a, b)(\Omega_1', \Omega_2')}$$

for any $(\alpha, \beta) \in \Omega_1 \times \Omega_2$ and $(a, b) \in K(\Omega_1, \Omega_2)$.

2) The map f is a homeomorphism from $\Omega_1 \times \Omega_2$ onto $\Omega_1' \times \Omega_2'$ and therefore the map ξ_i is a homeomorphism from Ω_i onto Ω_i' for $i = 1, 2$.

Theorem 2. The ternary semigroup $K(\Omega_1, \Omega_2)$ is a minimal ideal (the kernel) of the ternary semigroup $OH(\Omega_1, \Omega_2)$.

Theorem 3. Let Ω_1 and Ω_2 be open subsets of a finite-dimensional Euclidean space R and Ω_1' and Ω_2' be open subsets of a finite-dimensional Euclidean space R'. The ternary semigroups $OH(\Omega_1, \Omega_2)$ and $OH(\Omega_1', \Omega_2')$ are isomorphic if and only if the spaces Ω_i and Ω_i' are homeomorphic $(i = 1, 2)$.

Properties of the ternary semigroup of topological maps

Let G be a group and let $A = G \cup \{0\}$ be a zero adjoint semigroup. Let I, Λ be non-empty sets and let P be a $\Lambda \times I$ matrix over A such that every row and every column of P contain at least one non-zero entry. The set $S = A \times I \times \Lambda$ with ternary multiplication

$$[(a; i, \lambda)(b; j, \mu)(c; k, \nu)] = (ap_{\lambda j}bp_{\mu k}c; i, \nu)$$

is a ternary semigroup with zero $0 = (0, i, \lambda)$. Let (A, i, λ) denote a subset of S consisting of all triples (a, i, λ), where $a \in A$ and i, λ are fixed elements, then

$$S = \underset{i \in I, \lambda \in \Lambda}{\cup}(A, i, \lambda).$$

From the definition of the ternary operation it follows that S is the union of its nonzero minimal right ideals R_i and S is the union of its nonzero minimal left ideals L_λ, where

$$R_i = \underset{\lambda \in \Lambda}{\cup}(A, i, \lambda), \quad L_\lambda = \underset{i \in I}{\cup}(A, i, \lambda).$$

The ternary semigroup S does not contain any proper two sided ideals, in particular, S does not contain any proper ideals. We denote S by $M^0(G, I, \Lambda, P)$.

Let R and R' be finite-dimensional Euclidean spaces, Ω_1 and Ω_2 be open subsets of a finite-dimensional Euclidean space R and let Ω_1' and Ω_2' be open subsets of a finite-dimensional Euclidean space R'. Suppose that $\xi_1 : \Omega_1 \to \Omega_1'$ is a homeomorphism of Ω_1 onto Ω_1' and $\xi_2 : \Omega_2 \to \Omega_2'$ is a homeomorphism of Ω_2 onto Ω_2'. Then, the mapping $\varphi_{\xi_1,\xi_2} : K(\Omega_1, \Omega_2) \to K(\Omega_1', \Omega_2')$ defined by

$$\varphi_{\xi_1,\xi_2}(a, b) = \left(\xi_2 a \xi_1^{-1}, \xi_1 b \xi_2^{-1}\right)$$

is a homeomorphism from $K(\Omega_1, \Omega_2)$ onto $K(\Omega_1', \Omega_2')$.

Introduce the following symmetric and reflexive relation σ_r in the ternary semigroup $OH(\Omega_1, \Omega_2) : (a_1, b_1)$, $(a_2, b_2) \in \sigma_r$ if and only if $(a_1, b_1) = (a_2, b_2)$ or $(a_1, b_1), (a_2, b_2) \in R_{(a,b)}$ for some $(a, b) \in OH(\Omega_1, \Omega_2)$, where $R_{(a,b)}$ is a right ideal of $OH(\Omega_1, \Omega_2)$ generated by (a, b). Since the relation σ_r is stable, its transitive closure $\overline{\sigma_r}$ is a congruence on $OH(\Omega_1, \Omega_2)$. Each equivalence class R_α of $\overline{\sigma_r}$ either consists of one element of $OH(\Omega_1, \Omega_2)$ not contained in $K(\Omega_1, \Omega_2)$ or is a right ideal of $OH(\Omega_1, \Omega_2)$. Then,

$$K(\Omega_1, \Omega_2) = \bigcup_{\alpha \in I} R_\alpha$$

is the most fractional partition of $K(\Omega_1, \Omega_2)$ into pairwise disjoint union of the distinct right ideals of $OH(\Omega_1, \Omega_2)$.

Let A_i be some component of the set Ω_1, B_μ be some component of the set Ω_2 and $R_{i\mu}$ be a subset of the ternary semigroup $K(\Omega_1, \Omega_2)$ consisting of (a, b) such that $a\Omega_1 \subset B_\mu, b\Omega_2 \subset A_i$. If (a, b) is any element of $R_{i\mu}$ and $(x_1, y_1), (x_2, y_2)$ are the elements of the ternary semigroup $OH(\Omega_1, \Omega_2)$, then from $a\Omega_1 \subset B_\mu, b\Omega_2 \subset A_i$ it follows that

$$a y_1 x_2 \Omega_1 \subset B_\mu, \quad b x_1 y_2 \Omega_2 \subset A_i.$$

Thus, $(a y_1 x_2, b x_1 y_2) \in R_{i\mu}$ and $R_{i\mu}$ is a right ideal of $OH(\Omega_1, \Omega_2)$. Consequently, the partition

$$K(\Omega_1, \Omega_2) = \bigcup_{i \in I, \mu \in M} R_{i\mu}$$

is the presentation of $K(\Omega_1, \Omega_2)$ as the pairwise disjoint union of the distinct right ideals of $OH(\Omega_1, \Omega_2)$.

Lemma 6. If E_i is any closed ball contained in Ω_i, and α_i, β_i are any points in $IntE_i$, then there is $(b_1, b_2) \in K(\Omega_1, \Omega_2)$ such that

$$\alpha_i, \beta_i \in b_j \Omega_j, \quad b_j \Omega_j \subset IntE_i,$$

where $i, j = 1, 2$ $(i \neq j)$.

Proof. Let (a_1, a_2) be an arbitrary element in $K(\Omega_1, \Omega_2)$, C_i be a closed ball in $a_j \Omega_j$ and A_i, B_i, D_i be any closed balls such that

$$\Omega_i \subset IntE_i, \quad B_i \subset IntD_i, \quad D_i \subset IntE_i, \quad \alpha_i, \beta_i \in IntB_i.$$

Futher, let f_i be the homeomorphisms of A_i onto D_i such that $f_i(C_i) = B_i$. Then $(b_1, b_2) = (f_2 a_1, f_1 a_2)$ is the requiered element of $K(\Omega_1, \Omega_2)$.

Lemma 7. The partition

$$K(\Omega_1, \Omega_2) = \bigcup_{i \in I, \mu \in M} R_{i\mu}$$

is the most fractional partition of $K(\Omega_1, \Omega_2)$ into pairwise disjoint right ideals of $OH(\Omega_1, \Omega_2)$.

Proof. It is sufficient to show that for any $i \in I, \mu \in M$ the condition

$$(a_0, b_0), (a, b) \in R_{i\mu}$$

implies

$$(a_0, b_0), (a, b) \in \overline{\sigma_r}.$$

Let's prove this for the maps b_0 and b from Ω_2 into A_i, where A_i is a component of Ω_1. Let $\xi \in b_0 \Omega_2, \quad \xi' \in b\Omega_2$. Since $\xi, \xi' \in A_i$, they can be connected by a simple arc l contained in A_i. We have $d(F_r(A_i), l) = m > 0$, where $F_r(A_i)$ is a boundary of A_i. Then, it can be found finite covering of l with open balls of radius $r < m$ centered on l. Denote these balls by E_{2k} $(k = 1, 2, ..., s)$, numerated in order of their centers positions on l. Choose points

$$\xi_k \in E_{2k} \cap E_{2k+2}, (k = 1, 2, ..., s - 1)$$

and denote $\xi_0 = \xi, \xi_s = \xi'$. Since $\overline{E_{2k}} \subset A_i$, there are closed balls D_{2k} in $\overline{E_{2k}}$ such that $\xi_{k-1}, \xi_k \in IntD_{2k}$. According to Lemma 1 there exists a homeomorphism b_{2k} from Ω_2 to Ω_1 such that $b_{2k}\Omega_2 \subset IntD_{2k}$ and

$$\xi_{k-1}, \xi_k \in b_{2k}\Omega_2, (k = 1, 2, ..., s-1).$$

Denote $b = b_{2s+2}$. There are closed sets E_{2i+1} and E'_{2i+1} centered at ξ_i such that $E_{2i+1} \subset E'_{2i+1} \subset b_{2i}\Omega_2 \cap \cap b_{2i+2}, (k = 1, 2, ..., s)$. By Lemma 1, there exists a homeomorphism b_{2k+1} from Ω_2 to Ω_1 such that $b_{2i+1}\Omega_2 \subset IntE_{2i+1}$. Let E be a closed n-ball containing the set Ω_1 and let $\widetilde{E}$ be a closed n-ball contained in Ω_1. If f_{2i+1} is a homeomorphism from E onto E'_{2i+1} for which $f_{2i+1}\left(\widetilde{E}\right) = E_{2i+1}$, then

$$\begin{aligned} b_{2i+1} &= b_{2i} \circ b_{2i}^{-1} f_{2i+1} \circ f_{2i+1}^{-1} b_{2i+1}, \\ b_{2i+1} &= b_{2i+2} \circ b_{2i+2}^{-1} f_{2i+1} \circ f_{2i+1}^{-1} b_{2i+1}, \end{aligned}$$

where $k = 0, 1, 2, ..., s$. Analogously, it can be shown that the following equations hold for some homeomorphism g_{2i+1}

$$\begin{aligned} a_{2i+1} &= a_{2i} \circ a_{2i}^{-1} g_{2i+1} \circ g_{2i+1}^{-1} a_{2i+1}, \\ a_{2i+1} &= a_{2i+2} \circ a_{2i+2}^{-1} g_{2i+1} \circ g_{2i+1}^{-1} a_{2i+1}, \end{aligned}$$

where $j = 0, 1, 2, ..., s'$ and $a_{2s'+2} = a$. Suppose that $s < s'$. Thus, $(a_{i-1}, b_{i-1}), (a_i, b_i) \in \overline{\sigma_r}$ for $i = 1, 2, ..., 2s+2$ and

$$(a_{j-1}, b_{2s+2}), (a_j, b_{2s+2}) \in \overline{\sigma_r} fori = 1, 2, ..., 2s+2.$$

This means that $(a_0, b_0), (a, b) \in \overline{\sigma_r}$.

Analogously, it can be shown that

$$K(\Omega_1, \Omega_2) = \bigcup_{i \in I, \mu \in M} R_{i\mu}$$

is the most fractional partition of $K(\Omega_1, \Omega_2)$ into pairwise disjoint right ideals.

Theorem 4. The quotient $OH(\Omega_1, \Omega_2)/\overline{\sigma_r}$ is a ternary semigroup with the minimal ideal $K(\Omega_1, \Omega_2)/\overline{\sigma_r}$.

Proof. Since $K(\Omega_1, \Omega_2)$ is an ideal of $OH(\Omega_1, \Omega_2)$ the quotient $K(\Omega_1, \Omega_2)/\overline{\sigma_r}$ is an ideal of $OH(\Omega_1, \Omega_2)/\overline{\sigma_r}$. It follows from Lemma 2 that the elements of $K(\Omega_1, \Omega_2)/\overline{\sigma_r}$ are the classes $R_{i\lambda}$. Let $G = \{e\}$ be the unit group, and let P be a $\Lambda \times I$ matrix over G. Denote by T the completely simple ternary semigroup over $G = \{e\}$. To each element $R_{i\lambda}$ of $K(\Omega_1, \Omega_2)/\overline{\sigma_r}$ assign an element $(e; i, \lambda)$ of T. The map f is the isomorphism from $K(\Omega_1, \Omega_2)/\overline{\sigma_r}$ onto T. Indeed,

$$f([R_{i\lambda} R_{j\mu} R_{k\nu}]) = f(R_{i\nu}) = (e; i, \nu)$$
$$= [(e; i, \lambda)(e; j, \mu)(e; k, \nu)] = [f(R_{i\lambda}) f(R_{j\mu}) f(R_{k\nu})].$$

Theorem 5. The ternary semigroup $K(\Omega_1, \Omega_2)/\overline{\sigma_r}$ is a topological invariant of the pair (Ω_1, Ω_2).

Proof. Let Ω_1 and Ω_2 be open subsets of a finite-dimensional Euclidean space R^n and let Ω'_1 and Ω'_2 be open subsets of a finite-dimensional Euclidean space R^m. If $\xi_i : \Omega_i \to \Omega'_i \, (i = 1, 2)$ is a homeomorphism, then the mapping $f : K(\Omega_1, \Omega_2) \to K(\Omega'_1, \Omega'_2)$ defined by

$$f(a, b) = \left(\xi_2 a \xi_1^{-1}, \xi_1 b \xi_2^{-1}\right)$$

is an isomorphism from $K(\Omega_1, \Omega_2)$ onto $K(\Omega'_1, \Omega'_2)$. In the case of ξ_1 the component $A_i \subset \Omega_1$ is mapped onto the component $A'_{i'} \subset \Omega'_1$, in the case of ξ_2 the component $B_\lambda \subset \Omega_2$ is mapped onto the component $B'_{\lambda'} \subset \Omega'_2$. Therefore $K(\Omega_1, \Omega_2)/\overline{\sigma_r}$ is isomorphic to $K(\Omega'_1, \Omega'_2)/\overline{\sigma_r}$.

The following three theorems are immediate consequences of Lemma 7.

Theorem 6. The spaces Ω_1 and Ω_2 are connected if and only if the ternary semigroup $K(\Omega_1, \Omega_2)$ cannot be represented as the pairwise disjoint union of its distinct right ideals.

Theorem 7. If the quotient $K(\Omega_1, \Omega_2)/\overline{\sigma_r}$ is finite and its order is a prime number, then one of the spaces Ω_1 and Ω_2 is connected.

Theorem 8. Let Ω_1 and Ω_2 be open subsets of a finite-dimensional Euclidean space. The space Ω_2 is connected if and only if the quotient $K(\Omega_1, \Omega_2)/\overline{\sigma_r}$ is a ternary semigroup of left zeros, the space Ω_1 is connected if and only if the quotient $K(\Omega_1, \Omega_2)/\overline{\sigma_r}$ is a ternary semigroup of right zeros.

Theorem 9. If at least one of the ternary semigroups $R_{i\mu}$ in the partition $K(\Omega_1, \Omega_2) = \bigcup_{i \in I, \mu \in M} R_{i\mu}$ is simple, then the spaces Ω_1 and Ω_2 are connected.

Proof. Introduce the following relation $\rho_{i\mu}$ in the ternary semigroup $R_{i\mu} : \forall (\varphi_1, \phi_1)$, $(\varphi_2, \phi_2) \in R_{i\mu}$, $((\varphi_1, \phi_1), (\varphi_2, \phi_2)) \in \rho_{i\mu}$ if and only if $\forall \xi \in A_i, \eta \in B_\mu, \varphi_1(\xi) = \varphi_2(\xi), \phi_1(\eta) = \phi_2(\eta)$. The relation $\rho_{i\mu}$ is a congruence on $R_{i\mu}$. Indeed, if $(\varphi_1, \phi_1), (\varphi_2, \phi_2)$ are any two elements of $R_{i\mu}$ such that $((\varphi_1, \phi_1), (\varphi_2, \phi_2)) \in \rho_{i\mu}$ and $(x_1, y_1), (x_2, y_2) \in R_{i\mu}$, then

$$\begin{gathered}(\varphi_1, \phi_1) \overset{\rho_{i\mu}}{\sim} (\varphi_2, \phi_2) \rightarrow \{\forall \xi \in A_i, \eta \in B_\mu, (x_i, y_i) \in R_{i\mu} \\ \varphi_1 y_2 x_1(\xi) = \varphi_2 y_2 x_1(\xi), y_1 x_2 \phi_1(\eta) = y_1 x_2 \phi_2(\eta)\} \rightarrow \\ \forall (x_i, y_i) \in R_{i\mu}, \\ ([(x_1, y_1)(x_2, y_2)(\varphi_1, \phi_1)], [(x_1, y_1)(x_2, y_2)(\varphi_2, \phi_2)]) \in \rho_{i\mu}.\end{gathered}$$

Analogously, it can be shown that

$$\begin{gathered}([(x_1, y_1)(\varphi_1, \phi_1)(x_2, y_2)], [(x_1, y_1)(\varphi_2, \phi_2)(x_2, y_2)]) \in \rho_{i\mu}, \\ ([(\varphi_1, \phi_1)(x_1, y_1)(x_2, y_2)], [(\varphi_2, \phi_2)(x_1, y_1)(x_2, y_2)]) \in \rho_{i\mu}.\end{gathered}$$

Consider the mapping $f(\varphi, \phi) = (\varphi', \phi')$ from $R_{i\mu}$ to $K(A_i, B_\mu)$ such that

$$\varphi|_{A_i} = \varphi', \phi|_{B_\mu} = \phi'.$$

Clearly, the mapping f is a homomorphism of $R_{i\mu}$ to $K(A_i, B_\mu)$. Now, let the ternary semigroup $R_{i\mu}$ be simple and let at least one of the spaces Ω_1 or Ω_2 be connected. Suppose that Ω_1 is disconnected but Ω_2 is connected. The elements $(\varphi_1, \phi_1), (\varphi_2, \phi_2)$ can be found in $R_{i\mu}$ such that $((\varphi_1, \phi_1), (\varphi_2, \phi_2)) \in \rho_{i\mu}$ (φ_1, φ_2 can map Ω_1 to two disjoint balls $E_1, E_2 \subset \Omega_2$ and because of this $\varphi_1(\xi) \neq \varphi_2(\xi), \xi \in \Omega_1$). Besides, there are more than one element in each class of $\rho_{i\mu}$. Indeed, if φ is a homeomorphism of Ω_1 to the closed ball $E_1 \subset \Omega_2$ and g is a homeomorphism of Ω_1 to the closed ball $E_2 \subset \Omega_2$ such that $E_1 \cap E_2 = \emptyset, E_1, E_2 \subset E_3$, where E_3 is a closed ball in Ω_2, then the mapping

$$\chi(\alpha) = \begin{cases} \varphi(\alpha), \text{ if } \alpha \in A_i, \\ g(\alpha), \text{ if } \alpha \notin A_i, \end{cases}$$

where A_i is a component of Ω_1, is a homomorphism of Ω_1 to Ω_2. Clearly, $\chi \neq \varphi$ and $\chi|_{A_i} = \varphi|_{A_i}$. Therefore $((\chi, \phi), (\varphi, \phi)) \in \rho_{i1}$. Here ϕ denotes some homeomorphism of Ω_2 into the interior of some n-element containd in Ω_1 and ρ_{i1} is some congruence on R_{i1} ($B_1 = \Omega_2$). Then it follows that f is a homomorphism from R_{i1} onto $f(R_{i1}) \subset K(A_i, \Omega_2)$, which contains more than one element. But f is not an isomorphism, which contradicts the simplicity of R_{i1}.

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Топологиялық түрлендірудің тернарлы жартылайтоптары

Тернарлы жартылайтоп — бұл ассоциативті тернарлы операциясы бар бос емес жиын. Мақаланың мақсаты — ақырлы өлшемді евклид кеңістіктерінің ашық жиынтығын гомеоморфты қайта құру жұптарының тернарлы жартылайтоптарымен сипаттау және Л.М. Глушкиннің ақырлы өлшемді евклид кеңістіктерінің гомеоморфты түрлендірудің жартылайтоптарына қатысты кейбір нәтижелерін теріс жартылайтоптарға тарату.

Кілт сөздер: Евклид n-кеңістік, тернарлы жартылайтоп, гомеоморфты түрлендіру.

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Тернарные полугруппы топологических преобразований

Тернарная полугруппа — это непустое множество с ассоциативной тернарной операцией. Цель настоящей статьи — охарактеризовать открытые множества конечномерных евклидовых пространств тернарными полугруппами пар гомеоморфных преобразований и распространить на тернарные полугруппы некоторые результаты Л.М. Глушкина, касающиеся полугрупп гомеоморфных преобразований конечномерных евклидовых пространств.

Ключевые слова: евклидово n-пространство, тернарная полугруппа, гомеоморфные преобразования.

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Transmission dynamics and control strategies of COVID-19: a modelling study

In this paper a mathematical model is proposed, which incorporates quarantine and hospitalization to assess the community impact of social distancing and face mask among the susceptible population. The model parameters are estimated and fitted to the model with the use of laboratory confirmed COVID-19 cases in Turkey from March 11 to October 10, 2020. The partial rank correlation coefficient is employed to perform sensitivity analysis of the model, with basic reproduction number and infection attack rate as response functions. Results from the sensitivity analysis reveal that the most essential parameters for effective control of COVID-19 infection are recovery rate from quarantine individuals (δ_1), recovery rate from hospitalized individuals (δ_4), and transmission rate (β). Some simulation results are obtained with the aid of mesh plots with respect to the basic reproductive number as a function of two different biological parameters randomly chosen from the model. Finally, numerical simulations on the dynamics of the model highlighted that infections from the compartments of each state variables decreases with time which causes an increase in susceptible individuals. This implies that avoiding contact with infected individuals by means of adequate awareness of social distancing and wearing face mask are vital to prevent or reduce the spread of COVID-19 infection.

Keywords: COVID-19, mathematical modelling, basic reproduction number, transmission dynamics, sensitivity analysis.

Introduction

The coronavirus disease 2019 (COVID-19), previously recognized as "2019-nCoV", from the family of *Coronaviridae*, which includes the Middle East respiratory syndrome coronavirus (MERS-CoV) and the severe acute respiratory syndrome coronavirus (SARS-CoV), is a lethal virus that mostly transmits via human-to-human route [1–7]. The disease emerged from Wuhan, China, in late December 2019 and the outbreaks are still ongoing worldwide [8–15]. The disease can be transmitted from person-to-person through droplets when breathing, coughing or through contact with infected person [16]. During the early phase of the outbreak COVID-19 displayed comparable signs and symptoms with pneumonia, and spread throughout China and later to other part of the world [11, 12]. As of October 17, 2020, there were more than 39 million cases including over 1 million deaths of COVID-19 worldwide [1, 9, 11, 12]. Although COVID-19 displayed similar symptoms to MERS-CoV and SARS-CoV, the severity appears not as high as these two coronaviruses [2–4, 9, 17].

The natural reservoir of COVID-19 and intermediate host that at first spread the virus to humans (zoonotic transmission) have nevertheless now not been established [1, 5, 18]. Recent studies show that some animals such as bats, hedgehogs, pangolins and snakes are suspected to spread the virus to humans [2, 5, 9, 12], after which human-to-human transmissions continues through air droplets or contact with infected individual [2–4, 9, 18, 19]. The most common symptoms of COVID-19 include respiratory disorder, fever, common cold, cough and pneumonia in severe cases [3, 4, 20].

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As stated in [21], the latency period of the COVID-19 is found to be between 2–11 days that ought to be used in suggesting the time for quarantine of the exposed individual. Lately quite a few epidemiological modeling researches [13, 18, 22, 23] estimated the basic reproduction ratio and analysed the patterns of the virus in the early phase of the epidemic. Some of these latest researches [13, 18] found the estimated reproduction number to be extensively larger than unity which is a bit higher than that of MERS and SARS [24, 25]. It by implication shows that greater infections may additionally occur, which suggests that serious measures need to be taken to downsize the unfold of the virus. Zhao et al. [25] revealed that there is a sturdy affiliation between journey through train and the extend in the range of COVID-19 instances than journey via flight or road. Additionally, it is highlighted that ailment prevention and management measures are to be preferred for travelers by means of trains to curtail the COVID-19 spread. Also, Wu et al. [22] studied a meta-population compartmental model, which estimates and forecasts the COVID-19 outbreak, and recommended that the novel coronavirus infection can enlarge exponentially in more than one Chinese city with a lag time behind the Wuhan outbreak of about 1 to 14 days.

Motivated by some of these recent studies [13, 18, 22], in this paper a mathematical model is employed, which incorporates quarantine and hospitalization to explore the dynamical behavior of the COVID-19 transmission, and also to show the trends of its epidemics in order to notify policymakers and to suggest ways to curtail the spread of the virus.

The rest of the paper is organized in the following way: firstly, the model formulation is provided and the model analysis has been presented subsequently. Then, parameter estimation is presented. Afterwards, sensitivity analysis and numerical simulations are performed. Lastly, concluding remarks are given.

Model formulation

A new mathematical model is proposed to monitor the transmission dynamics of the novel corona virus (COVID-19). The total human population at time t, given by $N(t)$, is divided into sub-populations containing susceptible individuals $S(t)$, exposed individuals $E(t)$, individuals with mild infection $I_1(t)$, individuals with severe infection $I_2(t)$, asymptomatic individuals under quarantine $Q(t)$, hospitalized individuals $H(t)$, and recovered individuals $R(t)$, such that $N(t) = S(t) + E(t) + I_1(t) + I_2(t) + Q(t) + H(t) + R(t)$. The diagram of the model is given in Figure 1 to present the transmission between compartments.

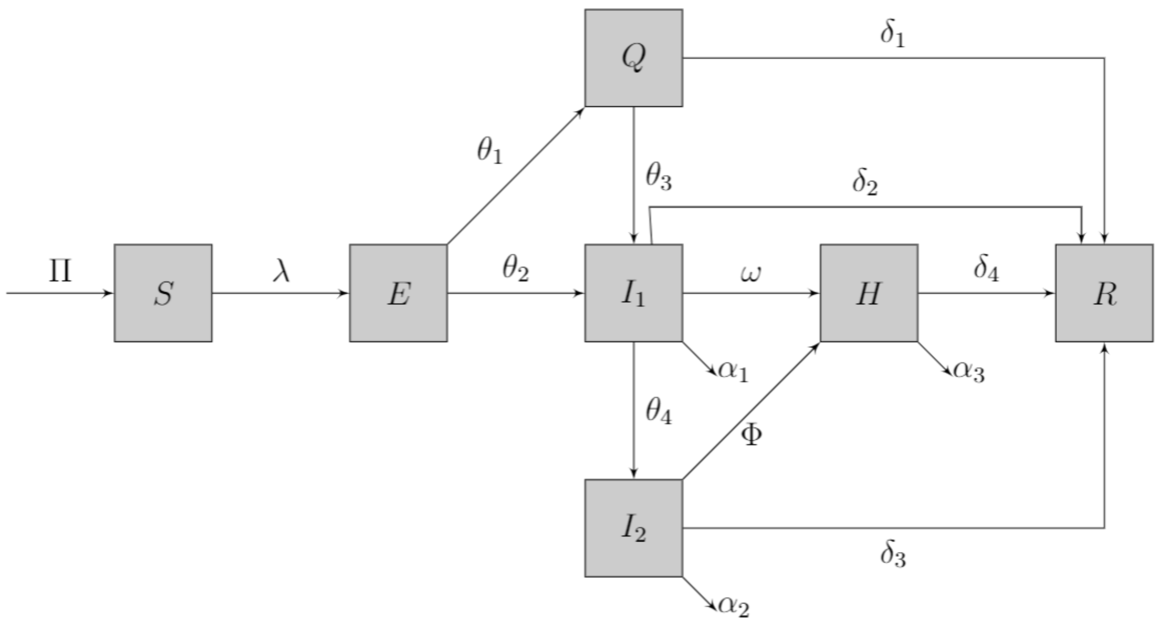

Figure 1. Flow diagram of COVID-19 model.

The system is constructed as follows:

$$\begin{aligned}\frac{dS}{dt} &= \Pi - \lambda S,\\ \frac{dE}{dt} &= \lambda S - (\theta_1 + \theta_2)E,\\ \frac{dQ}{dt} &= \theta_1 E - (\delta_1 + \theta_3)Q,\\ \frac{dI_1}{dt} &= \theta_2 E + \theta_3 Q - (\delta_2 + \omega + \theta_4 + \alpha_1)I_1,\\ \frac{dI_2}{dt} &= \theta_4 I_1 - (\Phi + \alpha_2 + \delta_3)I_2,\\ \frac{dH}{dt} &= \omega I_1 + \Phi I_2 - (\delta_4 + \alpha_3)H,\\ \frac{dR}{dt} &= \delta_1 Q + \delta_2 I_1 + \delta_3 I_2 + \delta_4 H,\end{aligned} \tag{1}$$

where $\lambda = \frac{\beta(\tau_1 I_1 + \tau_2 I_2 + \tau_3 Q + \tau_4 H)}{N}$ is the force of infection. The variables and parameters that are used in the model (1) are explained in Table 1 and Table 2, respectively.

Table 1

Interpretation of the State Variables Used in the Model (1)

Variables	Descriptions
N	Total population of individuals
S	Susceptible individuals at the risk of having COVID-19 infection
E	Exposed individuals
I_1	Infected individuals with mild infection
I_2	Infected individuals with severe infection
Q	Individuals under quarantine/isolated
H	Hospitalized individuals
R	Recovered individuals

Table 2

Interpretation of the State Parameters Used in the Model (1)

Parameters	Descriptions
Π	Recruitment rate
β	Transmission rate
τ_i $(i = 1, 2, 3, 4)$	Parameters for increase/decrease on infectiousness in individuals
θ_i $(i = 1, 2, 3, 4)$	Progression rates
ω	Hospitalization rate from I_1 class
φ	Hospitalization rate from I_2 class
α_i $(i = 1, 2, 3)$	Disease induced death rates
δ_i $(1 = 1, 2, 3, 4)$	Recovery rates

Model analysis

The model is non-negative with respect to the human population, each of its parameters and state variables for each $t \geq 0$. Thus, one can easily prove that for each non-negative initial prerequisite the state variables of the model are non-negative.

Theorem 1. Let $(S, E, Q, I_1, I_2, H, R)$ be the solution to the system (1) with initial conditions $S \geq 0, E \geq 0$, $Q \geq 0, I_1 \geq 0, I_2 \geq 0, H \geq 0, R \geq 0$. Then, the set

$$\Upsilon = \left\{(S, E, Q, I_1, I_2, H, R) \ \in R^7_+ / S + E + Q + I_1 + I_2 + H + R \leq \Pi\right\}$$

is invariant, positive, and all the solutions in R^7_+ stay in Υ with respect to (1).

Proof. Addition of all of the system (1) gives

$$\frac{dN}{dt} = \Pi - \alpha_1 I_1 - \alpha_2 I_2 - \alpha_3 H,$$

so that, $\frac{dN}{dt} \leq \Pi$, and integrating both sides gives $Ne^t \leq \Pi e^t + c$. By the use of theorem of Rota and Birkhoff regarding differential inequalities [26], we can easily obtain $0 \leq N \leq \Pi$ as $t \longrightarrow \infty$. Thus N approaches Π as $t \longrightarrow \infty$, and so the set of the solutions of the model (1) enters the region $\{\Upsilon = (S, E, Q, I_1, I_2, H, R) \in R^7_+ / S \geq \geq 0, E \geq 0, Q \geq 0, I_1 \geq 0, I_2 \geq 0, H \geq 0, R \geq 0, N \leq \Pi\}$, this guarantees the biological feasibility of the model (1).

Hence, it is enough to consider the model's dynamic in Υ [27].

Stability of disease-free equilibrium

The model exhibit a unique disease-free equilibrium point "DFE", which is obtained by equating the right-hand sides of (1) to zero, then

$$C^0 = (S, E, Q, I_1, I_2, H, R) = (\Pi, 0, 0, 0, 0, 0, 0)$$

and it can clearly be seen that C^0 attracts the region, so that

$$C^0 = \left\{(S, E, Q, I_1, I_2, H, R) \in C^0 : E = Q = I_1 = I_2 = H = R = 0\right\}.$$

The basic reproduction number R_0 is computed using the next generation matrix (NGM) method, which represents the number of secondary cases produced by an infected individual with COVID-19 infection throughout his/her entire period of infection in an absolutely susceptible population [27–31], which is given as follows:

$$f = \begin{bmatrix} \frac{\beta(\tau_2 I_1 + \tau_3 I_2 + \tau_4 Q + \tau_5 H)}{N} S_o \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \ v = \begin{bmatrix} (\theta_1 + \theta_2)E \\ -\theta_1 E + (\delta_1 + \theta_3)Q \\ -\theta_2 E - \theta_3 Q + (\delta_2 + \omega + \theta_4 + \alpha_1)I_1 \\ -\theta_4 I_1 + (\varphi + \alpha_2 + \delta_3)I_2 \\ -\omega I_1 - \varphi I_2 + (\delta_4 + \alpha_3)H \end{bmatrix},$$

$$F = \begin{bmatrix} 0 & \beta\tau_1 & \beta\tau_2 & \beta\tau_3 & \beta\tau_4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \ V = \begin{bmatrix} k_1 & 0 & 0 & 0 & 0 \\ -\theta_1 & k_2 & 0 & 0 & 0 \\ -\theta_2 & -\theta_3 & b_1 & 0 & 0 \\ 0 & 0 & -\theta_4 & b_2 & 0 \\ 0 & 0 & -\omega & -\varphi & k_3 \end{bmatrix},$$

where $k_1 = (\theta_1 + \theta_2)$, $k_2 = (\delta_1 + \theta_3)$, $k_3 = (\delta_4 + \alpha_3)$ $b_1 = (\delta_2 + \omega + \theta_4 + \alpha_1)$, $b_2 = (\varphi + \alpha_2 + \delta_3)$. Then V^{-1} is obtained as

$$V^{-1} = \begin{bmatrix} {k_1}^{-1} & 0 & 0 & 0 & 0 \\ \frac{\theta_1}{k_2 k_1} & {k_2}^{-1} & 0 & 0 & 0 \\ \frac{\theta_2 k_2 + \theta_3 \theta_1}{b_1 k_2 k_1} & \frac{\theta_3}{b_1 k_2} & {b_1}^{-1} & 0 & 0 \\ \frac{\theta_4(\theta_2 k_2 + \theta_3 \theta_1)}{b_2 b_1 k_2 k_1} & \frac{\theta_4 \theta_3}{b_2 b_1 k_2} & \frac{\theta_4}{b_2 b_1} & {b_2}^{-1} & 0 \\ \frac{(\theta_2 k_2 + \theta_3 \theta_1)(\omega\, b_2 + \varphi\, \theta_4)}{k_1 k_2 b_1 b_2 k_3} & \frac{\theta_3(\omega\, b_2 + \varphi\, \theta_4)}{b_2 b_1 k_2 k_3} & \frac{\omega\, b_2 + \varphi\, \theta_4}{b_2 b_1 k_3} & \frac{\varphi}{b_2 k_3} & {k_3}^{-1} \end{bmatrix}.$$

Thus, the basic reproduction number $R_0 = \rho(FV^{-1})$ equals to

$$R_0 = \frac{\left(\left(\left(b_1\beta\tau_1 + \beta\tau_2\theta_3\right)\theta_1 + \beta\tau_2 k_2\theta_2\right)k_3 + \omega\,\beta\tau_4\left(\theta_2 k_2 + \theta_3\theta_1\right)\right)b_2 + \theta_4\left(\beta\tau_4\varphi + \beta\tau_3 k_3\right)\left(\theta_2 k_2 + \theta_3\theta_1\right)}{k_1 k_2 k_3 b_1 b_2},$$

with $S_0 = N$ and ρ symbolizing the next generation matrix's spectral radius. Following [27], relying on the local stability of the disease-free equilibrium of the model (1), the theorem below is arrived.

Theorem 2. For the model (1) the disease-free equilibrium is locally asymptotically stable whenever $R_0 < 1$ and unstable if $R_0 > 1$.

Proof. The Jacobian matrix evaluated at $\mathcal{C}^0$, denoted by J_0 is

$$J(C^0) = \begin{pmatrix} -\lambda & 0 & 0 & 0 & 0 & 0 & 0 \\ \lambda & -\theta_1-\theta_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & \theta_1 & -\delta_1-\theta_3 & 0 & 0 & 0 & 0 \\ 0 & \theta_2 & \theta_3 & -\delta_2-\omega-\theta_4-\alpha_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \theta_4 & -\varphi-\alpha_2-\delta_3 & 0 & 0 \\ 0 & 0 & 0 & \omega & \varphi & -\delta_4-\alpha_3 & 0 \\ 0 & 0 & \delta_1 & \delta_2 & \delta_3 & \delta_4 & 0 \end{pmatrix}.$$

Then, the eigenvalues of this matrix are $0, -\theta_1-\theta_2, -\delta_1-\theta_3, -\delta_4-\alpha_3, -\varphi-\alpha_2-\delta_3, -\delta_2-\omega-\theta_4-\alpha_1$, and $-\lambda$, which are obtained by deleting the first row and the first column of J_0 as well as its last row and the last column. Thus, the DFE, C^0 is locally asymptotically stable if $R_0 < 1$ and unstable if $R_0 > 1$. Hence, the proof is complete.

Parameter estimation

This section explains the fitting of parameters involved in the proposed COVID-19 model based upon the real cases of the pandemic throughout Turkey. Daily cases of the pandemic are taken between March 11–October 10, 2020, while preparing this research paper. For initial conditions the total population of Turkey is noted to be $N(0) = 83.3 \times 10^6$, the initial exposed and quarantined population is taken as $E(0) = Q(0) = 3 \times 10^6$ and this helped us to determine rest of the initial values for the state variables using the relation $N(0) = S(0) + E(0) + Q(0) + I_1(0) + I_2(0) + H(0) + R(0)$. In this connection, $S(0) = 78338132$, $I_1(0) = 300$, $I_2(0) = 470$, $H(0) = 150$, and $R(0) = 15$ are obtained. There are 19 biological parameters which have been estimated with the aid of least-square fitting method leading to produce a best fit of the COVID-19 model's solution to the real pandemic cases as depicted in the Figure 2. By reducing the average absolute relative error between the real COVID-19 cases and the solution of the model, the best values of the biological parameters are obtained. The objective function yields to relatively small error having the value 9.8748×10^{-2}. The Figure 2 shows the real COVID-19 cases by red solid squares whereas the best fitted curve of the model is shown by the black solid line. The biological parameters included in the model are listed in Table 3 along with their best estimated values obtained via least-squares technique. These parameters have finally produced the value of the basic reproduction number equivalent to $\mathcal{R}_0 = 2.82$ for the real COVID-19 cases in Turkey from March 11 to October 10, 2020.

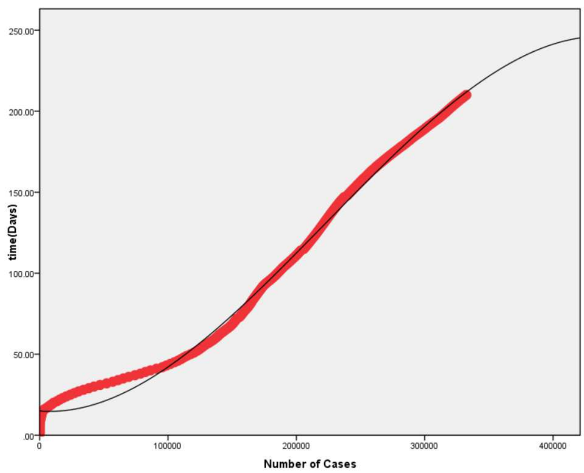

Figure 2. Data fitting for the real COVID-19 cases in Turkey from March 11 to October 10, 2020

Sensitivity analysis

Table 3

Baseline Values of the Parameters Used in (1)

Parameter	Value	Units/Remarks	Sources
$N(0)$	Turkey	Constant	[32]
$S(0)$	$0.95 \times N(0)$	Constant	Assumed
Π	9.9991	Day^{-1}	Fitted
β	0.641	Day^{-1}	Assumed
τ_1	0.647	Day^{-1}	Assumed
τ_2	0.456	Day^{-1}	Assumed
τ_3	0.567	Day^{-1}	Assumed
τ_4	0.334	Day^{-1}	Assumed
α_1	0.0378	Day^{-1}	Fitted
α_2	0.0.0324	Day^{-1}	Fitted
α_3	0.0289	Day^{-1}	Fitted
Φ	0.00213	Day^{-1}	Estimated by [32]
ω	0.00004	Day^{-1}	Estimated by [32]
δ_1	0.24	Day^{-1}	Fitted
δ_2	0.133	Day^{-1}	Fitted
δ_3	0.00389	Day^{-1}	Fitted
δ_4	0.00527	Day^{-1}	Fitted
θ_1	0.632	Day^{-1}	Fitted
θ_2	0.0034	Day^{-1}	Fitted
θ_3	0.0023	Day^{-1}	Fitted
θ_4	0.00512	Day^{-1}	Fitted

Since parameters of an epidemiological system are either evaluated or fitted along these lines are conveying some vulnerability with respect to their qualities utilized for reaching conclusions about the fundamental pestilence. Consequently, it is essential to survey the singular effects of every parameter on the dynamics of the pestilence, subsequently finding the parameters with the most significant impact towards decrease or diminishing the scourge. In the current study we employed a partial rank correlation coefficients (PRCCs) of the basic reproduction number and attack rate to investigate the most significant parameters for curbing the dissemination of COVID-19 in a community. Here, the most effective parameters are recovery rate from isolated people δ_1, recovery rate from hospitalized people δ_4, and transmission rate (β) [33–37].

Furthermore, since the basic reproductive number R_0 is the most important quantity to comprehend the extent for the spread of an epidemic, R_0 has been investigated by varying different kinds of biological parameters of the proposed COVID-19 model. Using mesh plot and the parameter values in Table 3, some numerical results are obtained. The results as depicted in Figure 4 showed a significant increase with the variation in the progression rates of asymptomatic individuals under quarantine to the mild infection θ_3 and that of mild infection individuals to the severe infection θ_4, while R_0 decreases/increases with the decreasing/increasing value of transmission rate β and the rate of increase of infectiousness in human τ_4 and increases with decrease of recovery rates from isolated people δ_1, mild infection individuals δ_2, severe infection individuals δ_3, and hospitalized individuals δ_4.

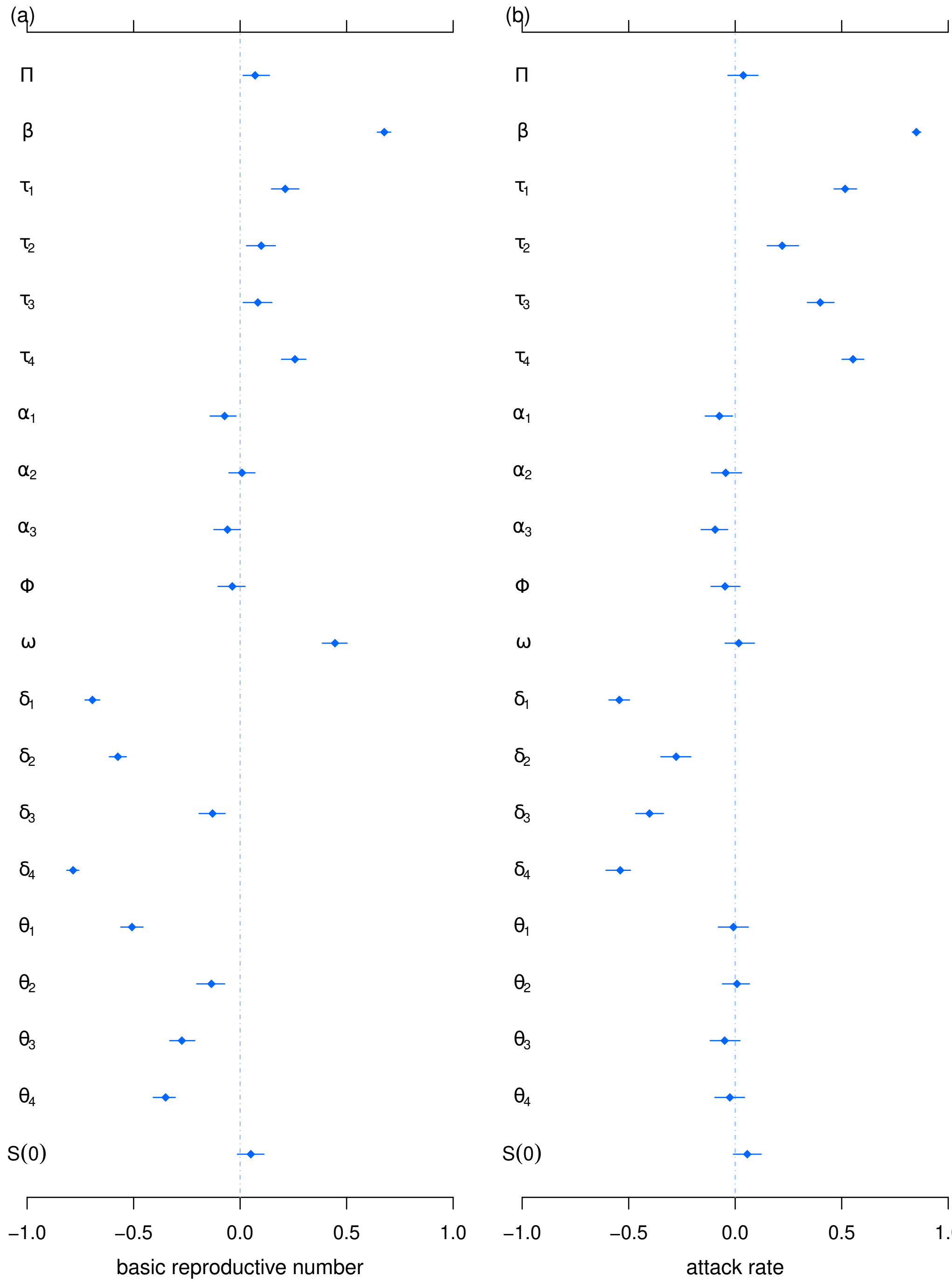

Figure 3. The partial rank correlation coefficient of the basic reproduction number R_0 with respect to model parameters. The dots are the estimated correlation and the bars represent the 95% confidence interval. The parameter values used for sensitivity analysis are summarised in Table 3

Numerical simulations

This is the position in which we got a deep insight into the complex behavior of the model. The present section provided the model's numerical simulations while using the biological parameters as previously mentioned. The Euler technique is used to get the solution of the proposed model and to obtain the graphical results based on parameters that are taken in Table 3.

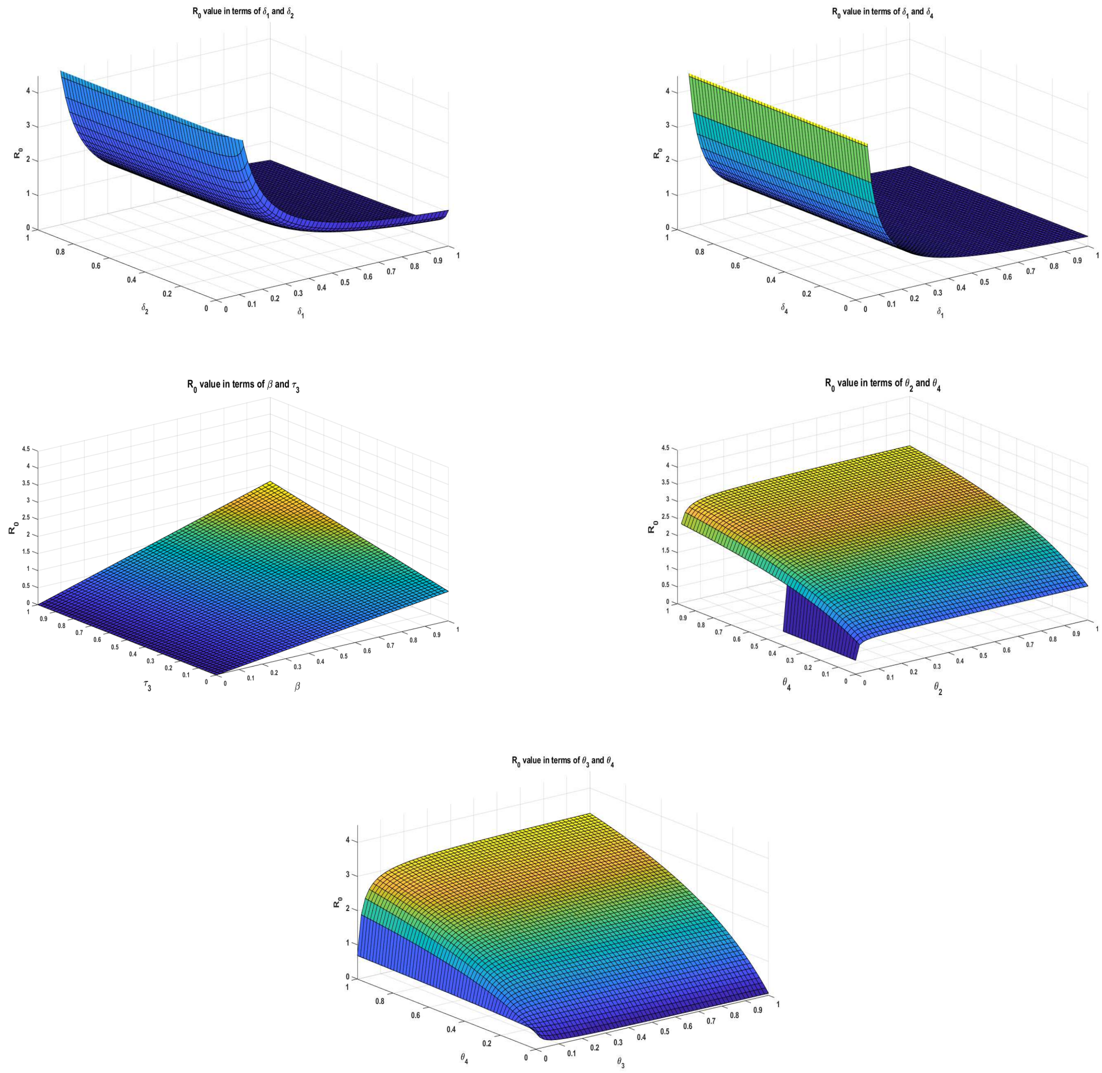

Figure 4. Mesh grid plots of the basic reproduction number in terms of the controllable parameters with basic reproduction number R_0 as a response function

In the absence of an exact solution for the proposed model we need to establish an approximate solutions to show the behaviour of the model. With this purpose we employ one of the effective numerical scheme called Euler method. The method is as follows: assume that a well-posed initial-value condition is given by

$$\frac{dy}{dt} = f(t, y),\ a \leq t \leq b \text{ and } y(a) = \chi.$$

A sequence of approximation point $(t, w) \approx (t, y(t))$ is established by Euler method to the exact solutions of ODE by $t_{i+1} = t_i + h$ and $w_{i+1} = w_i + hf(t_i, w_i), i = 0, 1, \ldots, N-1$, and $t_0 = a, w_0 = \alpha, h = \frac{b-a}{N}$.

The following figures are obtained by using the MATLAB version R2020a and the parameter values from Table 3. From the Figure 5 it is observed that there is a significant decrease in both the compartments of exposed individuals, isolated/quarantined individuals, infected individuals with mild infection, individuals with severe infections and hospitalized individuals while the susceptible and recovery compartments increases. These signified that the estimated parameter values taking from Table 3 gives the required results in controlling the spread of COVID-19 infection. The results depicted in Figures 6, 7, 8, and 9 with the decreasing/increasing values of λ (force of infection) showed that adequate awareness of social distancing and wearing of face masks in most vulnerable communities play significant role in the spread of the COVID-19 infection.

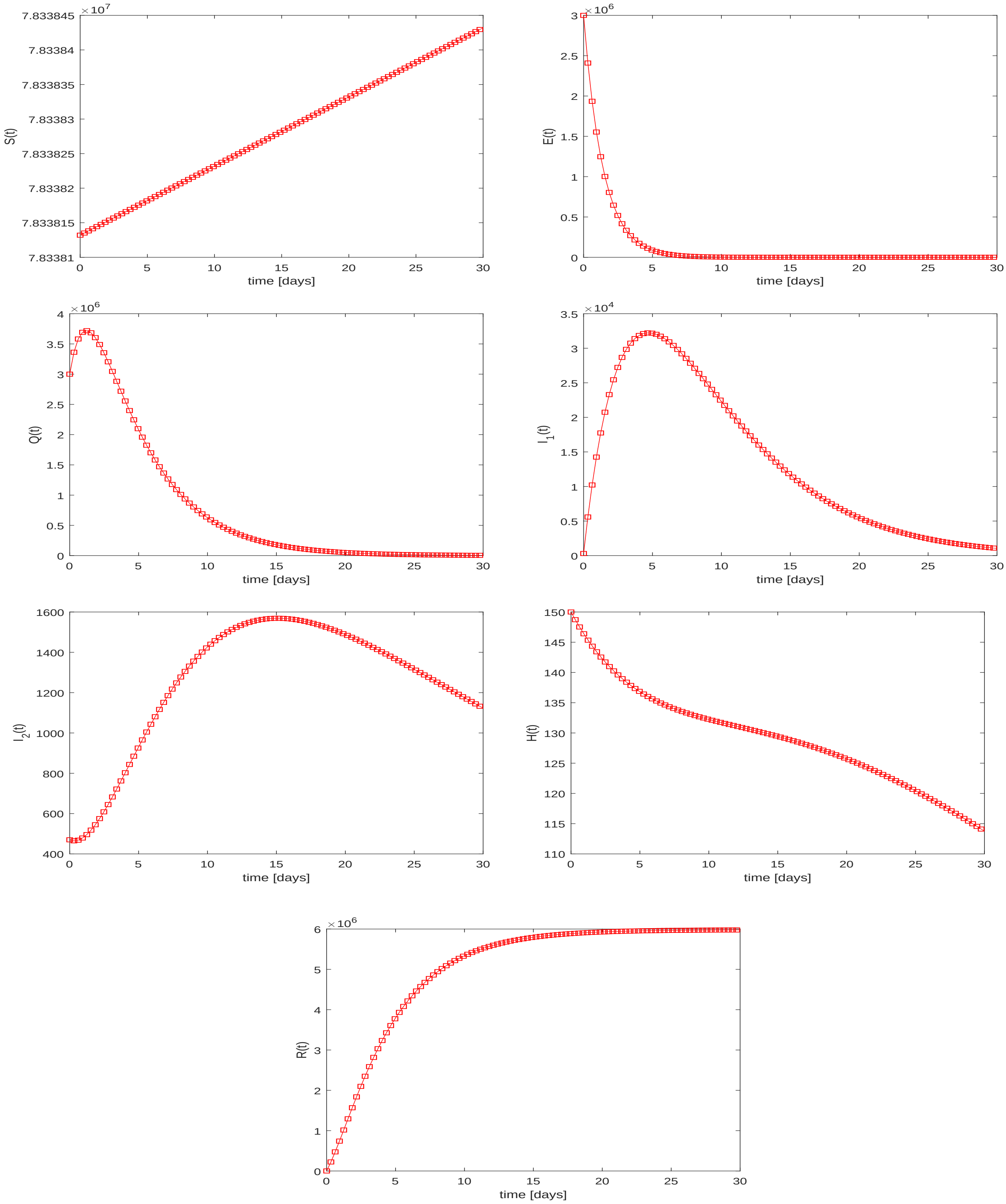

Figure 5. Dynamical behavior of each state variables of the proposed model (1) while taking parameters' values from the Table 3

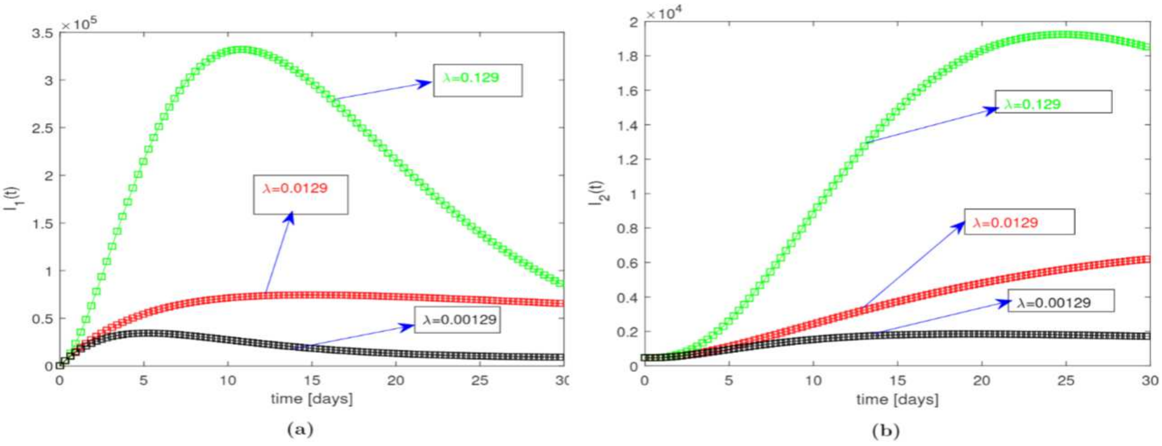

Figure 6. (a) Profile of I_1 (individuals with mild infection) and (b) profile of I_2 (individuals with severe infection), with decreasing values of λ (force of infection).

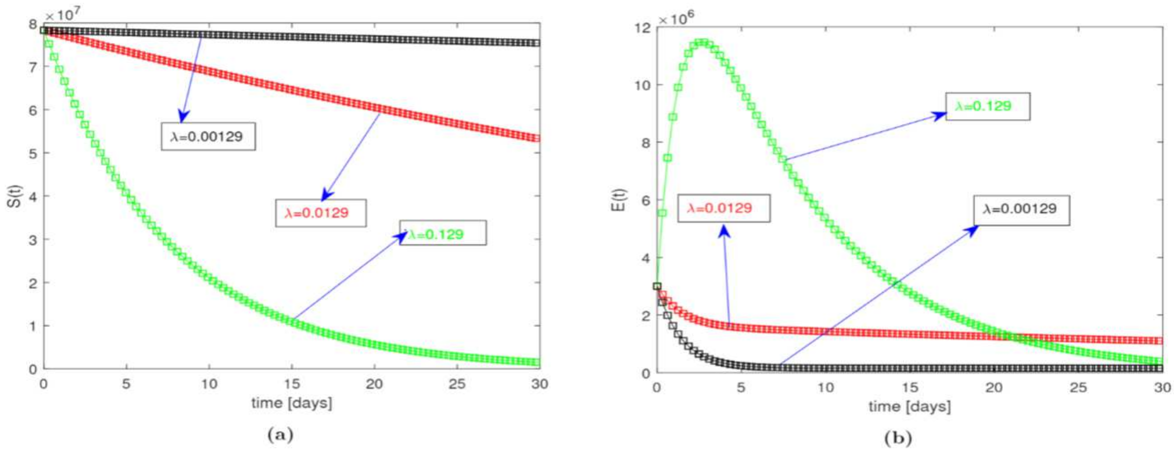

Figure 7. (a) Profile of S (susceptible individuals) and (b) profile of E (exposed individuals), with decreasing values of λ (force of infection).

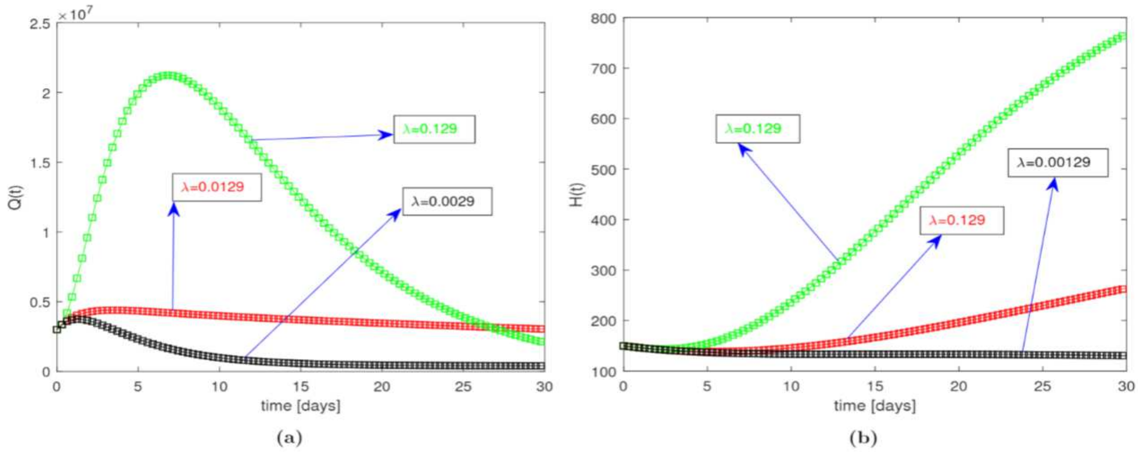

Figure 8. (a) Profile of Q (quarantined individuals) and (b) profile of H (hospitalized individuals), with increasing values of λ (force of infection)

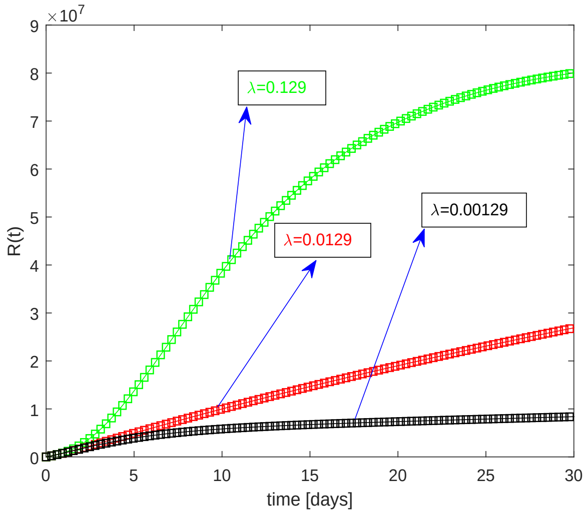

Figure 9. Profile of R (recovered individuals) with increasing values of λ (force of infection).

Conclusion

In this paper a mathematical model is proposed, which incorporates quarantine and hospitalization to study the dynamical behavior of the COVID-19 transmission. The parameters of the model are estimated and fitted to the model with the use of laboratory confirmed COVID-19 data cases of Turkey from March 11 to October 10, 2020, using least-square fitting method. The threshold quantity known as basic reproduction number is obtained by using the next generation matrix techniques. Some simulation results are obtained with the aid of mesh plots for the reproductive number as a function of two different biological parameters. Using partial rank correlation coefficients of the basic reproduction number and infection attack rate as a response functions, we revealed the most essential parameters for effectively controlling the COVID-19 infection. It is found that the epidemiological parameters that should be given emphasis in controlling the spread of COVID-19 are the recovery rate from quarantine individuals δ_1, recovery rate from hospitalized individuals δ_4 and transmission rate (β). Finally, numerical simulations on the dynamics of the model showed that the infections in the compartments of each state variables decreases with time which causes an increase in susceptible individuals. This implies that avoiding contact with infected individuals by means of adequate awareness of social distancing and wearing of face mask are vital to prevent or reduce the spread of COVID-19 infection.

Furthermore, it should also be emphasized that the present research study will be strengthened in future research by analyzing and investigating the modern fractional operators and optimal control strategies. To the unknown characters and characteristics of this pandemic of COVID-19 this is a significant and decisive step remaining to be accomplished.

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COVID-19-дың берілу динамикасы мен бақылау стратегиясы: модельді зерттеу

Бұл зерттеуде халықтың осал топтарының масканы тағуы және әлеуметтік ара қашықтықты сақтаудың әсерін бағалау, пациенттерді карантин мен ауруханаға жатқызуды қамтитын математикалық модель ұсынылған. Модель параметрлері Түркияда 2020 жылдың 11 наурызынан 10 қазанына дейін зертханалық расталған COVID-2019 жағдайларын қолдана отырып бағаланды және модельге бейімделді. Дәрежелік корреляцияның ішінара коэффициенті модельдің сезімталдығын негізгі көбею санымен және жауап беру функциясы ретінде инфекция жылдамдығымен талдау үшін қолданылды. Сезімталдықты талдау нәтижелері COVID-19 инфекциясын тиімді бақылаудың маңызды параметрлері карантиндегі адамдардың қалпына келу жылдамдығы (δ_1), ауруханаға жатқызылған адамдардың қалпына келу жылдамдығы (δ_4) және жұқпаның берілу жылдамдығы (β) екенін көрсетті. Модельдеудің кейбір нәтижелері модельден кездейсоқ таңдалған екі түрлі биологиялық параметрлердің функциясы ретінде негізгі репродуктивті санға қатысты торлы графиктер арқылы алынады. Соңында, модель динамикасын сандық модельдеу әр айнымалы бөлімдеріндегі инфекциялар саны уақыт өте келе азайып, ауруға шалдыққан адамдардың көбеюіне әкелетінін көрсетті. Бұл індет жұқтырған адамдардан аулақ болу, әлеуметтік арақашықтықты сақтау, маска кию және т.б. талаптарды орындау COVID-19 инфекциясының таралуын болдырмау немесе азайту үшін өте маңызды екенін білдіреді.

Кілт сөздер: COVID-19, математикалық модельдеу, базалық репродуктивті сан, берілу динамикасы, сезімталдықты талдау.

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Динамика передачи и стратегии контроля COVID-19: модельное исследование

В статье предложена математическая модель, которая включает карантин и госпитализацию пациентов, чтобы оценить влияние социального дистанцирования и ношение маски среди уязвимых групп населения. Параметры модели оценивались и подгонялись к модели с использованием лабораторно подтвержденных случаев COVID-19 в Турции с 11 марта по 10 октября 2020 г. Частичный коэффициент ранговой корреляции использован для проведения анализа чувствительности модели с базовым числом репродукции и скорости заражения как функции ответа. Результаты анализа чувствительности показывают, что наиболее важными параметрами для эффективного контроля за инфекцией COVID-19 являются скорость выздоровления лиц, находящихся на карантине (δ_1), скорость выздоровления госпитализированных лиц (δ_4) и скорость передачи инфекции (β). Некоторые результаты моделирования получены с помощью сеточных графиков относительно основного репродуктивного числа как функции двух различных биологических параметров, случайно выбранных из модели. Наконец, численное моделирование динамики модели показало, что количество инфекций из отделов каждой переменной состояния уменьшается со временем, что вызывает увеличение числа восприимчивых людей. Это означает, что избегание контактов с инфицированными людьми посредством адекватного понимания социального дистанцирования и ношения лицевых масок жизненно важно для предотвращения или уменьшения распространения инфекции COVID-19.

Ключевые слова: COVID-19, математическое моделирование, базовое репродуктивное число, динамика передачи, анализ чувствительности.

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Modelling the effect of horizontal and vertical transmissions of HIV infection with efficient control strategies

In this paper a mathematical model is developed to study the transmission dynamics of HIV infection and the effect of horizontal and vertical transmission in Turkey is analyzed. Model is fitted with the use of confirmed HIV cases of both vertical and horizontal transmission from 2011 to 2018. Using the next generation operator the basic reproduction number of the model is obtained, which shows whether the disease persists or dies out in time. Further analysis shows that the model is locally asymptotically stable when the basic reproduction number $\mathcal{R}_0 < 1$ and is unstable when $\mathcal{R}_0 > 1$. The most sensitive parameters efficient for the control of the infection are obtained using forward normalized sensitivity index. Lastly, the results are obtained with the aid of mesh and contour plots, which show that decreasing the values of transmission rate diseases induced mortality rates and progression rates play a significant role in controlling the spread of HIV transmission.

Keywords: HIV, mathematical modelling, control strategies, sensitivity analysis.

Introduction

Human Immune-deficiency Virus (HIV) reduces or destroys the human defense mechanisms, also known as the immune system, preventing fighting with infections or any other diseases and the progression of this virus occurres as a result of infecting the CD4+ T-cells of the organism [1, 2]. The number of these cells mainly shows how active and functioning the immune system is [3, 4]. The number of CD4+ T-cells must be in the range of 800 to 1200 cells/mm^3 for a healthy person. If this number of CD4+ T-cells goes down below 200 cells/mm^3 for any HIV patient, this patient is then considered to be an AIDS patient [5]. In other words, HIV is the virus that causes AIDS (Acquired Immune Deficiency Syndrome) which is the most advanced phase of the HIV infection [6].

HIV can be transmitted through direct contact with contaminated blood products, such as syringes or needles, contaminated transfusion, unprotected sexual intercourse, and breastfeeding or as a vertical transmission during birth [7]. However, not all HIV cases necessarily result in AIDS infection. It is clinically confirmed that an HIV patient may live a healthy life without progressing to severe stage (AIDS) [8].

HIV/AIDS was first discovered in the United States of America in the early 1980s in two homosexual men and it continues to progress with time [9]. 2003 was the year with the greatest number increase in an epidemic, where approximately 5 million additional infected individuals were discovered, which raised the global prevalence of the virus to 38 million people living with HIV/AIDS, and in the same year approximately 3 million patients passed away [10]. This virus happened to be the death cause of almost 25 million people as of 2005 and became one of the most devastative epidemics in history [11]. According to the statistics taken from the World Health Organization (WHO), in 2013 2.1 million people were infected and approximately 1.5 million people died because of AIDS [12]. Furthermore, in 2014 it was reported that the number of people that were living with HIV was 35 million [6].

According to the data taken from WHO, 2.3 million children were living with HIV and about 380,000 children passed away because of HIV in 2005 and approximately 2.1 million children were living with HIV/AIDS in 2007. In 2015, with 150000 newly infected children, 1.8 million children were living with HIV according to the UNAIDS

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and 110000 children died because of AIDS-related diseases [13]. This data shows that AIDS has become one of the major death causes. Each day about 1500 children get newly infected [14].

Several mathematical models have been developed and used to gain insight into the transmission dynamics of HIV in human population (see, for instance, [2, 6, 11, 13, 15] and some of the references therein). However, none of these studied the dynamics of HIV transmission with effect of both vertical and horizontal transmission. The purpose of the current study is to design and analyse a new realistic model (which extends some of the aforementioned studies in the literature) for HIV transmission dynamics.

This paper is organized as follows. The epidemic model is developed and analyzed in sections 2 and 3, respectively. Model fitting is presented in section 4. Section 5 contains sensitivity analysis and numerical simulation while section 6 presents the conclusions.

Model formulation

In this section a mathematical model is proposed to monitor the dynamics of both vertical and horizontal transmissions of HIV infections at time t. The total population $N(t)$ is divided into four different classes; susceptible adults, $S(t)$, infected adults, $I(t)$, newborn children with no HIV infection, $C(t)$, and newborn children with HIV infection, $I_c(t)$. That is, $N(t) = S(t) + I(t) + C(t) + I_c(t)$. Flow diagram of the model is presented in Fig. 1.

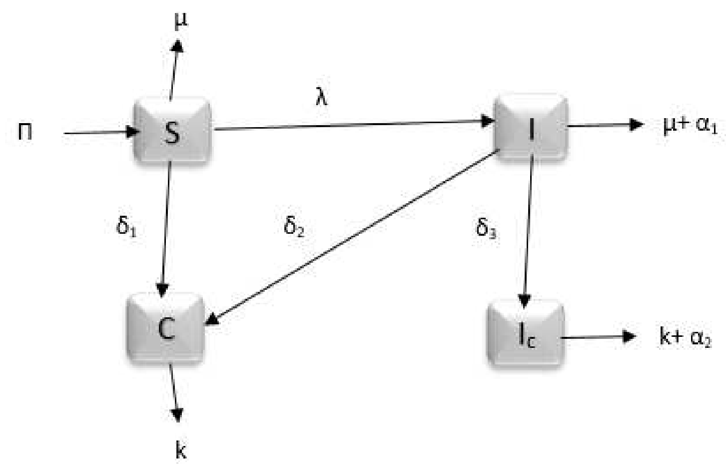

Figure 1. Flow diagram of the model.

By using the constructed model, system of ODE's is obtained as

$$\begin{aligned} \frac{dS}{dt} &= \Pi - \lambda S - (\delta_1 + \mu)S, \\ \frac{dI}{dt} &= \lambda S - (\mu + \alpha_1 + \delta_2 + \delta_3)I, \\ \frac{dC}{dt} &= \delta_1 S + \delta_2 I - kC, \\ \frac{dI_c}{dt} &= \delta_3 I - (k + \alpha_2)I_c, \end{aligned} \tag{1}$$

where $\lambda = \frac{\beta I}{N}$ is the force of infection.

Table 1

Interpretation of the State Variables Used in the Model (1).

Variables	Descriptions
N	Total human population
S	Susceptible adults (both male and female)
I	Infected adults with HIV (both male and female)
C	Newborn children
I_c	Newborn children with HIV infection

Table 2

Interpretation of the State Parameters Used in the Model (1).

Parameters	Descriptions
ϕ	Recruitment rate of both adults and new born children
β	Transmission or successful contact rate
α_1	HIV induced mortality rate of adults
α_2	HIV induced mortality rate of newborn children
μ	Natural death for adults
k	Natural death for newborn children
δ_j $(j = 1, 2, 3)$	Progression rates

Fundamental properties of the model

This section will highlight the quantitative analysis of HIV model (1) and briefly explain the relationship between the horizontal and vertical transmission dynamics. The persistence or elimination of HIV, which is determined by the threshold parameters, are studied. Thus, at first, the positivity and boundedness of the solutions of the model are verified for $t \geq 0$, and then the invariant region is studied.

Positivity of the solutions and boundedness

In this study to say that the model (1) is epidemiologically meaningful we need to verify the positivity of all the state variables of the model at $t > 0$. This means that every solution of the system (1) together with the positive initial conditions shall remain positive at any time $t > 0$.

Theorem 1. Suppose that we have initial data $S(0) > 0$, $I(0) > 0$, $C(0) > 0$, $I_c(0) > 0$. Then, the solutions of the model (S, I, C, I_c) are positive for all time $t > 0$.

Proof. It can easily be seen from the first equation of system (1) that

$$\begin{aligned} \frac{dS}{dt} =& \Pi - [\lambda(t) + \delta_1 + \mu]S \\ & \geq -[\lambda(t) + \delta_1 + \mu]S(t). \end{aligned}$$

Applying integrating factor method to the obtained inequality it is found that

$$S(t) \geq S_0 e^{-\int_0^t (\lambda(u)+\delta_1+\mu)du} \geq 0.$$

By using the equations given in (1) and applying the same method to the equations it can be easily seen that $I(t) \geq 0$, $C(t) \geq 0$ and $I_c(t) \geq 0$ whenever $t > 0$.

The invariant region

To obtain the region the following theorem is considered.

Theorem 2. The solutions of the system (1) are said to be feasible for all $t \geq 0$ whenever they enter the invariant region Ω. That is,

$$\Omega = \left\{ (S, I, C, I_c) \in R_+^4 : S + I + C + I_c \leq \frac{\Pi}{\mu} \right\}, \text{ where } N = S + I + C + I_c.$$

Proof. Let $\Omega = \left\{(S, I, C, I_c) \in R_+^4 : S + I + C + I_c \leq \frac{\Pi}{\mu}\right\}$ be the solutions of the system and assume that initial conditions are all non-negative. Then, the sum of equations of the system (1) gives

$$\frac{dN}{dt} = \Pi - \mu S - (\mu + \alpha_1)I - kC - (k + \alpha_2)I_c.$$

From the above equation it is clear that $\frac{dN}{dt} \leq \Pi$ and integrating both sides it is obtained that $Ne^t \leq \Pi e^t + c$, for some arbitrary constant c. With the use of Rota and Birkhoff [16] it can be seen that $0 \leq N \leq \frac{\Pi}{\mu}$ as $t \to \infty$.

This reveals that all the solutions together with the initial conditions in Ω stay inside the region for all cases when $t > 0$ (i.e., the set happen to be positively invariant). It is consequently adequate enough to study the dynamics of the generated flow by system (1) within the region Ω, which guarantees the mathematical and epidemiological well-posedness of the model [2, 15, 17].

Disease-free equilibrium (DFE) and local stability

Let $\chi^0 = (S_0, I_0, C_0, I_{c,0})$ be the disease-free equilibrium (DFE) of the model (1). DFE exists when the disease dies out. So, at this point there is no infection and hence, no infected individuals, i.e., $I_0(t) = I_{c,0}(t) = 0$. Here, it is enough to show χ^0 attraction on the region

$$\chi^0 = \left\{(S_0, I_0, C_0, I_{c,0}) \in \chi^0 : I_0 = I_{c,0} = 0\right\}.$$

S_0 and C_0 are obtained by equating the right hand side of the first and third equations in the system (1), and plugging 0 instead of I_0 and $I_{c,0}$. Therefore,

$$S_0 = \frac{\Pi}{\delta_1 + \mu}$$

and

$$C_0 = \frac{\delta_1 S_0}{k} = \frac{\Pi\delta_1}{k(\delta_1 + \mu)}.$$

The DFE point of the constructed system is

$$\chi^0 = \left(\frac{\Pi}{\delta_1 + \mu}, I_0, \frac{\Pi\delta_1}{k(\delta_1 + \mu)}, I_{c,0}\right).$$

Using the next generation matrix method [18], the basic reproduction number of the HIV model (1) (denoted by $\mathcal{R}_0 = \rho(FV^{-1})$, ρ is the spectral radius of the next generation matrix, FV^{-1}) is obtained, where F stands for the matrix of new infection terms and V stands for the matrix containing the remaining transition terms of the model. Thus,

$$f = \begin{bmatrix} \frac{\beta I}{N} S_o \\ 0 \\ 0 \end{bmatrix}, \quad v = \begin{bmatrix} (\alpha_1 + \delta_2 + \delta_3 + \mu)I \\ -\delta_2 I + kC \\ -\delta_3 I + (\alpha_2 + k)I_c \end{bmatrix},$$

$$F = \begin{bmatrix} \beta & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad V = \begin{bmatrix} \alpha_1 + \delta_2 + \delta_3 + \mu & 0 & 0 \\ -\delta_2 & k & 0 \\ -\delta_3 & 0 & k + \alpha_2 \end{bmatrix}.$$

Then, V^{-1} is obtained as

$$V^{-1} = \begin{bmatrix} (\alpha_1 + \delta_2 + \delta_3 + \mu)^{-1} & 0 & 0 \\ \frac{\delta_2}{(\alpha_1+\delta_2+\delta_3+\mu)k} & k^{-1} & 0 \\ \frac{\delta_3}{(\alpha_1+\delta_2+\delta_3+\mu)(\alpha_2+k)} & 0 & (\alpha_2 + k)^{-1} \end{bmatrix} \text{ and } FV^{-1} = \begin{bmatrix} \frac{\beta}{\alpha_1+\delta_2+\delta_3+\mu} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus, $\mathcal{R}_0 = \rho(FV^{-1})$ the basic reproduction number is given by

$$R_0 = \frac{\beta}{\alpha_1 + \delta_2 + \delta_3 + \mu}.$$

Endemic equilibrium

The endemic equilibrium (EE) of the model exists only when $I \neq 0$, $C \neq 0$, and $I_c \neq 0$. This means that there is a persistence of the HIV infection in the populace, and it is denoted by $\chi^* = (S^*, I^*, C^*, I_c^*)$: $(S^*, I^*, C^*, I_c^*) > 0$. Thus, the endemic equilibrium point is derived by solving the system (1) in terms of $(\lambda) = \beta\frac{I}{N}$, where (λ) is the force of infection. Then,

$$\begin{aligned}
S^* &= \frac{\pi}{\lambda + \mu + \delta_1}, \\
I^* &= \frac{\lambda\pi}{(\mu + \delta_3 + \delta_2 + \alpha_1)(\lambda + \mu + \delta_1)}, \\
C^* &= \frac{\pi(\lambda\delta_2 + \mu\delta_1 + \alpha_1\delta_1 + \delta_1\delta_2 + \delta_1\delta_3)}{k(\mu + \delta_3 + \delta_2 + \alpha_1)(\lambda + \mu + \delta_1)}, \\
I_c^* &= \frac{\delta_3\lambda\pi}{(\mu + \delta_3 + \delta_2 + \alpha_1)(k + \alpha_2)(\lambda + \mu + \delta_1)}.
\end{aligned}$$

Model fitting

This section explains the fitting of parameters involved in the proposed HIV model based upon the real cases of HIV (CD4+) in Turkey for both vertical and horizontal cases. Yearly cases are taken from 2011 to 2018 while preparing this research paper. The objective function yields to relatively small error value $9 * 10^{-6}$. The Fig. 2 shows the real HIV (CD+4) cases by black cycles whereas the best fitted curve of the model is shown by the black solid line. The biological parameters included in the model are listed in Table 3 along with their best estimated values obtained via least-squares technique. These parameters have finally produced the value of the basic reproduction number equivalent to $\mathcal{R}_0 = 1.23$.

Table 3

Values of the Parameters of the Proposed HIV Model

Parameter	Values	Source
Π	35	Estimated
β	0.0071	Estimated
α_1	0.000129	Fitted
α_2	0.000234	Fitted
μ	0.0052	[2]
k	0.0092	[13]
δ_1	0.00011	Fitted
δ_2	0.00000011	Fitted
δ_3	0.00044	Fitted

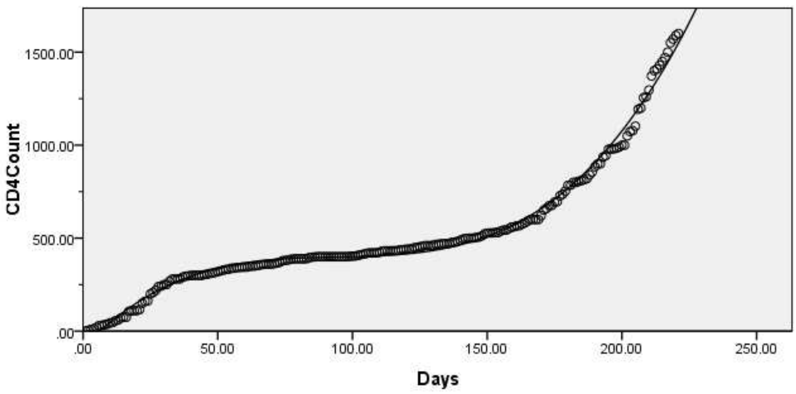

Figure 2. Data fitting for the real cases of TB (CD4+) in Turkey for both vertical and horizontal cases from 2011 to 2018

Sensitivity analysis

In this section the local sensitivity analysis method is used to outline the sensitivity of the basic reproduction number $\mathcal{R}_0$ to certain key associated parameters of the proposed HIV model. The basic reproduction number was obtained and described as a parameter-dependent output of the model and the severity indicator of the HIV infection, the main way of curtailing and spreading the HIV infection in the population is to lower this reproduction number below unity.

Therefore it became crucially important to investigate the relationship between the parameters of the model and the basic reproduction number. Our main concern here is to explain the sensitivity of the basic reproduction number with respect to the significant parameters used in the model. The set of input parameters relative to $\mathcal{R}_0$ is

$$\sigma = \{\beta, \mu, \delta_1, \delta_2, \delta_3, \alpha_1\}.$$

Typically, if a model has different parameters, variations in parameters might not always influence the outcome due to variance in the sensitivity of the parameters, those with positive sign are considered as highly and proportionally sensitive for increasing the value of $\mathcal{R}_0$ while those with negative sign are sensitive for the decrease of $\mathcal{R}_0$ value and the other category are neutrally sensitive (with zero relative sensitivity) [19, 20]. We denote by $\Omega_\gamma^{R_0}$ the normalized local sensitivity index of the output R_0 with respect to a parameter (γ), where $\gamma \in \sigma$, and it is defined as [21–23]

$$\dot{\Psi}_\gamma = \Omega_\gamma^{R_0} = \frac{\gamma}{R_0}\frac{\partial R_0}{\partial \gamma} = \frac{\partial ln(R_0)}{\partial ln(\gamma)}.$$

Using the above definition, the following indices shown in Table 4 are computed for the output $\mathcal{R}_0$ with respect to every parameter presented in Table 4.

Table 4

Forward Normalized Sensitivity Indices

Parameter	Elasticity Indices	Values of the Elasticity Indices
β	$\dot{\Omega}_\beta^{R_0}$	1.000
μ	$\dot{\Omega}_\mu^{R_0}$	-0.002
δ_2	$\dot{\Omega}_{\delta_2}^{R_0}$	-0.285
δ_3	$\dot{\Omega}_{\delta_3}^{R_0}$	-0.585
α_1	$\dot{\Omega}_{\alpha_1}^{R_0}$	-0.109

Numerical simulation

Some numerical simulation results were obtained with the use of mesh and contour plots for the reproductive number as a function of two different parameters chosen from the Table 3. The results given in Fig. 3, 4, and 5 show that the value of $\mathcal{R}_0$ increases when the values of transmission rates increases.

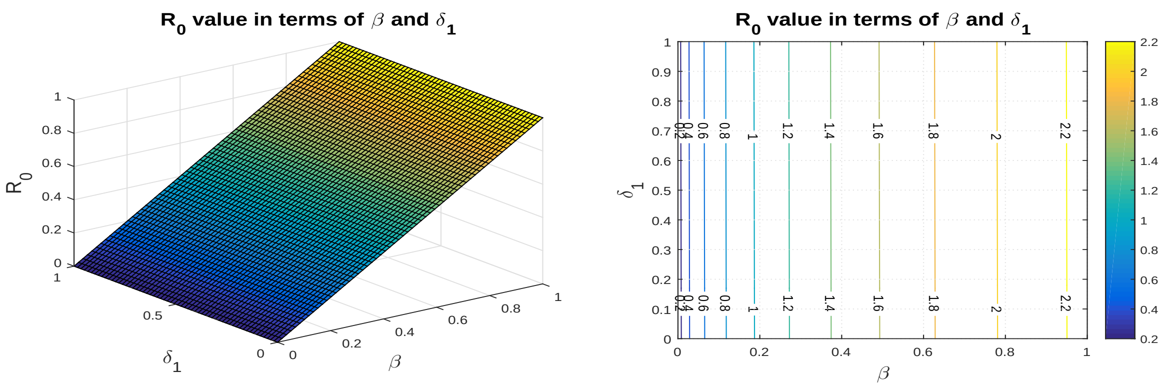

Figure 3. Profile of reproductive number in terms of transmission rate β and progression rate δ_1.

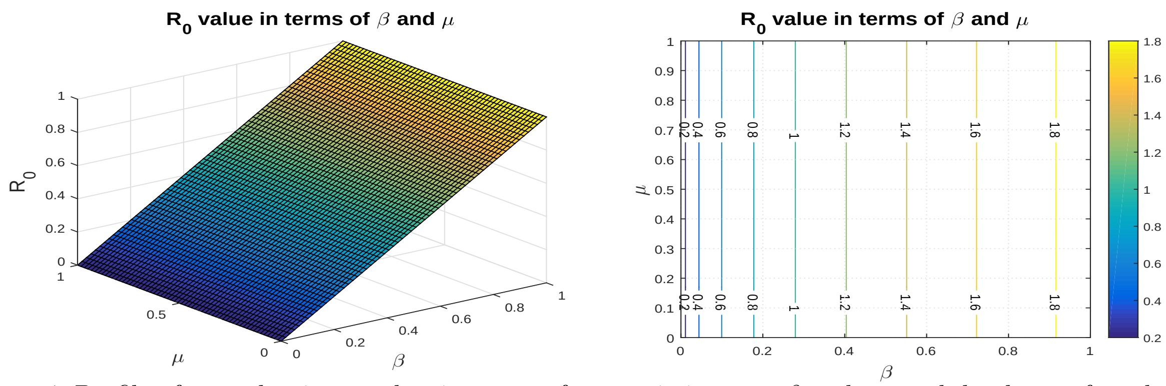

Figure 4. Profile of reproductive number in terms of transmission rate β and natural death rate for adults μ.

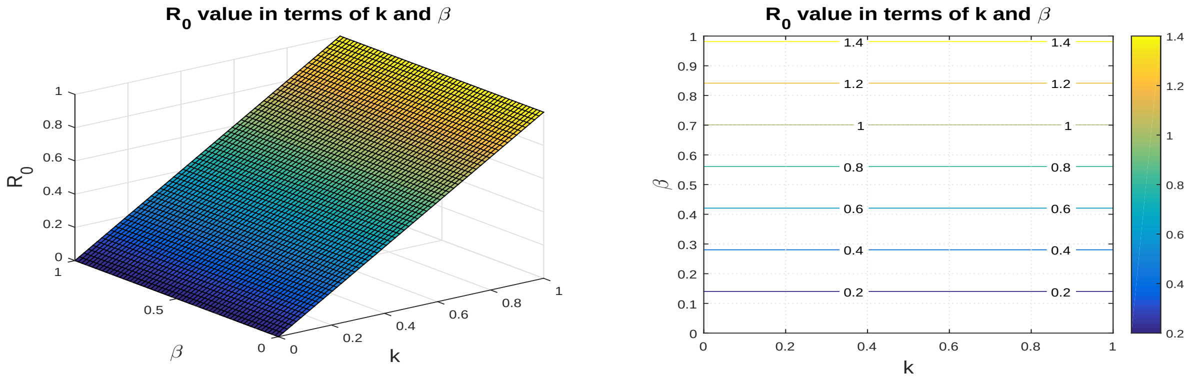

Figure 5. Profile of reproductive number in terms of transmission rate β and natural death rate for newborn children k.

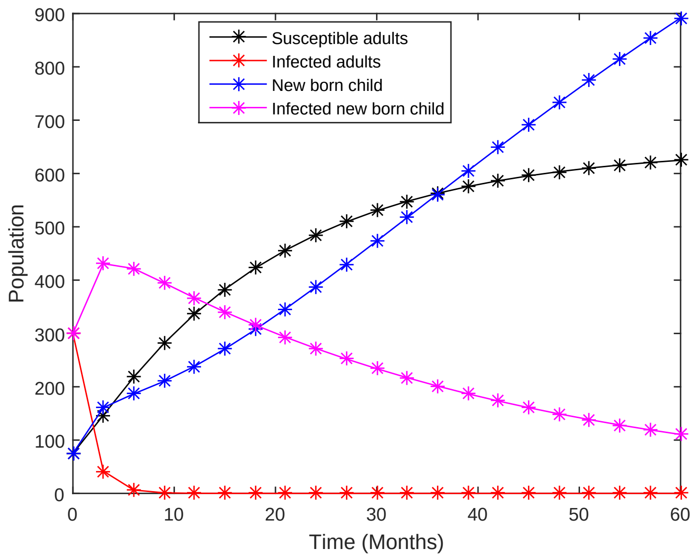

Figure 6. Profile of the total population dynamics with the respect to the parameter values in Table 2.

Conclusions

Human Immune-deficiency Virus (HIV) reduces or destroys the human defense mechanisms known as the immune system to prevent it fighting infections and any other diseases. In this study a mathematical model is developed to study the transmission dynamics of HIV infection and the effect of horizontal and vertical transmission in Turkey is analyzed. The model is fitted with the use of confirmed HIV cases of both vertical and horizontal transmission from 2011 to 2018. Using the next generation matrix method, the basic reproduction number of the model is obtained, which shows whether disease persists or dies out in time.

Further analysis showed that the model is locally asymptotically stable when the basic reproduction number $\mathcal{R}_0 < 1$ and is unstable when $\mathcal{R}_0 > 1$. The most sensitive parameters efficient for the control of the infection are obtained using forward normalized sensitivity index. The results obtained with the aid of mesh and contour plots showed that decreasing the values of transmission rate, disease induced mortality rates and progression rates play a significant role in controlling the spread of HIV transmission.

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Тиімді бақылау стратегияларының көмегімен АИТВ-инфекциясының көлденең және тік берілу әсерін модельдеу

Мақалада АИТВ-инфекциясының таралу динамикасын зерттеу үшін математикалық модель әзірленді және Түркияда инфекцияның көлденең және тік берілуінің әсері талданды. Модель 2011 жылдан бастап 2018 жылға дейін АИТВ-ның тік және көлденең берілуінің расталған жағдайларын пайдалана отырып, зерттелген. Келесі буын операторының көмегімен аурудың сақталатындығын немесе уақыт өте келе жоғалатынын көрсететін модельдің негізгі репродуктивті нөмірі алынады. Қосымша талдау көрсеткендей, базалық репродуктивті санында $\mathcal{R}_0 < 1$ моделі локалды асимптотикалық тұрақты және $\mathcal{R}_0 > 1$ кезінде тұрақсыз. Инфекциямен күресу үшін тиімді ең сезімтал параметрлер тікелей қалыпқа келтірілген сезімталдық индексін қолдану арқылы алынады. Торлы және контурлық графиктер арқылы алынған нәтижелер берілу жылдамдығының, аурудың, өлім-жітімнің және прогрессия көрсеткіштерінің төмендеуі АИТВ-ның таралуын бақылауда маңызды рөл атқаратынын көрсетеді.

Кілт сөздер: АИТВ, математикалық модельдеу, басқару стратегиялары, сезімталдықты талдау.

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Моделирование эффекта горизонтальной и вертикальной передачи ВИЧ-инфекции с помощью эффективных стратегий контроля

В статье разработана математическая модель для изучения динамики передачи ВИЧ-инфекции, и проанализировано влияние горизонтальной и вертикальной передачи инфекции в Турции. Модель адаптирована с использованием подтвержденных случаев как вертикальной, так и горизонтальной передачи ВИЧ с 2011 по 2018 годы. С помощью оператора следующего поколения получается базовый репродуктивный номер модели, который показал, сохраняется ли болезнь или исчезает со временем. Дальнейший анализ выявил, что модель локально асимптотически устойчива при базовом воспроизводственном числе $\mathcal{R}_0 < 1$ и нестабильна при $\mathcal{R}_0 > 1$. Наиболее чувствительные параметры, эффективные для борьбы с инфекцией, получены с использованием прямого нормализованного индекса чувствительности. Наконец, результаты, полученные с помощью сетчатых и контурных графиков, показывают, что снижение значений скорости передачи, показателей смертности от болезней и прогрессирования играет важную роль в контроле распространения передачи ВИЧ.

Ключевые слова: ВИЧ, математическое моделирование, стратегии управления, анализ чувствительности.

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Grid method for solution of 2D Riemann type problem with two discontinuities having an initial condition

This study aims to obtain the numerical solution of the Cauchy problem for 2D conservation law equation with one arbitrary discontinuity having an initial profile. For this aim, a special auxiliary problem allowing to construct a sensitive method is developed in order to get a weak solution of the main problem. Proposed auxiliary problem also permits us to find entropy condition which guarantees uniqueness of the solution for the auxiliary problem. To compare the numerical solution with the exact solution theoretical structure of the problem under consideration is examined, and then the interplay of shock and rarefaction waves is investigated.

Keywords: 2D nonlinear scalar conservation law, Riemann problem, finite differences scheme in a class of discontinuous functions.

Introduction

In the half plane $R^3_+ = R^2 \times [0,T)$ we consider the following problem for the function $u = u(x,y,t)$

$$\frac{\partial u}{\partial t} + \frac{\partial f_1(u)}{\partial x} + \frac{\partial f_2(u)}{\partial y} = 0, \tag{1}$$

$$u\left(r\cos\varphi, r\sin\varphi, 0\right) = u_0(\varphi) \equiv \begin{cases} u_1, & \varphi_1 < \varphi < \varphi_2, \\ u_2, & \varphi \in [0,\ 2\pi] \setminus [\varphi_1, \varphi_2]. \end{cases} \tag{2}$$

Here u_1 and u_2 are known constants. The existence and uniqueness of the global weak solution verifying the entropy condition of problem (1), (2) is proved in [1–4]. But the methods in these articles do not give enough information about the qualitative nature of the solution. A similar one-dimensional problem has been examined in detail in [5–8].

Obtaining exact solutions of the problems in terms of (1), (2) for the cases $f_1(u) = \frac{u^2}{2}$ and $f_2(u) = \frac{u^3}{3}$ was first discussed in [9]. This technique is referred to as Guckenheimer structure in the literature. In [10–18] Guckenheimer structure is developed under conditions

$$f_1(u), \quad f_2(u) \in C^3\left(R_1\right), \quad f_1''(u) > 0, f_2''(u) > 0, \left(\frac{f_1''(u)}{f_2''(u)}\right)' > 0 \tag{3}$$

and generalized characteristic analysis method has been suggested. In [19–23] various numerical methods have been developed for problem (1), (2).

In [22–25] a new method is proposed in a class of discontinuous functions to obtain the exact solution of problem (1), (2) in which $f_1(u) = \frac{u^2}{2}$ and $f_2(u) = \frac{u^2}{2}$ have both continuous initial functions with compact support and two-piece constant initial function. The same method is also included in [26] for the Cauchy problem for the one-dimensional Hopf equation.

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In this paper for problem (1), (2) we introduce a special auxiliary problem with some advantages over the main problem. Using the proposed auxiliary problem, a new finite difference method is developed to find the numerical solution of the main problem. To compare the resulting solution with exact solution, and for sake of simplicity, we first consider the exact solution of the problem with $f_1(u) = \frac{u^2}{2}$ and $f_2(u) = \frac{u^2}{2}$. Proposed method is also valid for other $f_1(u)$ and $f_2(u)$ functions satisfying condition (3).

As is well known, equation (1) is invariant to the transformation $(x, y, t) \to (cx, cy, ct)$ for any $c > 0$, and due to the existence and uniqueness theorem the problem has a self-similar solution as follows

$$u(x, y, t) = u\left(\frac{x}{t}, \frac{y}{t}, 1\right), \quad t > 0.$$

For this reason, equation (1) will be considered in the domain $\left(\xi = \frac{x}{t},\ \eta = \frac{y}{t}\right)$. Under the coordinates (ξ, η) equation (1) can be written as follows for a continuously differentiable u

$$(\xi - u)\, u_\xi + (\eta - u)\, u_\eta = 0. \tag{4}$$

The initial condition for (4) becomes

$$\lim_{\substack{\tan\varphi = \frac{\eta}{\xi},\\ \xi^2+\eta^2 \to \infty}} u(\xi, \eta) = u_0(\varphi), \quad \varphi \in [0, 2\pi].$$

So, outside a sufficient distance on (ξ, η) domain, the discontinuities of $u_0(\varphi)$ function produces two types of simple waves; shock waves and rarefaction waves (including semi-contact discontinuity). The importance of Riemann solution is the examination of the interaction of these waves in the region that includes the coordinate origin.

Since equation (4) is a first order differentiable equation, the following is equivalent to an ordinary differentiable system of equations

$$\begin{cases} \frac{d\eta}{d\xi} = \frac{\xi - u}{\eta - u}, \\ \frac{du(\xi, \eta(\xi))}{d\xi} = 0. \end{cases} \tag{5}$$

Thus, the function u takes constant values over the following characteristics

$$\frac{d\eta}{d\xi} = \frac{\xi - u}{\eta - u}.$$

It is clear that the characteristics are the following lines for any k

$$\eta = u + k(\xi - u).$$

As it is seen, the characteristic lines are the lines that end at the singular points $(\xi, \eta) = (u, u)$ of the integral curve of (5). These singular points match the characteristic lines defined as follows after the transformation of u

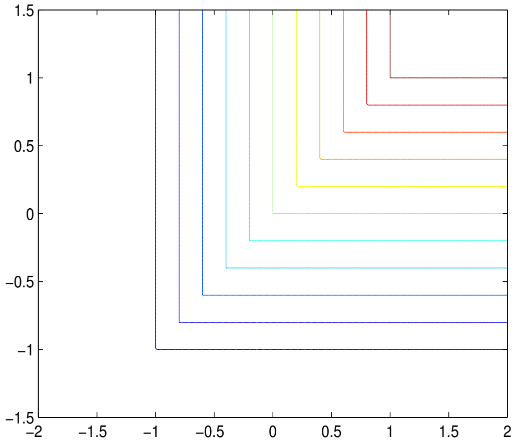
$$\begin{cases} x = ut, \\ y = ut. \end{cases}$$

Thus, the singular curve of the characteristics becomes $\Gamma(u) : \{\eta = \xi\}$. The singular curves of the singularities, similar to the singular curve of characteristics, are the same as $\Gamma(u)$, $\Gamma_S(u, \overline{u}) \subseteq \{(\xi, \eta) \in R^2, \quad \eta = \xi\}$, since each of the lines originating from the origin of singularity of Riemann data are the edges of the characteristic domain.

When $u_1 = 1$, $u_2 = -1$, two noninteracting rarefaction waves originating from $\{x > 0, y = 0\}$ and $\{x = 0, y > 0\}$ occur sufficiently outside of the region that includes the coordinate origin. The solution at $t = 1$ becomes as follows [19]

$$u(x, y, 1) = v(\xi, \eta) = \begin{cases} 1, & \xi > 1 \text{ and} \eta > 1, \\ -1, & \xi < -1 \text{ or } \eta < -1, \\ \eta, & \xi > \eta \text{ and } -1 < \eta < 1, \\ \xi, & \xi < \eta \text{ and } -1 < \xi < 1 \end{cases}$$

as shown in Figure 1a.



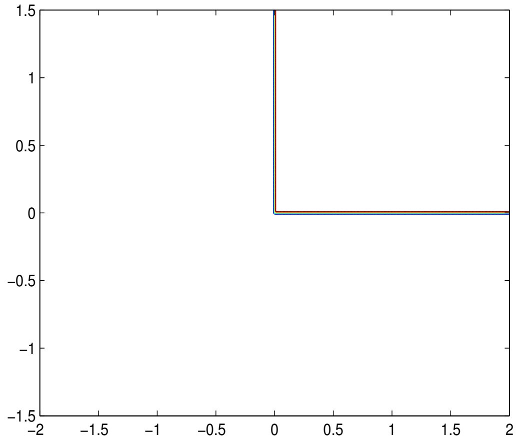


Figure 1. a) The exact solution to (1), (2) when $u_1 = 1,\ \ u_2 = -1,\ t = 1$.
b) The exact solution to (1), (2) problem, $u_1 = -1,\ \ u_2 = 1,\ t = 1$, [19].

Since these waves do not interact, we can expand them to (ξ, η) domain.

When $u_1 = -1,\ u_2 = 1$ in the solution of problem (1), (2) two shock waves occur along the lines $\{x > 0,\ y = 0\}$ and $\{x = 0,\ y > 0\}$ sufficiently outside the region that include the origin. We can expand these waves without interaction until the coordinate origin, preserving the entropy condition and reach the solution as follows

$$u(\xi, \eta) = \begin{cases} -1, & \xi > 0,\ \ \eta > 0, \\ 1, & \text{otherwise} \end{cases}$$

as seen in Figure 1b.

Auxiliary problem and numerical solution

To find the numerical solutions to the aforementioned problems various finite difference methods have been investigated in the literature [1, 9, 21]. As it is known, the solution to the one dimensional Riemann problem, even when the initial function is sufficiently smooth, contains discontinuities whose locations are unknown beforehand. In two-dimensional problems the number of discontinuities may be infinite. Special investigation is required to see which of these discontinuities are physically meaningful. Thus, directly using finite differencing for the problems which have discontinuities in the solution may spread the discontinuities to several points.

Due to the difficulties of working with the discontinuous functions, the paper proposes an original method to find the numerical solution of the problem that expresses the physical properties of the problem correctly. To this end, no problem arise in using the familiar methods in the literature to find the numerical solution to the auxiliary problem, which has advantages over the original problem but also is conventionally equivalent to the original problem. By using the numerical solution to the auxiliary problem one can find the numerical solution to the original problem.

As mentioned before, a classical solution may not exist for problem (1), (2). In this case, we define a weak solution as follows.

Definition 1. A function $u(x, y, t)$ satisfying initial condition (2) is called as a weak solution of the problem (1), (2), if the following integral relation

$$\int_{R^2 \times R^+} \left\{ u(x,y,t)\frac{\partial f(x,y,t)}{\partial t} + \frac{u^2(x,y,t)}{2}\left(\frac{\partial f(x,y,t)}{\partial x} + \frac{\partial f(x,y,t)}{\partial y}\right)\right\} dxdydt$$
$$+ \int_{R^2} f(x,y,0)u(x,y,0)dxdy = 0$$

holds for any test function $\varphi : R^2 \times R^+ \to R,\ f(x, y, t) \in \mathring{C}^\infty$.

To obtain the weak solution following [22], if we integrate equation (1) for any a and c over

$$D_{xy} = \{a < \xi < x,\ c < \eta < y\} \subset R^2,$$

we find the following

$$\frac{\partial}{\partial t}\int_a^x\int_c^y u(\xi,\eta,t)d\xi d\eta+\frac{1}{2}\left(\int_c^y u^2(x,\eta,t)d\eta+\int_a^x u^2(\xi,y,t)d\xi\right)$$
$$=\frac{1}{2}\left(\int_c^y u^2(a,\eta,t)d\eta+\int_a^x u^2(\xi,c,t)d\xi\right). \quad (6)$$

Here a and c may also be $\mp\infty$. Let us include the following function

$$w(x,y,t)=\int_a^x\int_c^y u(\xi,\eta,t)d\xi d\eta. \quad (7)$$

We can show

$$Mw(x,y,t)=u(x,y,t). \quad (8)$$

Here $M(\cdot)=\frac{\partial^2(\cdot)}{\partial x\partial y}$ is a differential operator. In the notation of (7) and (8) we can write equation (6) as follows

$$\frac{\partial w(x,y,t)}{\partial t}+\frac{1}{2}\int_a^y\left(u^2(x,\eta,t)-u^2(a,\eta,t)\right)d\eta+\frac{1}{2}\int_c^x\left(u^2(\xi,y,t)-u^2(\xi,c,t)\right)d\xi=0. \quad (9)$$

For equation (9) the initial condition will be

$$w(x,y,0)=w_0(x,y). \quad (10)$$

Here $w_0(x,y)$ is defined as any continuous and differentiable solution of the following equation

$$Mw_0(x,y)=u_0(x,y).$$

We will call problem (9), (10) auxiliary problem.

The auxiliary problem has the following advantages:

1 The differentiability property of the function $w(x,y,t)$ the solution to equation (9) is better than the function $u(x,y,t)$.

2 To find $u(x,y,t)$ its derivative with respect to any variable is not used, as aforementioned derivatives do not even exist.

From (6) we get

$$v(x,y,t)=-\frac{1}{2}\int_0^t\Big[\int_c^y u^2(x,\eta,t)d\eta-\int_c^a u^2(a,\eta,t)d\eta$$
$$+\int_a^x u^2(\xi,y,t)d\xi-\int_a^x u^2(\xi,c,t)d\xi\Big]d\tau.$$

Now, for any positive numbers a_1 and b_1 we consider the sum

$$\frac{v(x,y,t)-v(x-a_1,y,t)}{a_1}+\frac{v(x,y,t)-v(x,y-b_1,t)}{b_1}$$

$$=\frac{1}{a_1}\Bigg\{\int_0^t\Big[\int_c^y u^2(\xi-a_1,\eta,t)d\eta-\int_a^{x-a_1}u^2(\xi,y,t)d\xi\Big]d\tau$$
$$-\int_0^t\Big[\int_c^y u^2(x,\eta,t)d\eta-\int_a^x u^2(\xi,y,t)d\xi\Big]d\tau\Bigg\}$$
$$+\frac{1}{b_1}\Bigg\{\int_0^t\Big[\int_c^{y-b_1}u^2(x,\eta,t)d\eta-\int_a^x u^2(\xi,\eta-b_1,t)d\eta\Big]d\tau$$
$$-\int_0^t\Big[\int_c^y u^2(x,\eta,t)d\eta-\int_c^x u^2(\xi,y,t)d\xi\Big]d\tau\Bigg\}.$$

According to the consideration (3), we can obtain the following estimate

$$\frac{v(x,y,t)-v(x-a_1,y,t)}{a_1}+\frac{v(x,y,t)-v(x,y-b_1,t)}{b_1}$$
$$\le \frac{1}{t}\Big[\frac{2M(d-c)T+2M(b-a)T}{a_1}\Big]+\frac{1}{t}\Big[\frac{2MT(d-c)+2MT(b-a)}{b_1}\Big]$$
$$=\frac{1}{t}\Big[2MT\frac{(d-c)}{a_1}+2MT\frac{(b-a)}{b_1}\Big]+\frac{1}{t}\Big[2MT\left(\frac{(d-c)}{a_1}+\frac{(b-a)}{b_1}\right)\Big]=\frac{E}{t}, \tag{11}$$

where E is a constant. Condition (11) is called entropy condition. This condition implies that, if we get fixed $t>0$ and let x and y tend to $-\infty$ and ∞, respectively, then we can get only jump down in both direction across one discontinuity.

Now we will describe the finite difference algorithm to find the numerical solution of problem (9), (10). To this end we build a grid on D_{xy} region as follows.

Assuming L is a sufficiently big positive integer, we cover $D_{(-L,L)}$ region with $x=x_i$, $y=y_j$ lines and build a grid

$$\Omega^{(-\ell,\ell)}_{(h_x,h_y)}=\{(x_i,y_j)\ :\ x_i=-\ell+ih_x, y_j=-\ell+jh_y, i=0,1,2,...,n, j=0,1,2,...,m\}.$$

Now, let us divide $[x_i,x_{i+1}]$ and $[y_j,y_{j+1}]$ into p and q pieces, respectively. Then, let us define the grid formed by these points as

$$\Omega^{(x_i,y_j)}_{(h_\xi,h_\eta)}=\{\xi_\nu=x_i+\nu h_\xi;\ \ \eta_\mu=y_j+\mu h_\nu;\ \ \nu=0,1,2,...,np,\ \mu=0,1,2,...,mq\}.$$

It is clear that $\bigcup_{i,j}\Omega^{(x_i,y_j)}_{(h_\xi,h_\eta)}=\Omega^{(-\ell,\ell)}_{(h_x,h_y)}$. Let us write the integrals in (9) in quadratic form as follows

$$\int_{-\ell}^{y_i}u^2(x,\eta,t)d\eta\approx\frac{h_\eta}{2}\sum_{\mu=0}^{q\cdot j}U^2(x_i,\eta_\mu,t_k),\ \frac{1}{2}\int_{-\ell}^{x_i}u^2(\xi,y_j,t_k)d\xi\approx\frac{h_\xi}{2}\sum_{\nu=0}^{p\cdot i}U^2(\xi_\nu,y_j,t_k),$$

$$\int_{-\ell}^{x_i}\int_{-\ell}^{y_j}u(\xi,\eta,t)d\xi d\eta\equiv h_\xi h_\eta\sum_{\nu=0}^{i}\sum_{\mu=0}^{j}U(x_\nu,y_\mu,t_k).$$

If we take these into account (9), we obtain the following systems of finite difference equations

$$W_{i,j,k+1}=W_{i,j,k}+\frac{h_th_\eta}{2}\sum_{\mu=0}^{q\cdot j}\left(U^2(x_i,\eta_\mu,t_k)-U^2(x_0,\eta_\mu,t_k)\right)$$
$$+\frac{h_th_\xi}{2}\sum_{\nu=0}^{p\cdot i}\left(U^2(\xi_\nu,y_j,t_k)-U^2(\xi_0,y_j,t_k)\right), \tag{12}$$
$$(i=0,1,2,...,n;\ j=0,1,2,...,m;\ k=0,1,2,...).$$

For (12), the initial condition is as follows

$$W_{i,j,0}=w_0(x_i,y_j),\qquad (i=0,1,2,...,n;\ j=0,1,2,...,m). \tag{13}$$

Corresponding difference analogy for (9) is

$$V(x_i,y_j,t_{k+1})=V(x_i,y_j,t_k)-\frac{h_t}{2}\sum_{\ell=0}^{k}\Bigg\{h_\eta\sum_{\mu=1}^{qj}\Big[U^2(x_i,\eta_\mu,t_k)-U^2(a,\eta_\mu,t_k)\Big]$$
$$+h_\xi\sum_{\nu=1}^{pi}\Big[U^2(\xi_\nu,y_j,t_k)-U^2(\xi_\nu,c,t_k)\Big]\Bigg\}.$$

As above

$$\frac{V(x_i,y_j,t_{k+1})-V(x_{i-p_1},y_j,t_{k+1})}{p_1}+\frac{V(x_i,y_j,t_{k+1})-V(x_i,y_{j-q_1},t_{k+1})}{q_1}$$

$$= \frac{V(x_i,y_j,t_k)-V(x_{i-p_1},y_j,t_k)}{p_1}+\frac{V(x_i,y_j,t_k)-V(x_i,y_{j-q_1},t_k)}{q_1}$$

$$-\frac{h_t}{2}\sum_{\ell=0}^{k}\left\{\sum_{\mu=1}^{qj}\left[\frac{h_\eta}{q_1}\Big[U^2(x_i,\eta_\mu,t_k)-U^2(a,\eta_\mu,t_k)\Big]+\frac{h_\eta}{p_1}\Big[U^2(x_i,\eta_\mu,t_k)\right.\right.$$

$$\left.\left.-U^2(x_{i-p_1},\eta_\mu,t_k)\Big]\right]+\sum_{\nu=1}^{pi}\left[\frac{h_\xi}{q_1}\Big[U^2(\xi_\nu,y_j,t_k)-U^2(\xi_\nu,y_{j-q_1},t_k)\Big]\right.$$

$$\left.+\frac{h_\xi}{p_1}\Big[U^2(\xi_\nu,y_j,t_k)-U^2(\xi_\nu,c,t_k)\Big]\right]-\frac{h_\eta}{q_1}\sum_{\mu=1}^{q(j-q_1)}\Big[U^2(x_i,\eta_\mu,t_k)-U^2(\xi_\nu,c,t_k)\Big]$$

$$\left.-\frac{h_\xi}{p_1}\sum_{\nu=1}^{p(i-p_1)}\Big[U^2(\xi_\nu,y_j,t_k)-U^2(\xi_\nu,c,t_k)\Big]\right\}\le\frac{E_1}{t},$$

where E_1 is a constant. So we have obtained entropy solution of the Cauchy problem for Burgers equation.

From (12) we obtain the following

$$U_{i,j,k+1}=h_xh_y\sum_{\nu=1}^{i}\sum_{\mu=1}^{j}W(x_\nu,y_\mu,t_{k+1}),(i=0,1,...,n;\ j=0,1,...,m;\ k=0,1,...). \tag{14}$$

In addition, by using equation (9), finite difference schemes on higher orders with respect to t (such as Runge-Kutta method) can be used.

Numerical experiments

Two sets of computer experiments are performed using the algorithm proposed above. Such as grid steps with respect to spatial and time variables and the size of the grid is $[0,T]\times[-4.0,4.0]$, $h_\xi=\frac{h_x}{p}$, $h_\eta=\frac{h_y}{q}$, $h_x=\frac{b-a}{h}$, $h_y=\frac{d-c}{m}$, $p=q=10$, $n=m=533$, $t_k=0.005$.

In the first of the series of experiments, as seen in Figures 2 and 3, it is assumed that the jump strip on the initial condition is in each quadrant of xy plane. The results obtained for the cases $u_1=1$, $u_2=0$ and $u_1=0$, $u_2=1$ are shown in Figure 2 and 3, respectively. Solutions of the auxiliary problem (12), (13) are shown in the first column of Figure 2 and Figure 3. By using (14) the solution of the auxiliary problem is obtained and the second column shows the solutions of the main problem (1), (2). In the third column, contours of shock and rarefaction waves are illustrated.

As seen in Figure 2, in the cases of the initial profile at $(1,0,0,0)$ and $(0,0,1,0)$ it is observed that two rarefaction and two shock waves occur. In the other cases, at $(0,1,0,0)$ and $(0,0,0,1)$, one shock and one rarefaction wave are detected (Figure 3). Comparing the results obtained with the solutions found in [15] with same data, the importance of the proposed auxiliary problem emerges.

In the second series of experiments, as seen in figures, it is assumed that the jump-fan is in each quadrant of the xy plane. In these cases, as shown in Figure 4, one shock and one rarefaction wave occur in all cases.

In Figure 5, we assume that the initial data has $u_1=0$, $u_2=1$ and the dynamics of the wave propagation at $T=1$ are shown. The results obtained for the cases $u_1=1$, $u_2=0$ and $u_1=0$, $u_2=1$ are shown in Figure 4 and 5, respectively. As seen from the figures one shock and one rarefaction wave always occur on the quadrants.

The solutions we obtained show that the proposed numerical algorithm is highly sensitive and describes all of the physical properties of the phenomenon correctly.

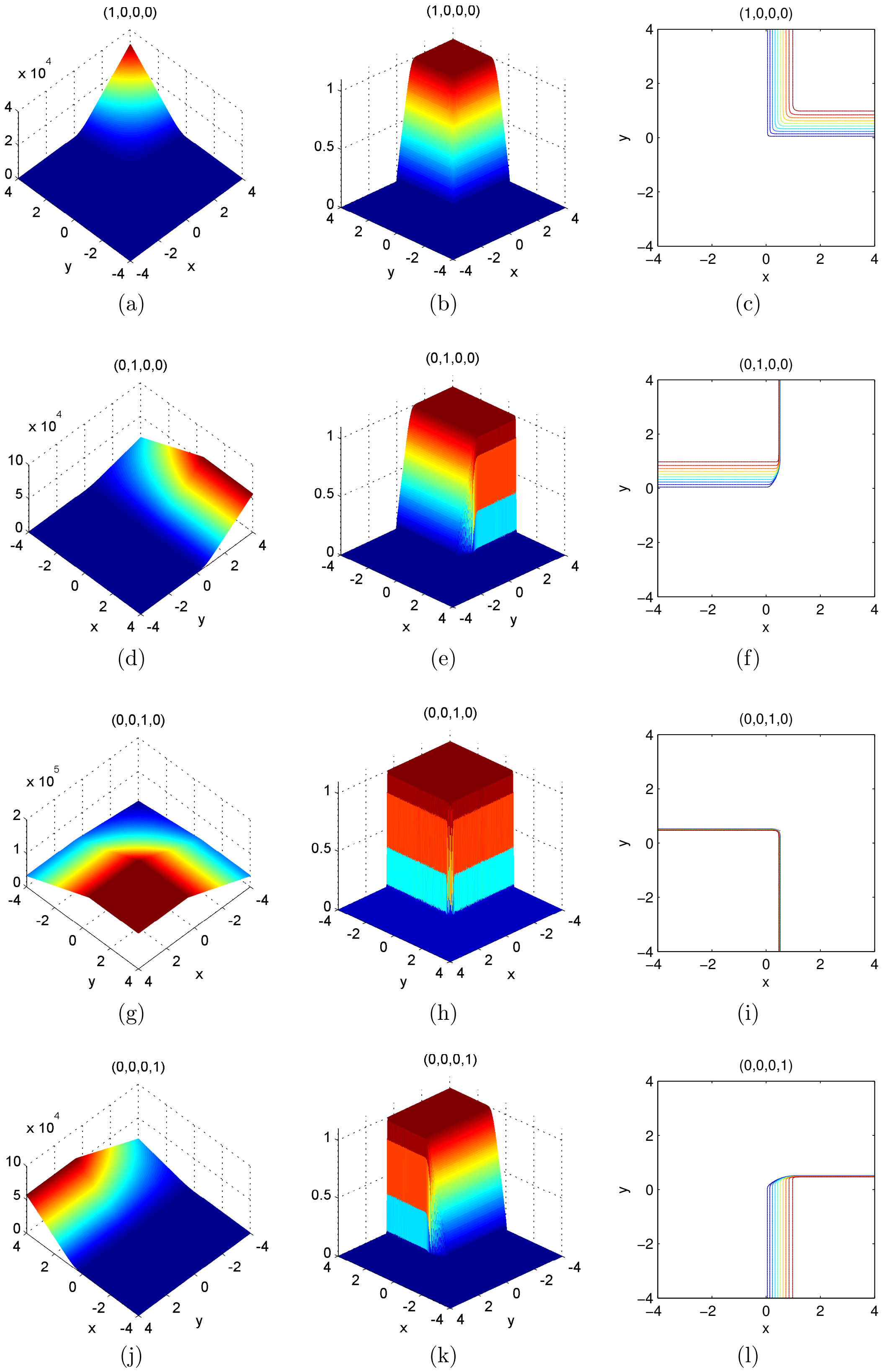


Figure 2. (a)–(c): $\varphi_1 = 0$, $\varphi_2 = \frac{\pi}{2}$; (d)–(f): $\varphi_1 = \frac{\pi}{2}$, $\varphi_2 = \pi$; (g)–(i): $\varphi_1 = \pi$, $\varphi_2 = \frac{3\pi}{2}$; (j)–(l): $\varphi_1 = \frac{3\pi}{2}$, $\varphi_2 = 2\pi$

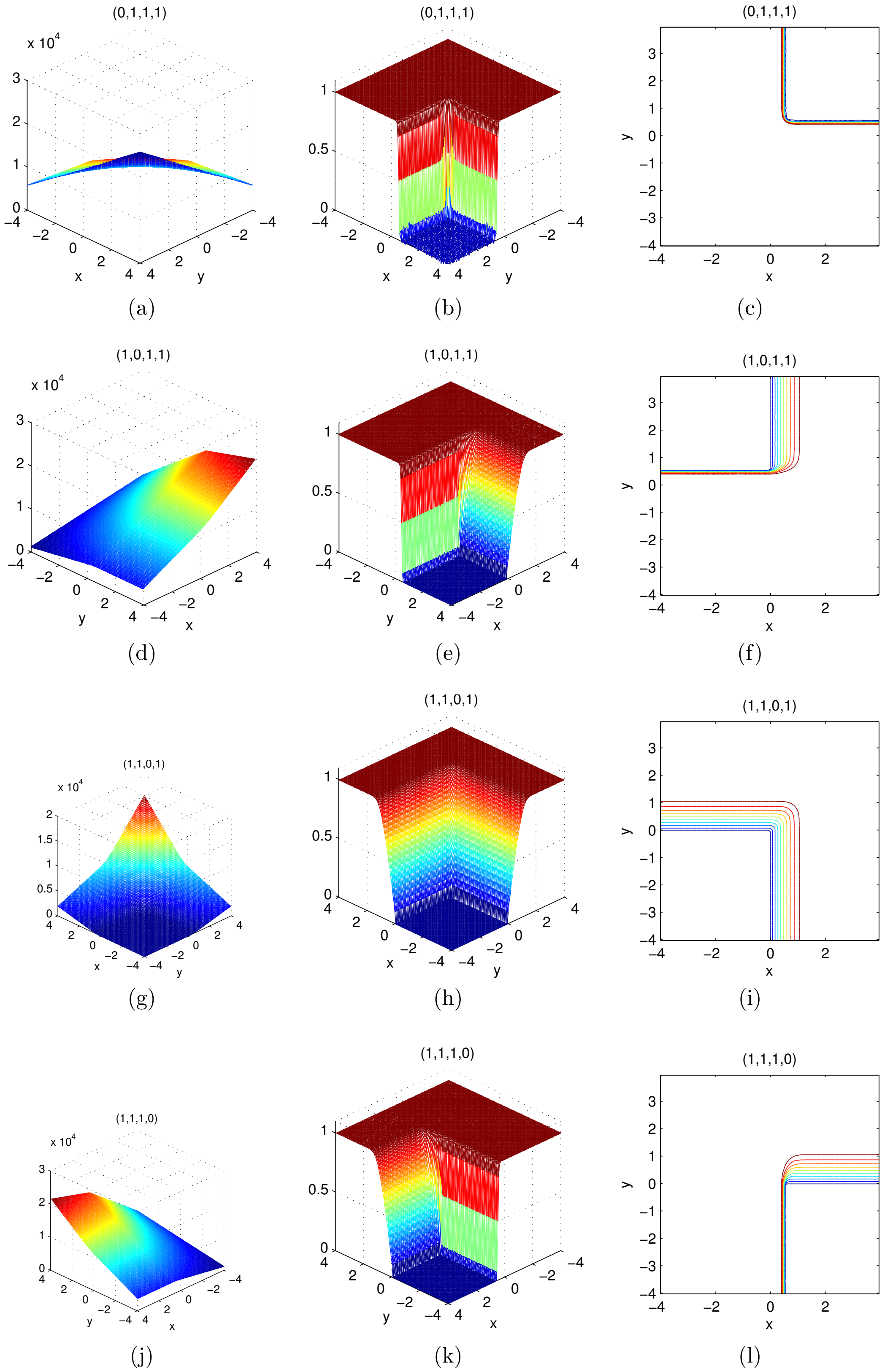


Figure 3. (a)–(c): $\varphi_1 = 0$, $\varphi_2 = \frac{\pi}{2}$; (d)–(f): $\varphi_1 = \frac{\pi}{2}$, $\varphi_2 = \pi$; (g)–(i): $\varphi_1 = \pi$, $\varphi_2 = \frac{3\pi}{2}$; (j)–(l): $\varphi_1 = \frac{3\pi}{2}$, $\varphi_2 = 2\pi$

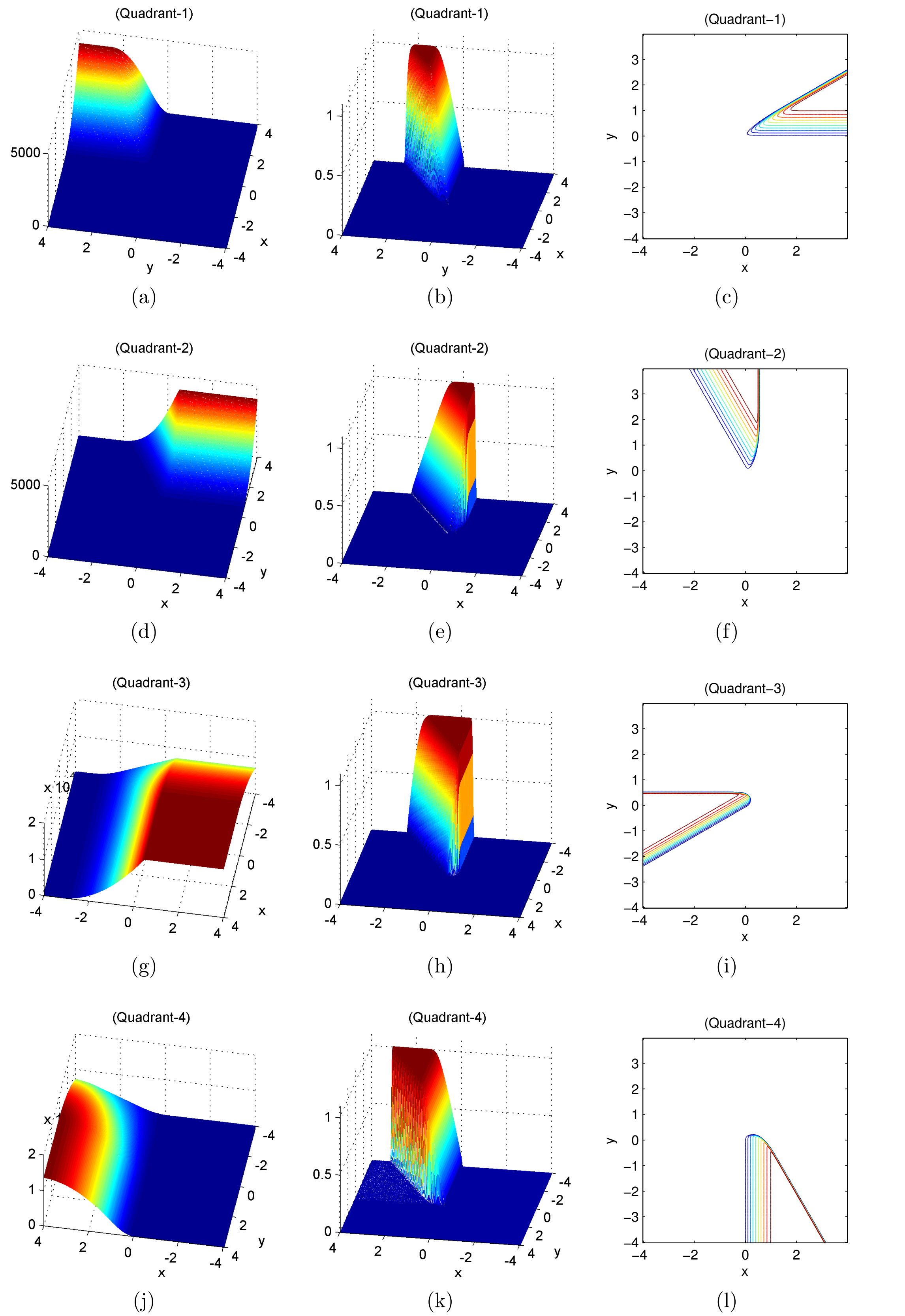


Figure 4. (a)–(c): $\varphi_1 = 0$, $\varphi_2 = \frac{\pi}{6}$; (d)–(f): $\varphi_1 = \frac{\pi}{2}$, $\varphi_2 = \frac{2\pi}{3}$; (g)–(i): $\varphi_1 = \pi$, $\varphi_2 = \frac{5\pi}{6}$; (j)–(l): $\varphi_1 = \frac{3\pi}{2}$, $\varphi_2 = \frac{7\pi}{6}$

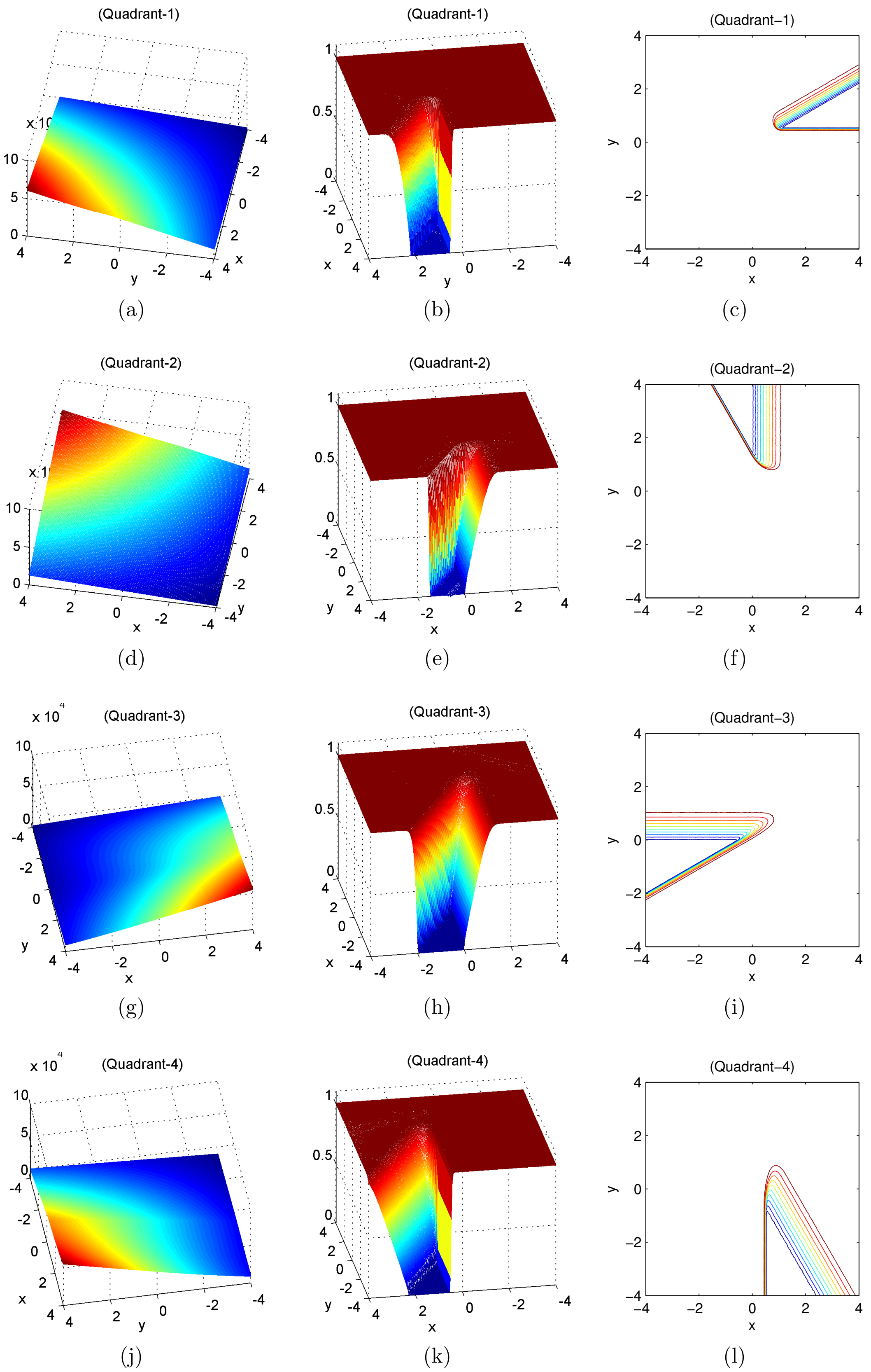


Figure 5. (a)–(c): $\varphi_1 = 0$, $\varphi_2 = \frac{\pi}{6}$; (d)–(f): $\varphi_1 = \frac{\pi}{2}$, $\varphi_2 = \frac{2\pi}{3}$; (g)–(i): $\varphi_1 = \pi$, $\varphi_2 = \frac{5\pi}{6}$; (j)–(l): $\varphi_1 = \frac{3\pi}{2}$, $\varphi_2 = \frac{7\pi}{6}$

Conclusion

In this study an original method for the numerical solution of the initial value problem for two-dimensional conservation law with two-piecewise constant discontinuous initial condition that accurately describes the all physical properties of the problem is proposed.

In addition to this, the auxiliary problem has provided the entropy condition, which provides the uniqueness of the solution. In establishing efficient algorithms for the numerical solution of the two-dimensional Riemann type problem the method incorporates an auxiliary problem that is equivalent to the original problem, and yet has advantages over it.

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Бастапқы күйде екі үзілісі бар Риман типті екі өлшемді есепті шешудің тор әдісі

Зерттеудің мақсаты — бастапқы профилі бар бір еркін үзілісті сақталу заңының екі өлшемді теңдеуі үшін Коши есебінің сандық шешімін алу. Ол үшін негізгі есептің әлсіз шешімін алуда сезімтал әдісті құруға мүмкіндік беретін арнайы көмекші есеп құрылады. Ұсынылған көмекші есеп, сонымен қатар, көмекші есепті шешімнің жалғыз болуына кепілдік беретін энтропияның шарттарын табуға мүмкіндік береді. Сандық шешімді дәл шешіммен салыстыру үшін алдымен есептің теориялық құрылымы, содан кейін соққы толқындары мен сирету толқындарының өзара әрекеттесуі зерттелген.

Кілт сөздер: бейсызықты скалярлы 2D сақтау заңы, Риманның есептері, үзілісті функциялар класындағы ақырлы-айырымды схема, екі өлшемді есептер.

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Сеточный метод решения двумерной задачи типа Римана с двумя разрывами, имеющими начальное состояние

Цель данного исследования — получение численного решения задачи Коши для двумерного уравнения закона сохранения с одним произвольным разрывом, имеющим начальный профиль. Для этого разработана специальная вспомогательная задача, позволяющая построить чувствительный метод для получения слабого решения основной задачи. Предлагаемая вспомогательная задача также позволяет найти условие энтропии, гарантирующее единственность решения вспомогательной задачи. Для сравнения численного решения с точным сначала исследована теоретическая структура рассматриваемой задачи, а затем изучено взаимодействие ударных волн и волн разрежения.

Ключевые слова: нелинейный скалярный 2D закон сохранения, задача Римана, конечно-разностная схема в классе разрывных функций, двумерное уравнение.

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On convergence of schemes of finite element method of high accuracy for the equation of heat and moisture transfer

In this paper difference schemes of the finite element method of a high order of accuracy for the non-stationary equation of moisture transfer of Aller are constructed and investigated. The increased order of accuracy is achieved through special sampling of temporal and spatial variables. The stability and convergence of the constructed numerical algorithms are proved, the corresponding a priori estimates are obtained in various norms, which are used later to obtain estimates of the accuracy of the scheme under weak assumptions on the smoothness of solutions to the differential problem.

Keywords: Aller's equation, finite element method, difference schemes, stability, a priori estimates, convergence, accuracy.

Introduction

As it is known, research in the field of heat and moisture transfer is fundamental in solving many applied problems, for example, problems of hydrogeology, agrophysics, ecology, building physics, etc. [1]. The interaction of heat fluxes in the soil-ground and snow cover determines the processes of infiltration, migration and frost heaving, evaporation and transpiration, metamorphism and snow melting. These processes determine conditions for overwintering and growing crops. In addition, the role of moisture migration and infiltration in the formation of productive moisture reserves in agricultural fields is of great importance. Mathematical models of these processes are described mainly by the Aller or Aller-Lykov moisture transfer equation [2]. This paper considers numerical methods for solving boundary value problems for the Aller moisture transfer equation written in a more general form. In this case, difference schemes of the finite element method of the fourth order of accuracy, constructed and investigated in [3], are used. These schemes have certain advantages over other schemes: a) high order accuracy scheme (above two); b) in addition to the solution itself, its derivative (velocity) is simultaneously found with the same accuracy; c) using interpolation representation

$$y(t) = y^n\phi_{00}^n(t) + \dot{y}^n\phi_{10}^n(t) + y^{n+1}\phi_{01}^n(t) + \dot{y}^{n+1}\phi_{11}^n(t), \tag{1}$$

$$\phi_{00}^n(t) = 2\xi^3 - 3\xi^2 + 1,\ \ \phi_{01}^n(t) = 3\xi^2 - 2\xi^3,\ \ \phi_{10}^n(t) = \tau(\xi^3 - 2\xi^2 + \xi),\ \ \phi_{11}^n(t) = \tau(\xi^3 - \xi^2),$$

if necessary, it is possible to get a solution and its derivative at any time; d) since the scheme is two-layer, it is possible to use a variable step without loss of accuracy; e) the scheme is conditionally stable and requires 4 times more arithmetic operations than the schemes of the finite difference method, but this scheme allows to choose large time steps to achieve a certain accuracy. To obtain an estimate of the accuracy a special technique for obtaining a priori estimates is used. The classical approach to the study of the convergence of difference schemes based on the Taylor formula imposes high requirements on the smoothness of the desired solution. Recently a number of results have been obtained on the estimation of the rate of convergence of difference schemes for equations of mathematical physics. These results can be found in [4–8]. Similar studies for various non-stationary problems were carried out by the authors in [3, 9–11].

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Statement of the problem

The problems of the thermal and water regime of the root layer of the soil, evaporation, transpiration, etc. are described by the following equation (the Aller equation in general form) [1]

$$\frac{\partial u}{\partial t} = L_1 u + \sigma \frac{\partial}{\partial t}(L_2 u) + f(x,t), \quad (x,t) \in Q_T = \{x \in \Omega,\ 0 < t \leq T\}. \tag{2}$$

Here $\Omega = \{x | x = (x_1, x_2),\ 0 < x_\alpha < l_\alpha,\ \alpha = 1,2\}$,

$$L_m u = \sum_{\alpha=1}^{p_m} \frac{\partial}{\partial x_\alpha}\left(k_\alpha^m(x)\frac{\partial u}{\partial x_\alpha}\right) - q^m(x), \quad x \in \Omega, \ \ p_m = 1,2,...,$$

$$0 < k_0 \leq k_\alpha^m(x) \leq k_1, \ \ q^m(x) \geq 0, m = 1,2,$$

where σ, k_0, k_1 are positive constants.

For equation (2) the initial condition

$$u(x,0) = u_0(x), \quad x \in \Omega \tag{3}$$

is set and some local or non-local boundary conditions are given.

Local conditions are classical boundary conditions, for example, the first boundary value condition

$$u(x,t) = 0, \quad x \in \partial\Omega, \ \ t \in (0,T]. \tag{4}$$

Conditions are called non-local if the boundary conditions are relations connecting the values of the sought solution and its derivatives at the boundary and interior points of the domain. Similar conditions arise in the mathematical modelling of processes of various natural phenomenon, for example, in the study of problems of moisture transfer, thermal conductivity, mathematical biology, control, etc. For example, for equation (1) in the one-dimensional case, the non-local boundary conditions

$$u(0,t) = \lambda u(l,t), \quad u_x(0,t) = \lambda u_x(l,t), \quad t \in [0,T]$$

are given in [1-2].

Let us formulate a generalized statement of problem (2)–(4). We call the generalized solution of problem (2)–(4) as the function $u(x,t)$, in which each $t \in [0,T]$ belongs to Sobolev space $H = \overset{\circ}{W}{}_2^1(\Omega)$, has a derivative $\frac{\partial u}{\partial t} \in L_2(0,T)$ and satisfies the relations [12]

$$\left(\frac{du(t)}{dt}, \vartheta\right) + \sigma a_1\left(\frac{du(t)}{dt}, \vartheta\right) + a_2(u(t),\vartheta) = (f(t),\vartheta), \ \forall\vartheta(x) \in H, \ \ u(0) = u_0 \tag{5}$$

almost everywhere on $(0,T)$. Here

$$a_m(u(t),\vartheta) = -(L_m u, \vartheta) = \int\limits_\Omega \sum_{\alpha=1}^{p_m}\left(k^m(x)\frac{\partial u}{\partial x_\alpha}\cdot\frac{\partial \vartheta}{\partial x_\alpha} + q^m(x)u\vartheta\right)dx, \ \ m = 1,2.$$

For bilinear form $a_m(u,\vartheta)$ there is an evaluation $a_m(\vartheta,\vartheta) \geq k\|\vartheta\|^2$.

Note that the dimension of the operators L_1, L_2 can be different, i.e., $p_1 \neq p_2$, and so L_1 can be strongly elliptical and L_2 can be a degenerate operator that does not contain all second derivatives of variables x_α.

Discretization in space and time

We discretize problem (2)–(3) with respect to spatial variables using the finite element method. Let $H_h \subset H$ be many elements of the form $\vartheta_h = \sum_{i=1}^N a_i\phi_i(x)$. Here $\{\phi_i = \phi_i(x)\}_{i=1}^N$ is the basis of piecewise polynomial functions that are polynomials of degree on each finite element (a segment in one-dimensional case, a triangle or rectangle in two-dimensional case, etc.).

Let us write relation (5) a semi-discrete problem for $t \in [0, T]$:

$$\left(\frac{du_h(t)}{dt}, \vartheta_h\right) + \sigma\ a_1 \left(\frac{du_h}{dt}, \vartheta_h\right) + a_2 \left(u_h(t), \vartheta_h\right) = \left(f(t), \vartheta_h\right), \quad \forall \vartheta_h \in H_h, \tag{6}$$

$$u_h(0) = u_{0,h}.$$

Problem (6) corresponds to the Cauchy problem in time for the system of ordinary differential equations of the first order for coefficients of the approximate solution $u_h(t) = \sum_{i=1}^{N} a_i(t)\phi_i$ from H_h :

$$M\frac{d\vec{a}(t)}{dt} + G\vec{a}(t) = \vec{\Phi}(t), \ \ \vec{a}(0) = \vec{a}^0.$$

Here, $\vec{a}(t) = \{a_i(t)\}_{i=1}^{N}$, $\{\vec{a}_i(0)\}_{i=1}^{N}$ are dimension vectors N; $M = \{(\phi_i, \phi_j)\}_{i,j=1}^{N}$ is the mass matrix, $G = \{a(\phi_i, \phi_j)\}_{i,j=1}^{N}$ is the stiffness matrix and $\vec{\Phi}(t) = \{\Phi(t)\}_{i=1}^{N}$ is a vector of the right side.

The same problem can also be written in the form of an operator equation

$$D\frac{du_h(t)}{dt} + Au_h(t) = f_h(t), \ \ u_h(0) = u_{0,h}, \tag{7}$$

$$u_h(0) = u_{0,h}.$$

Here $u_h(t)$ is the element of finite-dimensional space H_h for any moment in time t, operators D and A operate from H_h to H_h: $D = M + \sigma G_1, \ A = G_2, \ M = ((\phi_i, \phi_j))_{i,j=1}^{N}$ is a subspace coordinate system mass matrix H_h, and $G_m = (a_m(\phi_i, \phi_j))_{i,j=1}^{N}$ is a stiffness matrix corresponding to the operator $L_m u, \ m = 1, 2$ in H_h.

We approximate problem (7) with a three-parameter finite element method of the fourth order of accuracy in time [3]:

$$\begin{cases} D\frac{\hat{y}-y}{\tau} - \frac{\tau^2}{12}A\frac{\hat{\dot{y}}-\dot{y}}{\tau} + A\frac{\hat{y}+y}{2} = \phi_1, \\ \gamma D\frac{\hat{\dot{y}}-\dot{y}}{\tau} + \alpha A\frac{\hat{y}-y}{\tau} + \beta A\frac{\hat{\dot{y}}+\dot{y}}{2} = \phi_2, \\ \ y^0 = u_0, \ \dot{y}^0 = u_1, \end{cases} \tag{8}$$

where

$$y = y^n = y(t_n), \ \hat{y} = y^{n+1} = y(t_n + \tau), \ \dot{y} = \dot{y}^n = \frac{dy}{dt}(t_n), \ \hat{\dot{y}} = \dot{y}^{n+1} = \frac{dy}{dt}(t_n + \tau),$$

$$\phi_1 = \frac{1}{\tau}\int_{t_n}^{t_{n+1}} f(t)dt, \quad \phi_2 = \frac{12}{\tau^3}\int_{t_n}^{t_{n+1}} f(t)\left(s_1\vartheta_2^{(1)} + s_2\vartheta_2^{(3)}\right)dt,$$

$$s_1 = 15\gamma - 35\alpha/3, \ \ s_2 = 140\gamma - 350\alpha/3,$$

$$\upsilon_2^{(1)} = \tau(\xi + 1/2), \quad \upsilon_2^{(3)} = \tau(\xi^3 - 3\xi^2/2 + \xi/2), \quad \xi = (t - t_n)/\tau.$$

Scheme (8) obeys the condition of the fourth order of approximation in time

$$\alpha + \beta = \gamma, \ \ \alpha > 0, \ \ 0 < \beta \leq \alpha/(3\varepsilon), \ \ \alpha, \beta \ = \ O(\tau^2), \ \ 0 < \varepsilon < 1. \tag{9}$$

Circuit stability conditions are

$$\alpha = \tau^2/12, \quad \beta > 0, \quad \gamma > 0, \ \ , \ \ \bar{R} \geq ((1 + \varepsilon)/4)\bar{A},$$

where $\bar{R} = \frac{1}{\tau}\left(\gamma D^2 + \frac{\tau^2}{12}(3\beta + \alpha)A^2\right), \ \ \bar{A} = \tau\beta A^2$.

A high order of accuracy of the scheme is achieved due to a special discretization of temporal and spatial variables [3].

Investigation of the accuracy of discretization in space

Let us estimate the accuracy of the solution of problem (2)–(4). All notations are borrowed from [13]. The following theorem holds.

Theorem 1 Let $u(x,t) \in L_2\left\{[0,T];\ \ W_2^{k+1}(\Omega) \cap \overset{\circ}{W}{}_2^1(\Omega)\right\}$. If a narrowing of space H_h into a single finite element is a polynomial of degree k, then for the solution of problem (7) there is an estimation of accuracy

$$\sqrt{\int\limits_0^t \|u(x,t') - u_h(x,t')\|_0^2\, dt'} + \sigma \sqrt{\int\limits_0^t \|u(x,t') - u_h(x,t')\|_1^2\, dt'}$$

$$+ \left\| \int\limits_0^t [u(x,t') - u_h(x,t')]\, dt' \right\|_1 \le Mh^k \left\{ \|u(x,0)\|_k + \sigma \|u(x,0)\|_{k+1} \right.$$

$$\left. + \sqrt{(1+\sigma)\int\limits_0^t \|u(x,t')\|_{k+1}^2\, dt'} \right\}, \quad \forall t \in [0,T],\ M = M(k_0,k_1,T).$$

Proof. We integrate identity (5) over t from $t_n = n\tau,\ n = 0,1,...$ to $t_{n+1} = t_n + \tau$ and apply the formula for integration by parts

$$\int\limits_{t_n}^{t_{n+1}} \left[-(u(t),\dot{\vartheta}) - \sigma\, a_1(u(t),\dot{\vartheta}) + a_2(u(t),\vartheta) \right] dt + \left[(u(t),\vartheta) + \sigma\, a_1(u(t),\vartheta) \right] \Big|_{t_n}^{t_{n+1}}$$

$$= \int\limits_{t_n}^{t_{n+1}} (f(t),\vartheta)dt,\ \forall \vartheta(x) \in H. \tag{11}$$

Similar actions for identity (6) give

$$\int\limits_{t_n}^{t_{n+1}} \left[-(u_h,\dot{\vartheta}_h) - \sigma a_1(u_h,\dot{\vartheta}_h) + a_2(u_h,\vartheta_h) \right] dt + \left[(u_h,\vartheta_h) + \sigma a_1(u_h,\vartheta_h) \right] \Big|_{t_n}^{t_{n+1}} = \int\limits_{t_n}^{t_{n+1}} (f,\vartheta_h)dt,\ \forall \vartheta_h(x) \in H_h.$$

Here and further $\dot{u} = \partial u / \partial t$. Choosing $\vartheta = \vartheta_h \in H_h \subset H$ in (11) and subtracting both obtained identities, we have

$$\int\limits_{t_n}^{t_{n+1}} \left[-(u-u_h,\dot{\vartheta}_h) - \sigma\, a_1(u-u_h,\dot{\vartheta}_h) + a_2(u-u_h,\vartheta_h) \right] dt$$

$$+ \left[(u-u_h,\vartheta_h) + \sigma\, a_1(u-u_h,\vartheta_h) \right] \Big|_{t_n}^{t_{n+1}} = 0, \quad \forall \vartheta_h(x) \in H_h. \tag{12}$$

Let $z_h = u - u_h = e_h + \xi_h$. Let us choose a trial function

$$\vartheta_h(t) = -\int\limits_t^s \xi_h(t')dt' \in H_h,\ t < s;\ \ \vartheta_h(t) = 0,\ \ t \ge s,\ \dot{\vartheta}_h(t) = \ \xi_h(t),\ \vartheta_h(s) = 0. \tag{13}$$

Taking into account the introduced designations, identity (12) can be written in the form:

$$\int\limits_{t_n}^{t_{n+1}} \left[-(\xi_h,\xi_h) - \sigma\, a_1(\xi_h,\xi_h) + a_2(\dot{\vartheta}_h,\vartheta_h) \right] dt + \left[(\xi_h,\vartheta_h)\ + \sigma\, a_1(\xi_h,\vartheta_h) \right] \Big|_{t_n}^{t_{n+1}}$$

$$= \int\limits_{t_n}^{t_{n+1}} \left[(e_h,\xi_h) + \sigma\, a_1(e_h,\xi_h) - a_2(e_h,\vartheta_h) \right] dt - \left[(e_h,\vartheta_h) + \sigma a_1(e_h,\vartheta_h) \right] \Big|_{t_n}^{t_{n+1}}.$$

Since $a_2(\dot{\vartheta}_h, \vartheta_h) = \frac{1}{2}\frac{d}{dt}a_2(\vartheta_h, \vartheta_h)$, then the last identity can be written as:

$$-\int\limits_{t_n}^{t_{n+1}} (\xi_h, \xi_h)\, dt - \sigma \int\limits_{t_n}^{t_{n+1}} a_1(\xi_h, \xi_h)\, dt + \frac{1}{2}a_2(\vartheta_h, \vartheta_h)(t_{n+1}) + [(\xi_h, \vartheta_h) + \sigma a_1(\xi_h, \vartheta_h)]|_{t_n}^{t_{n+1}}$$

$$= -\left[(e_h, \vartheta_h) + \sigma a_1(e_h, \vartheta_h)\right]|_{t_n}^{t_{n+1}} + \frac{1}{2}a_2(\vartheta_h, \vartheta_h)(t_n) + \int\limits_{t_n}^{t_{n+1}} \left[(e_h, \xi_h) + \sigma a_1(e_h, \xi_h) - a_2(e_h, \vartheta_h)\right] dt. \qquad (14)$$

Let us sum up (14) by $n = \overline{1, m-1}$, where is the number m corresponds to the moment in time $s = m\tau$:

$$-\int\limits_0^s (\xi_h, \xi_h)dt - \sigma \int\limits_0^s a_1(\xi_h, \xi_h)dt + \frac{1}{2}a_2(\vartheta_h, \vartheta_h)(s) + [(\xi_h, \vartheta_h) + \sigma a_1(\xi_h, \vartheta_h)]|_0^s$$

$$-\left[(e_h, \vartheta_h) + \sigma a_1(_h, \vartheta_h)\right]|_0^s + \frac{1}{2}a_2(\vartheta_h, \vartheta_h)(0) + \int\limits_0^s \left[(e_h, \xi_h) + \sigma a_1(e_h, \xi_h) - a_2(e_h, \vartheta_h)\right] dt.$$

Considering the properties of the function $\vartheta_h(t)$ (see eq. (13)) and the initial condition $\xi_h(0) = 0$, from the last identity we have

$$\int\limits_0^s (\xi_h, \xi_h)dt + \sigma \int\limits_0^s a_1(\xi_h, \xi_h)dt + \frac{1}{2}a_2(\vartheta_h, \vartheta_h)(0)$$

$$= -\left[(e_h, \vartheta_h)(0) + \sigma a_1(e_h, \vartheta_h)(0)\right] - \int\limits_0^s \left[(e_h, \xi_h) + \sigma a_1(e_h, \xi_h) - a_2(e_h, \vartheta_h)\right] dt.$$

Let us introduce one more function

$$w_h(t) = \int\limits_0^t \xi_h(t')dt' \in H_h,\ t < s;\ w_h(t) = 0,\ \ t \geq s.$$

Then $\vartheta_h(t) = w_h(t) - w_h(s)$, and, finally, we have the energy identity:

$$\int\limits_0^s (\xi_h, \xi_h)dt + \sigma \int\limits_0^s a_1(\xi_h, \xi_h)dt + \frac{1}{2}a_2(w_h, w_h)(s) = (e_h(0), w_h(s))$$

$$+\sigma a_1(e_h(0), w_h(s)) - \int\limits_0^s \left[(e_h, \xi_h) + \sigma\, a_1(e_h, \xi_h) - a_2(e_h, w_h(t) - w_h(s))\right] dt. \qquad (15)$$

Let us estimate the terms on the right hand side of (15):

$$(e_h(0), w_h(s)) \leq \varepsilon_1(w_h(s), w_h(s)) + \frac{1}{4\varepsilon_1}(e_h(0), e_h(0)),$$

$$a_1(e_h(0), w_h(s)) \leq \varepsilon_2 a_1(w_h(s), w_h(s)) + \frac{1}{4\varepsilon_2}a_1(e_h(0), e_h(0)),$$

$$\left|\int\limits_0^s (e_h, \xi_h)dt\right| \leq \varepsilon_3 \int\limits_0^s (\xi_h, \xi_h)dt + \frac{1}{4\varepsilon_3}\int\limits_0^s (e_h, e_h)dt,$$

$$\left|\int\limits_0^s a_1\,(e_h, \xi_h)dt\right| \leq \varepsilon_4 \int\limits_0^s a_1\,(\xi_h, \xi_h)dt + \frac{1}{4\varepsilon_4}\int\limits_0^s a_1\,(e_h, e_h)dt,$$

$$\left|\int_0^s a_2(e_h, w_h(t) - w_h(s))dt\right|$$

$$\leq \varepsilon_5 \int_0^s a_2(w_h(t), w_h(t))dt + s\varepsilon_5 a_2(w_h(s), w_h(s)) + \frac{1}{2\varepsilon_5}\int_0^s a_2(e_h, e_h)dt.$$

Choosing $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \varepsilon_4 = 1/2$, and ε_5 from condition $\varepsilon_1/2 + \varepsilon_5 T \leq 1/4$, from (15) we get the estimate

$$\int_0^s (\xi_h, \xi_h)dt + \sigma\int_0^s a_1(\xi_h, \xi_h)dt + a_2(w_h, w_h)(s)$$

$$\leq M\left(\int_0^s a_2(w_h,\, w_h)(t)dt + (e_h(0), e_h(0)) + \sigma a_1(e_h(0), e_h(0)) + (w_h(s), w_h(s))\right.$$

$$\left. + \sigma a_1(w_h(s), w_h(s)) + \int_0^s (e_h, e_h)dt + \sigma\int_0^s a_1(e_h, e_h)dt + \int_0^s a_2(e_h, e_h)dt\right),$$

where $M = \max(8,\ 1/T,\ 16T)$. Applying Gronwall's lemma, we obtain the error estimate

$$\int_0^s (\xi_h, \xi_h)dt + \sigma\int_0^s a_1(\xi_h, \xi_h)dt + a_2(w_h, w_h)(s)$$

$$\leq M\left[(e_h(0), e_h(0)) + \sigma a_1(e_h(0), e_h(0))\right] + (w_h(s), w_h(s)) + \sigma a_1(w_h(s), w_h(s))$$

$$+ \int_0^s (e_h, e_h)dt + \sigma\int_0^s a_1\,(e_h, e_h)dt + \int_0^s a_2(e_h, e_h)dt).$$

It's obvious that $k_0\left\|w_h(s)\right\|_1^2 \leq a_m(w_h, w_h)(s) \leq k_1\left\|w_h(s)\right\|_1^2$, $(\xi_h, \xi_h)(s) = \left\|\xi_h(s)\right\|_0^2$. Therefore, we have the final estimate for the error

$$\int_0^s \|\xi_h(s)\|_0^2 dt + \sigma\int_0^s \|\xi_h(s)\|_1^2 dt + \left\|\int_0^s \xi_h(t)dt\right\|_1^2$$

$$\leq M\left(\left\|e_h(0)\right\|_0^2 + \sigma\left\|e_h(0)\right\|_1^2 + \int_0^s \left\|e_h(t)\right\|_0^2 dt + \sigma\int_0^s \left\|e_h(t)\right\|_1^2 dt + \int_0^s \left\|e_h(t)\right\|_1^2 dt\right). \qquad (16)$$

For solutions $u(x,t) \in W_2^{k+1}(\Omega),\ \forall t \in [0,T]$, there is an evaluation [13]:

$$\left\|e_h(0)\right\|_0 \leq Mh^{k+1}\|u_0\|_k,\ \left\|e_h(0)\right\|_1 \leq Mh^{k+1}\|u_0\|_{k+1},$$

$$\left\|e_h(t)\right\|_0 \leq Mh^{k+1}\|u(t)\|_k,\ \left\|e_h(t)\right\|_1 \leq Mh^{k}\|u(t)\|_{k+1}.$$

Therefore, based on (16) and the triangle inequality $\|z_h\| \leq \|e_h\| + \|\xi_h\|$ the statement of the theorem holds. The theorem uses the standart notation for the Sobolev space W_2^{k+1} from [13].

Accuracy research of discretization in time

Let us now turn to the estimation of the discretization error for problem (7) in time. Investigation of the error in approximating scheme (8) using the Taylor formula, as already mentioned, leads to overestimated requirements for the smoothness of the solution to the original problem. An alternative to this method of estimating the accuracy is the application of the Bramble-Hilbert lemma. This method of accuracy estimation is the main one in the theory of the finite element method for solving elliptic equations [13–16]. We also note the paper [8], in which the Bramble-Hilbert lemma is used to estimate the accuracy of solving difference schemes for elliptic problems.

Let us apply the Bramble-Hilbert lemma to estimate the accuracy of the solution to the original problem with respect to the time variable. Recall that the solution $u_h(t)$ of semi-discrete task (7) for each t is an element of the discrete subspace $u_h(t) \in H_h$.

The following theorem holds.

Theorem 2. Let $A^* = A > 0,\ \ D^* = D > 0,\ \ AD = DA$ and the conditions of approximation (9) and stability (10) are fulfilled. Then to solve the scheme (8) approximate solution to problem (7) such that $\frac{d^4 u_h}{dt^4}(t) \in L_2[0,T]$ and the accuracy estimate

$$\sqrt{\int\limits_0^t \|u_h(t') - y(t')\|_0^2\, dt'} + \sigma \sqrt{\int\limits_0^t \|u_h(t') - y(t')\|_1^2\, dt'} + \left\| \int\limits_0^t [u_h(t') - y(t')]\, dt' \right\|_1$$

$$\leq M\tau^4 \left\{ \|u_h(0)\|_0 + \sigma \|u_h(0)\|_1 + \sqrt{(1+\sigma) \int\limits_0^t \left\| \frac{d^4 u_h}{dt^4}(t') \right\|_1^2 dt'} \right\} \tag{17}$$

is correct.

Proof. Denote by H_τ argument function subspace t, which are a cubic Hermitian spline of the form (1) on the interval $[t_n, t_{n+1}]$, $n = 0, 1, 2, \ldots$. Consider scheme solution (8) to $y(t) \in H_\tau$. Simultaneously for each t, $y(t)$ is an element of the subspace H_h. Actually $y(x,t) \in H_h^\tau = H_h \otimes H_\tau$.

Difference scheme (8) corresponds to the following weak setting

$$\int\limits_{t_n}^{t_{n+1}} \Big[-(y(t), \dot{\vartheta}_\tau) - \sigma\, a_1(y(t), \dot{\vartheta}_\tau) + \, a_2(y(t), \vartheta_\tau) \Big]\, dt$$

$$+ \left[(y(t), \vartheta_\tau) + \sigma\, a_1(y(t), \vartheta_\tau) \right] \Big|_{t_n}^{t_{n+1}} = \int\limits_{t_n}^{t_{n+1}} (f, \vartheta_\tau) dt, \quad \forall \vartheta_\tau(x) \in H_h^\tau, \tag{18}$$

where $y(t)$ is the cubic Hermitian spline (1).

In (18) select

$$\vartheta_\tau(t) = -\int\limits_t^s \xi_\tau(t) dt', \ \ t < s; \ \ \vartheta_\tau(t) = 0, \ \ t \geq s.$$

It's clear that $\dot{\vartheta}_\tau(t) = \xi_\tau(t),\ t < s$ and $\vartheta_\tau(s) = 0$. Substituting the function $\vartheta_\tau(t)$ into (18) and performing same transformations with the resulting identity that we used when evaluating $z_h = u - u_h = e_h + \xi_h$, we get the following energy identity

$$\int\limits_0^s (\xi_\tau, \xi_\tau) dt + \sigma \int\limits_0^s a_1(\xi_\tau, \xi_\tau) dt + \frac{1}{2} a_2(\vartheta_\tau, \vartheta_\tau)(0)$$

$$= (e_\tau, \vartheta_\tau)(0) + \sigma a_1(e_\tau, \vartheta_\tau)(0) - \int\limits_0^s \left[(e_\tau, \xi_\tau) + \sigma a_1(e_\tau, \xi_\tau) - a_2(e_\tau, \vartheta_\tau) \right] dt.$$

Denote by

$$w_\tau(t) = \int\limits_0^t \xi_\tau(t') dt' \in H_h \ t < s, \ w_\tau(t) = 0 \ \ t \geq s$$

and note that $e_\tau(0) = u_h(0) - u_I^\tau(0) = u_{0,h} - u_{0,h} = 0$. Then the last identity becomes

$$\int\limits_0^t (\xi_\tau, \xi_\tau) dt + \sigma \int\limits_0^t a_1(\xi_\tau, \xi_\tau) dt + \frac{1}{2} a_2(w_\tau, w_\tau)(s)$$

$$= -\int_0^s \left[(e_\tau, \xi_\tau) + \sigma a_1(e_\tau, \xi_\tau) - a_2(e_\tau, w_\tau(t) - w_\tau(s))\right] dt. \tag{19}$$

Applying the Cauchy-Bunyakovsky inequality, ε-inequality and Gronwall's lemma, as in the estimate $\xi_h(t)$, we obtain from (19) the following estimate

$$\int_0^s \|\xi_\tau(t)\|_0^2\, dt + \sigma \int_0^s \|\xi_\tau(t)\|_1^2\, dt + \left\| \int_0^s \xi_\tau(t)dt \right\|_1^2$$

$$\leq M \left(\int_0^s \|e_\tau(t)\|_0^2\, dt + \sigma \int_0^s \|e_\tau(t)\|_1^2\, dt + \int_0^s \|e_\tau(t)\|_1^2\, dt \right). \tag{20}$$

Now let us estimate the error of the scheme (8) $\zeta_\tau(t) = \xi_\tau(t) + e_\tau(t)$. By the triangle inequality and $(a+b)^2 \leq 2(a^2+b^2)$ we have

$$\int_0^s \|\zeta_\tau(s)\|_0^2\, dt + \sigma \int_0^s \|\zeta_\tau(s)\|_1^2\, dt + \left\| \int_0^s \zeta_\tau(t)dt \right\|_1^2 \leq 2 \left(\int_0^s \|e_\tau(s)\|_0^2 dt + \sigma \int_0^s \|e_\tau(s)\|_1^2 dt \right.$$

$$\left. + \left\| \int_0^s e_\tau(t)dt \right\|_1^2 + \int_0^s \|e_\tau(s)\|_0^2\, dt + \sigma \int_0^s \|e_\tau(s)\|_1^2\, dt + \left\| \int_0^s e_\tau(t)dt \right\|_1^2 \right).$$

For the last term, we apply the Cauchy-Bunyakovsky inequality and get

$$\left\| \int_0^s e_\tau(t')dt' \right\|_1^2 \leq \left\| \sqrt{\int_0^s 1dt'}\ \sqrt{\int_0^s e_\tau^2(t')dt'} \right\|_1^2 \leq s \int_0^s \|e_\tau(t')\|_1^2\, dt'.$$

From this and (20) we have the estimate

$$\int_0^s \|\zeta_\tau(s)\|_0^2\, dt + \sigma \int_0^s \|\zeta_\tau(s)\|_1^2\, dt + \left\| \int_0^s \zeta_\tau(t)dt \right\|_1^2$$

$$\leq M \left(\|e_\tau(s)\|_0^2 + \sigma \|e_\tau(s)\|_1^2 + \int_0^s \|e_\tau(t)\|_0^2\, dt + (\sigma+1) \int_0^s \|e_\tau(t)\|_1^2\, dt \right). \tag{21}$$

Consider the linear functional $e_\tau(u_h) = u_h - u_I^\tau$. We introduce the change of variable $t = t_n + \eta\tau$, $0 < \eta < 1$. Then, we get

$$\tilde{e}_\tau(\tilde{u}_h(\eta)) = e_\tau(u_h) = u_h(t_n + \eta\tau) - u_I^\tau(t_n + \eta\tau) = \tilde{u}_h(\eta) - \tilde{u}_I^\tau(\eta).$$

This functionality is limited for continuous functions $\tilde{u}_h(\eta) \in C[0,1]$. Moreover, it is limited for $\tilde{u}_h(\eta) \in W_2^4[0,1]$. So it is written as

$$|\tilde{e}_\tau(\tilde{u}_h)| = |\tilde{u}_h(\eta) - \tilde{u}_I^\tau(\eta)| \leq M \sum_{m=0}^4 \left(\int_0^1 \left(\frac{d^m \tilde{u}_h}{d\eta^m} \right)^2 d\eta \right)^{1/2}.$$

This functional vanishes on polynomials up to the third degree inclusive in the variable η, i.e., on the segment $[0,1]$ $\tilde{u}_I^\tau$ a third-degree polynomial that interpolates $\tilde{u}_h$. Based on the Bramble-Hilbert lemma, from the last estimate one can obtain

$$|\tilde{e}_\tau(\tilde{u}_h)| = |\tilde{u}_h(\eta) - \tilde{u}_I^\tau(\eta)| \leq \bar{M} \left(\int_0^1 \left(\frac{d^4 \tilde{u}_h}{d\eta^4} \right)^2 d\eta \right)^{1/2}.$$

Returning to the previous variables we have the estimate

$$|e_\tau(u_h(t))| = |u_h(t) - u_I^\tau(t)| \le \bar{M}\tau^{7/2}\left(\int\limits_{t_n}^{t_{n+1}} \left(\frac{d^4u_h}{dt^4}\right)^2 dt\right)^{1/2}, \ \forall t \in [t_n, t_{n+1}].$$

Then

$$\int\limits_0^s \|e_\tau(t')\|_1^2\, dt' = \sum_{n=0}^{m-1} \int\limits_{t_n}^{t_{n+1}} \|e_\tau(t')\|_1^2\, dt' \le \sum_{n=0}^{m-1} \int\limits_{t_n}^{t_{n+1}} \bar{M}^2\tau^7 \int\limits_{t_n}^{t_{n+1}} \left\|\frac{d^4u_h}{dt^4}(t)\right\|_1^2 dtdt'$$

$$= \sum_{n=0}^{m-1} \bar{M}^2\tau^8 \int\limits_{t_n}^{t_{n+1}} \left\|\frac{d^4u_h}{dt^4}(t)\right\|_1^2 dt = \bar{M}^2\tau^8 \int\limits_0^s \left\|\frac{d^4u_h}{dt^4}(t)\right\|_1^2 dt.$$

Similarly, the estimate

$$\int\limits_0^s \|e_\tau(t)\|_0^2\, dt \le \bar{M}^2\tau^8 \int\limits_0^s \left\|\frac{d^4u_h}{dt^4}(t)\right\|_0^2 dt$$

holds. If limitations $\left\|\frac{d^4u_h}{dt^4}(t)\right\|_0$ are required for each t , then we obtain

$$\|e_\tau(s)\|_0^2 \le \bar{M}^2\tau^7 \left\| \int\limits_{t_{m-1}}^{t_m} \left(\frac{d^4u_h}{dt^4}\right)^2 dt \right\|_0 \le \bar{M}^2\tau^8 \max_t \left\|\frac{d^4u_h}{dt^4}\right\|_0^2,$$

$$\|e_\tau(s)\|_1^2 \le \bar{M}^2\tau^8 \max_t \left\|\frac{d^4u_h}{dt^4}\right\|_1^2.$$

Further, based on these estimates we obtain the statement of the theorem from (21).

On the convergence of the scheme

In order to estimate the approximation error we need to go from u_h to the solution u in the right-hand sides of $z = u - y = (u - u_h) - (y - u_h)$.

For $k = 0, 1$ the following estimate holds [13]:

$$\|u_h\|_k = \|u - u + u_h\|_k \le \|u\|_k + \|u - u_h\|_k \le \|u\|_k + Mh\|u\|_{k+1} \le \bar{M}\|u\|_{k+1}.$$

Therefore, estimate (17) has the form

$$\sqrt{\int\limits_0^t \|u_h(t') - y(t')\|_0^2\, dt} + \sigma\sqrt{\int\limits_0^t \|u_h(t') - y(t')\|_1^2\, dt} + \left\|\int\limits_0^t [u_h(t') - y(t')]\, dt'\right\|_1$$

$$\le M\tau^4 \left\{ \|u(0)\|_0 + \sigma\|u(0)\|_1 + \sqrt{(\sigma+1)\int\limits_0^t \left\|\frac{d^4u}{dt^4}(t')\right\|_2^2 dt'} \right\}.$$

Thus, we formulate an assertion about the convergence of the solution of the vector scheme (8) to the solution of the original problem (2)–(4).

Theorem 3. Let $A^* = A > 0, \ \ D^* = D > 0, \ AD = DA$ and the conditions of approximation (9) and stability (10) of scheme (8). Then for its solution, which approximates the solution to problem (2)–(4) such that

$$u(x,t) \in L_2\left\{[0,T]; \ \ W_2^{k+1}(\Omega) \cap \overset{\circ}{W}{}_2^1(\Omega)\right\}, \ \frac{\partial^4 u}{\partial t^4}(x,t) \in L_2\left\{[0,T]; \ \ W{}_2^2(\Omega)\right\},$$

the accuracy estimate

$$\sqrt{\int_0^t \|u(x,t')-y(x,t')\|_0^2\,dt'}+\sigma\sqrt{\int_0^t \|u(x,t')-y(x,t')\|_1^2\,dt'}+\left\|\int_0^t [u(x,t')-y(x,t')]\,dt'\right\|_1$$

$$\leq M\left\{\tau^4\left(\|u(x,0)\|_0+\sigma\|u(x,0)\|_1+\sqrt{(1+\sigma)\int_0^t \left\|\frac{\partial^4 u}{\partial t^4}(x,t')\right\|_2^2 dt'}\right)\right.$$

$$\left.+h^k\left(\|u(x,0)\|_k+\sigma\|u(x,0)\|_{k+1}+\sqrt{(1+\sigma)\int_0^t \|u(x,t')\|_{k+1}^2\,dt'}\right)\right\}$$

is correct.

Algorithm for implementing the scheme

We consider one of the possible algorithms for implementing the scheme (8). We rewrite it as

$$m_{11}\hat{y}+m_{12}\hat{\dot{y}}=\varphi_1,\quad m_{21}\hat{y}+m_{22}\hat{\dot{y}}=\varphi_2,\tag{22}$$

where

$$\varphi_1=\tau\phi_1+\left(D+\frac{\tau}{2}A\right)y-\frac{\tau^2}{12}A\dot{y},\quad \varphi_2=\tau\phi_2+\alpha Ay+\left(\gamma D+\frac{\tau}{2}\beta A\right)\dot{y},$$

$$m_{11}=D+\frac{\tau}{2}A,\quad m_{12}=-\frac{\tau^2}{12}A,\quad m_{21}=\alpha A,\quad m_{22}=\gamma D+\frac{\tau}{2}\beta A.$$

Integrals in ϕ_1 and ϕ_2, for example, are calculated by Simpson's formula. Assuming that the operators A and D commute and excluding $\hat{\dot{y}}$ from equation (22), we obtain

$$C\hat{y}=F.\tag{23}$$

Here $C=\gamma D^2+\frac{\tau}{2}(\beta+\gamma)AD+\frac{\tau^2}{12}(3\beta+\alpha)A^2,\quad F=m_{22}\varphi_1-m_{12}\varphi_2.$

Equation (23) can be solved either directly by inverting the operator C or by factoring it as

$$C=\gamma C_1C_2=\gamma\left[D^2-(x_1+x_2)\tau AD+x_1x_2\tau^2A^2\right],\quad C_k=(D-x_k\tau A),\quad k=1,2.$$

Then equation (23) is solved using an algorithm

$$\gamma_1 C_1\bar{y}=F,\quad C_2\hat{y}=\bar{y}.\tag{24}$$

After finding $\hat{y}$ from (24) solution $\hat{\dot{y}}$ is calculated, for example, from the equation $\left(\gamma D+\frac{\tau}{2}\beta A\right)\hat{\dot{y}}=\varphi_2-\alpha A\hat{y}$.

Conclusion

Problems for the Aller moisture transfer equation are considered. On the basis of the finite element method difference schemes of high order of accuracy are constructed and investigated. The high order of accuracy of the circuit is achieved through special discretization of temporal and spatial variables. The convergence of the constructed algorithms is proved. Estimates for the accuracy of the scheme are obtained under weak assumptions on the smoothness of solutions to differential problems. Other boundary value problems can be studied similarly, in particular, nonlocal boundary value problems for equation (1). In addition, these results can be carried over to loaded equations with nonlocal boundary conditions.

Remark

A separate article will be devoted to computational experiments for test problems with local and non-local boundary conditions.

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Жылу-ылғал тасымалдау теңдеуі үшін жоғары дәлдікті ақырлы элементтер әдісінің схемасының жинақтылығы туралы

Мақалада Аллердің ылғал тасымалдау бейстационарлық теңдеуі үшін жоғары дәлдіктегі ақырлы элементтер әдісінің айырымдық схемалары құрылып зерттелді. Дәлдіктің жоғарғы ретіне уақыт және кеңістіктік айнымалыларын арнайы дискретизациялау арқылы қол жеткізіледі. Құрылған сандық алгоритмдердің тұрақтылығы мен жинақтылығы дәлелденді, дифференциалдық есептің шешімдерінің тегістігі туралы әлсіз болжамдармен схеманың дәлдік бағалауларын алу үшін пайдаланылған, әртүрлі нормаларда сәйкес априорлық бағалаулар алынды.

Кілт сөздер: Аллер теңдеуі, ақырлы элементтер әдісі, айырымдық схемалары, тұрақтылық, априорлық бағалаулар, жинақтылық, дәлдік.

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О сходимости схемы метода конечных элементов повышенной точности для уравнения тепло-влагопереноса

В статье построены и исследованы разностные схемы метода конечных элементов высокого порядка точности для нестационарного уравнения влагопереноса Аллера. Повышенный порядок точности достигается за счет специальной дискретизации временных и пространственных переменных. Доказана устойчивость и сходимость построенных численных алгоритмов, получены соответствующие априорные оценки в различных нормах, которые использованы в дальнейшем для получения оценок точности схемы при слабых предположениях о гладкости решений дифференциальной задачи.

Ключевые слова: уравнение Аллера, метод конечных элементов, разностные схемы, устойчивость, априорные оценки, сходимость, точность.

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On the Lie symmetries of the boundary value problems for differential and difference sine-Gordon equations

In general, due to the nature of the Lie group theory, symmetry analysis is applied to single equations rather than boundary value problems. In this paper boundary value problems for the sine-Gordon equations under the group of Lie point symmetries are obtained in both differential and difference forms. The invariance conditions for the boundary value problems and their solutions are obtained. The invariant discretization of the difference problem corresponding to the boundary value problem for sine-Gordon equation is studied. In the differential case an unbounded domain is considered and in the difference case a lattice with points lying in the plane and stretching in all directions with no boundaries is considered.

Keywords: symmetry analysis, partial differential equations, difference equations, boundary value problems.

Introduction

There are many theoretical and numerical studies on the nonlinear wave equations such as sine-Gordon and Klein-Gordon equations in the literature (see [1–3] and the references given therein). Sine-Gordon equations are of particular interest since they attracted much attention in the recent decades due to the exitance of soliton solutions. Solitons are nonlinear waves and have been used in many mathematical models.

Lie symmetries are one of the most powerful methods in obtaining exact solutions of many partial differential equations (PDEs). Many researchers have been studying this field and publishing articles and books [1–21] which investigate the general theory of these applications. However, there is relatively small number of studies that deal with Lie symmetries of boundary value problems for the PDEs. There are some difficulties in the application of Lie symmetries to boundary value problems (BVPs). In symmetry analysis every symmetry of a BVP must be a symmetry of a given PDE, a mapping of the domain to itself and a symmetry of the boundary data. In general, the prescribed initial or boundary conditions are not invariant under the group transformation of the corresponding PDE.

To the extent of our investigation the study of Lie symmetries of BVPs were first done by V.V. Puknachov [19] and G.W. Bluman [7]. For the theoretical aspects we refer to books [6, 20, 21] In the recent studies R. Cherniha et al. [16, 17] defined a new formula. This formula applies for the invariance of BVPs in a wide range of boundary conditions including free (moving) boundaries and boundaries at infinity.

In the present paper BVPs for nonlinear sine-Gordon equation in the differential and difference forms are investigated. Under the transformation groups boundary curves and boundary conditions of the equations are obtained. The formula for the invariance of BVPs presented by Cherniha [16] is used. The main object of this work is to investigate the invariance of a BVPs for sine-Gordon equation in differential and discrete form under the Lie point symmetries of the corresponding equations. Note that some of the results of this work was presented, without proof, in [4].

Preliminaries
Symmetry analysis of differential and difference equations

In this section we present the theory and definitions in the Lie symmetry analysis. Let us consider the system of differential equations

$$F_{\lambda}(x, u, u_1, u_2, \ldots, u_s) = 0, \lambda = 1, 2, \ldots, m, \tag{1}$$

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where $x \in R^n, u \in R^m$ and u_s is the set of s-th partial derivatives. We can write the group of point transformations in the space (x, u) as

$$G_r = \{x^{i^*} = f^i(x, u, a); u^{k^*} = g^k(x, u, a), i = 1, 2, \dots, n, k = 1, 2, \dots, m\}.$$

Using the power series and expanding the transformations about some neighborhood of the parameter $a^\alpha = 0$ gives

$$x^{i^*} = x^i + a^\alpha \frac{\partial f^i(x, a)}{\partial a^\alpha}|_{a=0} + O((a^\alpha)^2), \alpha = 1, \dots, r,$$

$$u^{k^*} = u^k + a^\alpha \frac{\partial g^k(u, a)}{\partial a^\alpha}|_{a=0} + O((a^\alpha)^2), \alpha = 1, \dots, r.$$

The derivatives of f^i and g^k are smooth functions and are called infinitesimals of the group G_r and denoted by ξ^i_α and η^k_α.

Finding the Lie group of differential system (1) is equivalent to finding its infinitesimal operator(generator), thus we seek for the infinitesimal operators of G_r in the following form

$$X_\alpha = \xi^i_\alpha(x, u)\frac{\partial}{\partial x_i} + \eta^k_\alpha(x, u)\frac{\partial}{\partial u^k}, i = 1, \dots, n, k = 1, \dots, m, \alpha = 1, \dots, r.$$

The set of tangent vectors to the manifold G_r at the identity element $a = 0$ is $\{X_\alpha, \alpha = 1, \dots, r\}$ and is a basis of the Lie algebra of the infinitesimal operators of G_r. The determination of the infinitesimal functions ξ^i_α and η^k_α states the group of transformations. By X_α, one can determine the point transformations of the group G_r by solving the Lie equations

$$\frac{\partial f^i}{\partial a^\alpha} = \xi^i_\alpha(f), \frac{\partial g^k}{\partial a^\alpha} = \eta^k_\alpha(g), \alpha = 1, \dots, r, i = 1, \dots, n, k = 1, \dots, m \tag{2}$$

with the initial conditions

$$f^i|_{a=0} = x^i, g^k|_{a=0} = u^k.$$

These equations obtain a one-to-one correspondence between vector fields (2) and the group of transformations G_r. The Lie algebra vector field is prolonged to the derivatives in order to modify it with differential variables u^k_i,

$$u^k_i = \frac{\partial u^k}{\partial x^i}, i = 1, \dots, n, k = 1, \dots, m.$$

From that the extended infinitesimal operators are obtained as

$$\widetilde{X}_\alpha = \xi^i \frac{\partial}{\partial x_i} + \eta^k \frac{\partial}{\partial u^k} + \zeta^{(1)k}_i \frac{\partial}{\partial u^k_i} + \dots + \zeta^{(s)k}_{i_1 i_2 \dots i_s} \frac{\partial}{\partial u^k_{i_1 i_2 \dots i_s}}. \tag{3}$$

Here we denote

$$\zeta^{(1)k}_i = D_i(\eta^k) - u^k_j D_i(\xi^j)$$

and

$$\zeta^{(s)k}_{i_1 i_2 \dots i_s} = D_{i_s} \zeta^{(s-1)k}_{i_1 i_2 \dots i_{s-1}} - u^k_{i_1 i_2 \dots i_{s-1} j} D_{i_s}(\xi^j),$$

$$D_i = \frac{\partial}{\partial x_i} + u^k_i \frac{\partial}{\partial u^k} + u^k_{ij} \frac{\partial}{\partial u^k_{ij}} + \dots + u^k_{i_1 i_2 \dots i_n} \frac{\partial}{\partial u^k_{i_1 i_2 \dots i_n}} + \cdots .$$

Theorem 1. [13] Let the Lie group of point transformations in the space of independent variables $(x, u, u_1, u_2, \dots, u_s)$, dependent variables and all s-th order partial derivatives of dependent variables with respect to independent ones be $\tilde{G}_r$. Then system of differential equations (1) is invariant under the group $\tilde{G}_r$ if and only if

$$\tilde{X}_\alpha F_\lambda(x, u, u_1, u_2, \dots, u_s)|_{(1)} = 0, \lambda = 1, 2, \dots, m. \tag{4}$$

The invariance condition (4) is an overdetermined system of linear equations for the coordinates of infinitesimal operator (3) and is called the system of determining equations.

Now, let us introduce the Lie symmetry analysis of difference equations. The difference scheme for the solution of the system of differential equations (1) is denoted by

$$H_\lambda(x, u, h, Tu) = 0, \lambda = 1, 2, \dots, m. \tag{5}$$

Here $h = (h_1, h_2, \ldots, h_n)$ is the mesh space vector and $T = (T_1, T_2, \ldots, T_n)$ represents the shift operator along the axis of the independent variables and given by

$$T_i[u](x_1, \ldots, x_i, \ldots, x_n) = u(x_1, \ldots, x_i + h_i, \ldots, x_n).$$

We denote a group of transformations in the space of mesh variables (x, u, h) by G_r^h and define as

$$G_r^h = \{x^{i^*} = f^i(x, u, a); u^{k^*} = g^k(x, u, a); h^{i^*}\}$$
$$= \varphi^i(x, u, h, a), i = 1, 2, \ldots, n, k = 1, 2, \ldots, m$$

with the infinitesimal operator

$$X_\alpha^h = X_\alpha + \varsigma_\alpha^i(x, u, h)\frac{\partial}{\partial h_i}, \alpha = 1, \ldots, r.$$

Here

$$\varsigma_\alpha^i = \frac{\partial \varphi^i}{\partial a^\alpha}, \alpha = 1, \ldots, r.$$

In the space of differential and difference variables $(x, u, h, u_1, u_2, \ldots, u_s)$ the prolongation operator of the group of point transformations $\tilde{G}_\alpha^{(h)}$ is $\tilde{X}_\alpha^{(h)}$.

Theorem 2. [13] Finite difference scheme (5) is invariant under the group of transformations $\tilde{G}_\alpha^{(h)}$ if and only if

$$\tilde{X}_\alpha^{(h)} H_\lambda(x, u, h, Tu)|_{(5)} = 0, \lambda = 1, 2, \ldots, m.$$

Symmetry analysis of the boundary value problem for PDEs

In this section we consider the Lie symmetry properties of BVPs. The invariance conditions under a group of point transformations of a BVP for a scalar PDE satisfy if the group separately leaves invariant the boundary conditions and the PDE of the BVP. The solution of the BVP resulting from the admitted point symmetry is an invariant solution if the BVP is well-posed. On the other hand, the concerned boundary conditions are in general not invariant under the symmetry of the considered PDEs. In view of this issue, one of the early definitions on the invariance of a BVP was given by G.W. Bluman [5].

Let us consider a k-th order ($k \geq 2$) scalar PDE represented by

$$F(x, u, \partial u, \partial^2 u, \ldots, \partial^k u) = 0. \tag{6}$$

Here $x = (x_1, x_2, \ldots, x_n)$ represents the coordinates corresponding to its n independent variables, u represents its dependent variable, and $\partial^j u$ represents the coordinates with components

$$\partial^j u/\partial x_{i_1}\partial x_{i_2}\ldots\partial x_{i_j} = u_{i_1 i_2 \ldots i_j}, i_j = 1, 2, \ldots, n, j = 1, 2, \ldots, k$$

corresponding to all j-th order partial derivatives of u with respect to x.

We assume that PDE (6) can be written in the following form in terms of some specific component of the l-th order partial derivatives of u

$$F(x, u, \partial u, \partial^2 u, \ldots, \partial^k u) = u_{i_1 i_2 \ldots i_l} - f(x, u, \partial u, \partial^2 u, \ldots, \partial^k u) = 0, \tag{7}$$

where $f(x, u, \partial u, \partial^2 u, \ldots, \partial^k u)$ does not depend explicitly on $u_{i_1 i_2 \ldots i_l}$.

Now consider a BVP for PDE (7) defined on the domain Ω_x in x-space $[x = (x_1, x_2, \ldots, x_n)]$ with boundary conditions

$$B_a(x, u, \partial u, \ldots, \partial^{k-1} u) = 0 \tag{8}$$

described on boundary surfaces

$$\omega_a(x) = 0, a = 1, 2, \ldots, s. \tag{9}$$

Let us assume that problem (7)–(9) has a unique solution. We use an infinitesimal generator as follows

$$X = \xi_i(x)\frac{\partial}{\partial x_i} + \eta(x, u)\frac{\partial}{\partial u}. \tag{10}$$

This infinitesimal generator defines a point symmetry acting on both (x, u)-space and on its projection to x-space.

Definition 1. [5] The point symmetry X in the form (10) is admitted by BVP (7)–(9) if and only if:
1 $X^{(k)}F(x,u,\partial u,\partial^2 u,\ldots,\partial^k u)=0$ when $F(x,u,\partial u,\partial^2 u,\ldots,\partial^k u)=0$.
2 $X\omega_a(x)=0$ when $\omega_a(x)=0, a=1,2,\ldots,s$.
3 $X^{(k-1)}B_a(x,u,\partial u,\ldots,\partial^{k-1}u)=0$ when $B_a(x,u,\partial u,\ldots,\partial^{k-1}u)=0$ on $\omega_a(x)=0, a=1,2,\ldots,s$.

The above definition does not apply for BVPs with free boundaries or with boundary conditions given at infinity. Therefore R. Chernica et al. (see [16], [17]) proposed a new invariance definition for BVPs which extends Bluman's definition to all possible boundary conditions. They formulated the definition of invariance for BVPs at operators of conditional symmetry case expressing what kind of transformations can be applied to transform boundary conditions at infinity to those containing no conditions at infinity. Consider a BVP for PDE (7) with boundary conditions (8) and conditions defined at infinity:

$$\gamma_c(x)=\infty : \gamma_c(x,u,\partial u,\ldots,\partial^{k_c}u)=0, c=1,2,\ldots,p_\infty, \tag{11}$$

where $k_c<k$ and p_∞ are given numbers and $\gamma_c(x)$ are specified functions that extend the domain on which the BVP is defined at infinity. We assume that all functions arising in (7), (8), (9), and (11) are given such that a classical solution of this BVP exists. Let us assume that the operator

$$Q=\xi_i(x,u)\frac{\partial}{\partial x_i}+\eta(x,u)\frac{\partial}{\partial u} \tag{12}$$

is a Q-conditional symmetry of PDE (7) satisfying the criterion:

$$Q^{(k)}F(x,u,\partial u,\partial^2 u,\ldots,\partial^k u)|_{F(x,u,\partial u,\partial^2 u,\ldots,\partial^k u)=0}=0, \tag{13}$$

where $Q^{(k)}$ is the k-th prolongation of Q and $Q(u)=0$ with $Q(u)=\xi_i(x,u)u_{x_i}-\eta(x,u)$. Let us consider the manifold for each $c=1,2,\ldots,p_\infty$ as

$$M=\{\gamma_c(x)=\infty : \gamma_c(x,u,\partial u,\ldots,\partial^{k_c}u)=0\}$$

in the extended space of variables $x,u,u_x,\ldots,u_x^{(k_c)}$. Suppose that there exists a smooth bijective transformation

$$y=g(x), w=h(x,u), \tag{14}$$

where $h(x,u)$ is a smooth function, $g(x)$ is a smooth vector function that maps the manifold M into

$$M^*=\{\gamma_c^*(y)=0 : \gamma_c^*(y,u,\partial u,\ldots,\partial^{k_c^*}u)=0\}$$

of the same dimensionality in the extended variable space $y,w,w_y,\ldots,w_y^{(k_c)}(k_c^*\le k_c)$ and $y=y_1,\ldots,y_n$.

Definition 2. [17] BVPs (7), (8), and (11) are Q-conditionally invariant under operator (12) if:
1 Criterion (13) is satisfied;
2 $Q(\omega_a(x))=0$ when $\omega_a(x)=0, B_a|_{\omega_a(x)=0}=0, a=1,\ldots,s$;
3 $Q^{(k)}(B_a(x,u,\partial u,\ldots,\partial^{k-1}u))=0$ when $\omega_a(x)=0$ and $B_a|_{\omega_a(x)=0}=0, a=1,\ldots,s$;
4 There exists a smooth bijective transform (14) mapping M into M^* of the same dimensionality;
5 $Q^*(\gamma_c^*(y))=0$ when $\gamma_c^*(y)=0, c=1,2,\ldots,p_\infty$;
6 $(Q^*)^{(k_c^*)}(\gamma_c^*(y,u,\partial u,\ldots,\partial^{k_c^*}u))=0$ when $\gamma_c^*(y)=0$ and $\gamma_c^*|_{\gamma_c^*(y)=0}=0, c=1,2,\ldots,r$.

This definition coincides with Definition 1 when Q is a Lie symmetry operator and there is not any boundary condition defined at infinity.

Lie symmetry analysis of the problem with sine-Gordon equation

Let us consider the nonlinear hyperbolic problem for sine-Gordon equation

$$u_{tt}-u_{xx}=\sin u, t>0, -\infty<x<\infty, \tag{15}$$

$$u(0,x)=\varphi(x), \tag{16}$$

$$u_t(0,x)=\psi(x). \tag{17}$$

Equation (15) admits three-dimensional Lie group [8] spanned by the operators

$$X_1 = \frac{\partial}{\partial t}, X_2 = \frac{\partial}{\partial x}, X_3 = x\frac{\partial}{\partial t} + t\frac{\partial}{\partial x}.$$

The operators generate one-parameter Lie groups

$$T_1 : t^* = t + \epsilon_1, x^* = x, u^* = u,$$

$$T_2 : t^* = t, x^* = x + \epsilon_2, u^* = u,$$

$$T_3 : t^* = t + x\epsilon_3, x^* = x + t\epsilon_3, u^* = u,$$

respectively. Since the group T_1 corresponds to translation on the variable t, the invariance of the boundary curve $t = 0$ is not preserved. Thus BVP (15)–(17) is not invariant with respect to the group T_1. For the invariance of boundary condition (16) with respect to the symmetry group T_2, the equations

$$t^*|_{t=0} = 0, [u^* - \varphi(x^*)]|_{u-\varphi(x)=0} = 0 \tag{18}$$

must be satisfied. The first equation of (18) is an identity, while the second equation results

$$\varphi(x) = \varphi(x + \epsilon_2). \tag{19}$$

For the invariance of boundary condition (17) we need the first prolongation of the operator X_2. Using the prolongation formula for first-order derivatives

$$\begin{aligned} X^{(1)} = X &+ (\eta_t + u_t\eta_u - u_t(\xi^0_t + u_t\xi^0_u) - u_x(\xi^1_t + u_t\xi^1_u))\frac{\partial}{\partial u_t} \\ &+ (\eta_x + u_x\eta_u - u_t(\xi^0_x + u_x\xi^0_u) - u_x(\xi^1_x + u_x\xi^1_u))\frac{\partial}{\partial u_x}, \end{aligned} \tag{20}$$

where ξ^0, ξ^1 are infinitesimals with respect to the variables t and x respectively, we get

$$X_2^{(1)} = \frac{\partial}{\partial x}. \tag{21}$$

Applying this operator to condition (17), we have

$$t^*|_{t=0} = 0, [u^*_t - \psi(x^*)]|_{u_t-\psi(x)=0} = 0,$$

which gives

$$\psi(x) = \psi(x + \epsilon_2). \tag{22}$$

BVP (15)–(17) is invariant under the group of transformations T_2 if and only if equations (19) and (22) are satisfied. These equations result is that the functions $\varphi(x)$ and $\psi(x)$ are constant functions.

Following the same way, we obtain the invariance criterions of boundary condition (16) with respect to the symmetry group T_3 if the equations

$$\begin{gathered} t + x\epsilon_3 = 0 \text{ when } t = 0, \\ u - \varphi(x + t\epsilon_3) = 0 \text{ when } u - \varphi(x) = 0 \end{gathered}$$

are satisfied. The first equation results with $x = 0$ or $\epsilon_3 = 0$ that gives the trivial group. Hence we arrive at boundary condition (16), which is invariant under the group of transformations T_3 with restriction

$$x = 0, \varphi(x) = \varphi(x + t\epsilon_3). \tag{23}$$

To examine invariance of boundary condition (17) we apply the first prolongation of the operator X_3 which is obtained from formula (20)

$$X_3^{(1)} = x\frac{\partial}{\partial t} + t\frac{\partial}{\partial x} - u_x\frac{\partial}{\partial u_t} - u_t\frac{\partial}{\partial u_x} \tag{24}$$

to (17) and we get

$$x = 0, \psi(x) - \psi(x + t\epsilon_3) = u_x\epsilon_3. \tag{25}$$

Combining equations (23) and (25) we conclude that BVP (15)–(17) is invariant under the group of transformations T_3 with restriction $u_x(t, 0) = 0$ and conditions:

(i) when $t = 0$ the arbitrary functions are functions of x variable only, such that $\varphi(x)$ and $\psi(x)$,

(ii) when $t \neq 0$ then the arbitrary functions $\varphi(x)$ and $\psi(x)$ are constant functions.

Considering all situations presented above, we infer that BVP (15)–(17) admits two-parameter Lie group $T_2 \circ T_3$ that corresponds to symmetries $t^* = t + x\epsilon_3, x^* = x + t\epsilon_3 + \epsilon_2, u^* = u$ if and only if $\varphi(x)$ and $\psi(x)$ are constant functions.

Lie symmetry analysis of sine-Gordon equation in the difference scheme form

In this section we study the Lie point symmetries of difference model for nonlinear problem (15)–(17). Before we proceed, let us present some preliminaries and notations about transformation groups and prolongations in the space of discrete variables given in [13]. We denote the sequence space $(x, u, u_1, u_2, ...)$ by Z with

$$x = \left\{x^i \mid i = 1, 2, ..., n\right\},$$

$$u = \left\{u^k \mid k = 1, 2, ..., m\right\}.$$

We denote the set of mn first partial derivatives as $u_1 = \left\{u_i^k\right\}$, the set of second partial derivatives as $u_2 = \left\{u_{ij}^k\right\}$, etc. The formulas for the derivatives when $n = 2$ for $x = \left(x^1, x^2\right)$ are

$$D_1 = \frac{\partial}{\partial x^1} + u_1\frac{\partial}{\partial u} + u_{11}\frac{\partial}{\partial u_1} + u_{21}\frac{\partial}{\partial u_2} + ...,$$

$$D_2 = \frac{\partial}{\partial x^2} + u_2\frac{\partial}{\partial u} + u_{12}\frac{\partial}{\partial u_1} + u_{22}\frac{\partial}{\partial u_2} + ...,$$

where

$$u_1 = \frac{\partial u}{\partial x^1}, u_{11} = \frac{\partial^2 u}{\partial \left(x^1\right)^2}, u_{21} = \frac{\partial^2 u}{\partial x^2 \partial x^1}, ...$$

In the proofs, for simplicity, the superscript k on u^k is omitted. The two commuting Taylor groups [18] with finite transformations $T_a^1 = e^{aD_1}$ and $T_a^2 = e^{aD_2}$ are generated by the given operators. In one dimensional case the new coordinates

$$x^* = T_a\left(x\right) = x + a,$$

$$u^* = T_a\left(u\right) = \sum_{s=0}^{\infty} \frac{a^s}{s!} u_s,$$

$$u_1^* = T_a\left(u_1\right) = \sum_{s=0}^{\infty} \frac{a^s}{s!} u_{s+1},$$

$$\vdots$$

$$u_k^* = T_a\left(u_k\right) = \sum_{s=0}^{\infty} \frac{a^s}{s!} u_{s+k}$$

$$\vdots$$

are generated by the action of operator $T_a = e^{aD}$. Setting the arbitrary parameters $h_1, h_2 > 0$ the shift operators

$$\underset{\pm h}{S_1} = e^{\pm h_1 D_1} \equiv \sum_{s\geq 0} \frac{\left(\pm h_1\right)^s}{s!} D_1^s,$$

$$\underset{\pm h}{S_2} = e^{\pm h_2 D_2} \equiv \sum_{s\geq 0} \frac{\left(\pm h_2\right)^s}{s!} D_2^s$$

are obtained. Using these shift operators, two discrete differentiation operators

$$\underset{+h}{D_i} = \pm\frac{1}{h}(\underset{\pm h}{S_i} - 1), i = 1, 2$$

are obtained. In the (x^1, x^2)-plane, the set of points

$$\left\{\underset{\pm h_1}{S^\alpha}\left(x^1\right), \underset{\pm h_2}{S^\beta}\left(x^2\right)\right\}, \alpha, \beta = 0, 1, 2, ...$$

is called a uniform orthogonal difference mesh and denoted by $\underset{h}{\omega}$.

In two dimensional case with dependent variable u, independent variables t, x and mesh variables h_1, h_2 we denote the spaces of differential variables, difference variables and the product of those spaces which is the space of sequences of power series by

$$\tilde{Z} = (t, x, u, u_t, u_x, u_{tx}, \ldots),$$

$$\underset{h}{Z} = (t, x, u, \underset{h}{u_t}, \underset{h}{u_x}, \underset{h}{u_{tx}}, \ldots, h_1, h_2),$$

$$\underset{h}{\tilde{Z}} = (t, x, u, u_t, u_x, \ldots, \underset{h}{u_t}, \underset{h}{u_x}, \underset{h}{u_{tx}}, \ldots, h_1, h_2),$$

where

$$u_{ij} = \frac{\partial^2 u}{\partial x^i \partial x^j}, \underset{h}{u_{ij}} = \underset{+h}{D_j}\underset{+h}{D_i}(u), \ldots, \underset{h}{\omega} = \underset{h}{\omega_1} \times \underset{h}{\omega_2}$$

and $\underset{h}{\omega_i}$ is the difference mesh in the i-th direction, respectively.

Transformations in $\underset{h}{\tilde{Z}}$ is defined by the sequence of series with analytic coefficients,

$$z^{j*} = \sum_{s\geq 0} A_s^j(z)a^s, A_0^j = z^j,$$

where z^j is a coordinate of the vector $(t, x, u, u_t, u_x, \ldots, \underset{h}{u_t}, \underset{h}{u_x}, \underset{h}{u_{tx}}, \ldots)$ and these series form one-parameter groups generated by infinitesimal operators

$$X = \xi^t\frac{\partial}{\partial t} + \xi^x\frac{\partial}{\partial x} + \eta^k\frac{\partial}{\partial u^k} + \sum_{s\geq 1}\zeta_{i_1 i_2 \ldots i_s}\frac{\partial}{\partial u_{i_1 i_2 \ldots i_s}} + \sum_{l\geq 1}\zeta_{i_1 i_2 \ldots i_l}\frac{\partial}{\partial u_{i_1 i_2 \ldots i_l}}. \tag{26}$$

Prolongating the operator (26) for the variables h_1 and h_2 gives

$$\widetilde{X} = \cdots + h_1\underset{+h}{D_1}(\xi^t)\frac{\partial}{\partial h_1} + h_2\underset{+h}{D_2}(\xi^x)\frac{\partial}{\partial h_2}.$$

For first-order difference derivatives the coordinates of prolongation operator are given by the formulas

$$\underset{h}{\zeta_t} = \underset{+h}{D_1}(\eta) - \underset{h}{u_t}\underset{+h}{D_1}(\xi^t) - \underset{+h}{S_1}(u_x)\underset{+h}{D_1}(\xi^x), \tag{27}$$

$$\underset{h}{\zeta_x} = \underset{+h}{D_2}(\eta) - \underset{+h}{S_2}(u_t)\underset{+h}{D_2}(\xi^t) - \underset{h}{u_x}\underset{+h}{D_2}(\xi^x). \tag{28}$$

If the considered mesh is invariantly uniform or invariantly orthogonal, then the corresponding formulas for the invariant meshes must be satisfied in addition to prolongation formulas (27)–(28).

We presented the five-point difference scheme

$$\frac{\hat{u} - 2u + \check{u}}{h_1^2} - \frac{u_+ - 2u + u_-}{h_2^2} = \sin u \tag{29}$$

for sine-Gordon equation (15) on the uniform and orthogonal mesh

$$\hat{t} - 2t + \check{t} = 0, x_+ - 2x + x_- = 0$$

in our paper [3]. Here we denote mesh variables by h_1, h_2 and $\hat{t} = t + h_1, \check{t} = t - h_1, x_+ = x + h_2,$ $x_- = x - h_2, \hat{u} = u(\hat{t}, x), \check{u} = u(\check{t}, x), u_+ = u(t, x_+), u_- = u(t, x_-)$. We used the prolongation operator

$$prX = \xi^t \frac{\partial}{\partial t} + \xi^x \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial u} + \hat{\xi}^t \frac{\partial}{\partial \hat{t}} + \check{\xi}^t \frac{\partial}{\partial \check{t}} + \xi^x_+ \frac{\partial}{\partial x_+}$$

$$+\xi^x_- \frac{\partial}{\partial x_-} + \hat{\eta} \frac{\partial}{\partial \hat{u}} + \check{\eta} \frac{\partial}{\partial \check{u}} + \eta_+ \frac{\partial}{\partial u_+} + \eta_- \frac{\partial}{\partial u_-}$$

in the discrete subspace $(t, x, \hat{t}, \check{t}, x_+, x_-, u, \hat{u}, \check{u}, u_+, u_-)$ and obtained three-parameter transformation group generated by the operators

$$X_1 = \frac{\partial}{\partial t} + \frac{\partial}{\partial \hat{t}} + \frac{\partial}{\partial \check{t}}, X_2 = \frac{\partial}{\partial x} + \frac{\partial}{\partial x_+} + \frac{\partial}{\partial x_-},$$

$$X_3 = x \frac{\partial}{\partial t} + t \frac{\partial}{\partial x} + x \frac{\partial}{\partial \hat{t}} + x \frac{\partial}{\partial \check{t}} + t \frac{\partial}{\partial x_+} + t \frac{\partial}{\partial x_-}.$$

Difference equation (29) on the set of a finite number of points (x^k_n, t^k_n) can be expressed as

$$E_1 : \frac{u^{k+1}_n - 2u^k_n + u^{k-1}_n}{h_1^2} - \frac{u^k_{n+1} - 2u^k_n + u^k_{n+1}}{h_2^2} = \sin u^k_n \tag{30}$$

on the uniformly spaced orthogonal lattice

$$E_2 : t^{k+1}_n - t^k_n = h_1, E_3 : x^{k+1}_n - x^k_n = 0, \tag{31}$$

$$E_4 : t^k_{n+1} - t^k_n = 0, E_5 : x^k_{n+1} - x^k_n = h_2. \tag{32}$$

D. Levi et al. mentioned certain independence criteria for difference schemes in two dimensional case [14]. By this criteria one can calculate the values of (x, t, u) at all points beginning from the point (x^k_n, t^k_n) and a given number of neighboring points and assures the existence of solution of the system. The following condition on the Jacobian

$$|J| = \left| \frac{\partial(E_1, E_2, E_3, E_4, E_5)}{\partial(t^{k+1}_n, x^{k+1}_n, t^k_{n+1}, x^k_{n+1}, u^{k+1}_n)} \right| \neq 0 \tag{33}$$

is imposed by Levi et al. [14]. This condition allows to move upward and to the right along the curves passing through (x^k_n, t^k_n) (with either k or n fixed). Difference scheme (30)–(32) satisfy certain independence criteria (33) by

$$t^k_n = h_1 k + t_0, x^k_n = h_2 n + x_0.$$

In this step, using difference equation (29) for the BVP (15)–(17), we write the difference problem

$$\frac{\hat{u} - 2u + \check{u}}{h_1^2} - \frac{u_+ - 2u + u_-}{h_2^2} = \sin u, \tag{34}$$

$$u^0_n = \varphi^h(x), \tag{35}$$

$$\frac{\hat{u}^1_n - u^0_n}{\tau^+} = \psi^h(x). \tag{36}$$

In this paper the notation $(t, x, u, \underset{h}{u_t}, \underset{h}{u_x}, \underset{h}{u_{tx}}, \ldots, h_1, h_2)$ for difference variables in two-dimensional case is used for simplicity. Using these symbols, we rewrite the difference model (34)–(36) in the following form

$$\underset{h}{u_{tt}} - \underset{h}{u_{xx}} = \sin u, \tag{37}$$

$$u(0, x) = \varphi^h(x), \tag{38}$$

$$\underset{h}{u_t}(0, x) = \psi^h(x). \tag{39}$$

Difference equation (37) admits three-parameter groups generated by operators [3]

$$X_1 = \frac{\partial}{\partial t}, X_2 = \frac{\partial}{\partial x}, X_3 = x \frac{\partial}{\partial t} + t \frac{\partial}{\partial x}. \tag{40}$$

Lie symmetry groups corresponding to the translation about time variable is described by the operator X_1, to the translation in space variable is described by the operator X_2, and to the rotation is described by the operator X_3. The difference scheme (37)–(39) does not admit symmetry group generated by X_1, because time translation violates invariance of the boundary surface $t = 0$.

The invariance of boundary surface $t = 0$ generated by the operator X_2 with respect to the transformation group is trivial. Under the symmetry of space translation X_2, boundary condition (38) is invariant if the equation

$$u - \varphi^h(x + \epsilon_2) = 0 \text{ for } u - \varphi^h(x) = 0$$

is satisfied. From that it follows the condition

$$\varphi^h(x) = \varphi^h(x + \epsilon_2). \tag{41}$$

For the invariance of boundary condition (39) we require the first-order prolongation formulas in space of discrete variables. From (21) we know the coordinates for continuous derivatives in the prolongation of the operator X_2 are zero. Using formulas (27)-(28), we obtain the coordinates of first-order difference derivatives

$$\underset{h}{\zeta_t} = \underset{+h}{D_1}(0) - \underset{h}{u_t}\underset{+h}{D_1}(0) - \underset{+h}{S_1}(u_x)\underset{+h}{D_1}(1) = 0,$$

$$\underset{h}{\zeta_x} = \underset{+h}{D_2}(0) - \underset{+h}{S_2}(u_t)\underset{+h}{D_2}(0) - \underset{h}{u_x}\underset{+h}{D_2}(1) = 0$$

for the operator X_2 with $\eta = 0, \xi^t = 0, \xi^x = 1$. In this case $\underset{h}{X_2}^{(1)} = X_2$ where $\underset{h}{X_2}^{(1)}$ is the first prolongation of the operator X_2 in discrete space. Applying this prolongation to condition (39) we get the criterion

$$\psi^h(x) = \psi^h(x + \epsilon_2). \tag{42}$$

In consequence of combining criterions (41) and (42) one can say that difference scheme (37)–(39) is invariant with respect to the transformation group defined by the operator X_2 if and only if $\varphi^h(x)$ and $\psi^h(x)$ are constant functions. Using the same procedure, we obtain the invariance criterion of condition (38) under the rotation group spanned by the operator X_3 as

$$t + x\epsilon_3 = 0 \text{ when } t = 0,\ u - \varphi^h(x + t\epsilon_3) = 0 \text{ when } u - \varphi^h(x) = 0$$

which results

$$x = 0, \varphi^h(x) = \varphi^h(x + t\epsilon_3). \tag{43}$$

Under the symmetry group generated by this operator we need to prolong operator (24) for the first-order difference derivatives in order to analyze invariance of condition (39). Substituting $\eta = 0$,
$\xi^t = x, \xi^x = t$ in the operator X_3 in (27)–(28), we obtain the coefficients

$$\underset{h}{\zeta_t} = -u_x, \underset{h}{\zeta_x} = -u_t$$

and the prolongation operator

$$\underset{h}{X_3}^{(1)} = x\frac{\partial}{\partial t} + t\frac{\partial}{\partial x} - u_x\frac{\partial}{\partial u_t} - u_t\frac{\partial}{\partial u_x} - u_x\frac{\partial}{\partial \underset{h}{u_t}} - u_t\frac{\partial}{\partial \underset{h}{u_x}}.$$

This operator generates the group $t^* = t + x\epsilon_3, x^* = x + t\epsilon_3, u^* = u, u_t^* = u_t - u_x\epsilon_3, u_x^* = u_x - u_t\epsilon_3$,
$\underset{h}{u_t^*} = \underset{h}{u_t} - u_x\epsilon_3, \underset{h}{u_x^*} = \underset{h}{u_x} - u_t\epsilon_3$. Applying the operator to boundary condition (39) gives

$$t + x\epsilon_3 = 0 \text{ for } t = 0,$$
$$\underset{h}{u_t} - u_x\epsilon_3 - \psi^h(x + t\epsilon_3) = 0 \text{ when } \underset{h}{u_t} - \psi^h(x) = 0$$

and consequently

$$x = 0, \psi^h(x) - \psi^h(x + t\epsilon_3) = u_x\epsilon_3. \tag{44}$$

From equations (43) and (44) under the group of transformations X_3 and with the restriction $u_x(t, 0) = 0$ we conclude that difference scheme (37)–(39) is invariant in two cases:

(1) if $t = 0$ for all arbitrary functions $\varphi^h(x)$ and $\psi^h(x)$,
(2) if $t \neq 0$ then $\varphi^h(x)$ and $\psi^h(x)$ are constant functions.

Remark. Note that in the prolongation operators $\underset{h}{X_2}^{(1)}$ and $\underset{h}{X_3}^{(1)}$ we omit the coordinates for the mesh variables. Indeed, substituting the infinitesimals $\eta = 0, \xi^t = 0$ of the operator X_2 in $\underset{+h}{D_1}(\xi^t)$ and $\underset{+h}{D_2}(\xi^x)$ gives zero.

For the operator X_3 the infinitesimals are $\eta = 0, \xi^t = x, \xi^x = t$ and we calculate $\underset{+h}{D_1}(\xi^t)$ and $\underset{+h}{D_2}(\xi^x)$ as

$$\underset{+h}{D_1}(x) = \frac{1}{h}(\underset{h}{S_1} - 1) = (D_1 + \frac{h_1}{2!}D_1^2 + \cdots)(x) = 0,$$

$$\underset{+h}{D_2}(t) = \frac{1}{h}(\underset{h}{S_2} - 1) = (D_2 + \frac{h_1}{2!}D_2^2 + \cdots)(t) = 0,$$

where

$$D_1 = \frac{\partial}{\partial t} + u_t\frac{\partial}{\partial u} + u_{tt}\frac{\partial}{\partial u_t} + u_{xt}\frac{\partial}{\partial u_x} + ...,$$
$$D_2 = \frac{\partial}{\partial x} + u_x\frac{\partial}{\partial u} + u_{tx}\frac{\partial}{\partial u_t} + u_{xx}\frac{\partial}{\partial u_x} + ...$$

Conclusion

Some results discussed in this paper are as follows. We have investigated the BVP for sine-Gordon equation in differential and difference cases which are defined on an unbounded domain and lattice respectively. We obtain the invariance conditions for the problems under the group of transformations admitted by continuous and discrete sine-Gordon equation by applying the invariance definition in [16]. The transformations act on the difference scheme, lattices, and boundary conditions and preserve uniformity and orthogonality of the lattice. We used the prolongation formulas in discrete space which are formulated by Dorodnitsyn in [13] and analyze the invariance of the boundary conditions with derivative. On this basis we conclude that difference scheme (37)–(39) is invariant under the same restrictions of differential form (15)–(17) with respect to the symmetry groups generated by (40).

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Дифференциалдық және айырымдық теңдеулері үшін шеттік есептердегі Лидің симметриялары туралы

Жалпы Ли топтары теориясының сипатына байланысты симметрияны талдау шеттік есептерге емес, жеке теңдеулерге қолданылады. Мақалада Лидің нүктелік симметриялар тобына қатысты синус-Гордон теңдеулері үшін шеттік есептер дифференциалдық және айырымдық түрлерінде алынды. Шеттік есептердің және олардың шешімдерінің инварианттық шарттары анықталған. Синус-Гордон теңдеуі үшін шеттік есепке сәйкес келетін айырымдық есептің инвариантты дискретизациясы зерттелді. Дифференциалдық жағдайда шексіз облыс, ал айырымдық жағдайда — жазықтықта орналасқан және барлық бағытта шекарасыз созылатын нүктелері бар тор қарастырылған.

Кілт сөздер: симметрияны талдау, дербес туындылы теңдеулер, айырымдық теңдеулері, шеттік есептер.

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О лиевских симметриях в краевых задачах для дифференциальных и разностных уравнений

Ввиду природы теории групп Ли анализ симметрии применяется к отдельным уравнениям, а не к краевым задачам. В статье краевые задачи для уравнений синус-Гордон относительно группы точечных симметрий Ли получены как в дифференциальной, так и в разностной форме. Определены условия инвариантности краевых задач и их решений. Исследована инвариантная дискретизация разностной задачи, соответствующей краевой задаче для уравнения синус-Гордон. В дифференциальном случае рассмотрена неограниченная область, а в разностном — решетка с точками, лежащими в плоскости и тянущимися во всех направлениях без границ.

Ключевые слова: анализ симметрии, уравнения в частных производных, разностные уравнения, краевые задачи.

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