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МАТЕМАТИКА
MATHEMATICS

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Strong approximation of Fourier series on generalized periodic Morrey spaces

In recent years, a lot of attention has been paid to study of Morrey type spaces. Many applications in partial differential equation of Morrey spaces and Lizorkin-Triebel spaces have been given in work G.Di Fazioand, M. Ragusa and the book of T. Mizuhara. The theory of generalized Triebel-Lizorkin-Morrey spaces is developed. Generalized Morrey spaces, with T. Mizuhara and E. Nakai proposed, are equipped with a parameter and a function. First we give definition of Morrey and generalized Morrey spaces. Then we recall the boundedness of periodic Hilbert transform. This will be our main tool for all wtht follows. In a more or less elementary way, we carry over the known boundedness assertions for the Hilbert transform on Morrey spaces defined on $\mathbb{R}$ to periodic Morrey spaces. Boundedness of the Hilbert transform implies uniform estimates of the operator norms of the partial sumd of the Fourier series. Then we study vector-valued Fourier-multiplier theorem for smooth multipliers. Afterwards, we study vector valued version of famous Riesz theorem. Here we concentrate on Lizorkin representations. Finally, we get an interesting characterization of the space $\mathcal{E}^s_{\varphi,p,q}(T)$ by using differences of partial sums of the Fourier series Finally, we get an interesting characterization of the space $\mathcal{E}^s_{\varphi,p,q}(T)$ by using differences of partial sums of the Fourier series and consequence for strong approximation of Fourier series on Morrey space.

Keywords: Morrey spaces, generalized periodic Morrey spaces, strong approximation, vector-valued version of the Riesz theorem.

First we recall the definition of the generalized Morrey spaces (nonperiodic and periodic). As usual, $B(x,r)$ denotes the open interval $(x-r,x+r)$. $\mathbb{T}$ denotes the one-dimensional torus, usually identified with $[-\pi,\pi]$. For a measurable set $\Omega\subset\mathbb{R}$ we use $|\Omega|$ to denote the Lebesgue measure of Ω. As usual, $\mathbb{N}$ denotes the natural numbers, $\mathbb{N}_0$ the natural numbers including 0. All functions are assumed to be complex-valued, i.e., we consider functions $f:\ \mathbb{R}\to\mathbb{C}$. As usual, the symbols $C, C_1,\dots, A, B, C,\dots$ denote positive constants which depend only on the fixed parameters s,p,q and λ and probably on auxiliary functions, unless otherwise stated; its value may vary from line to line.

Definition 1. Let $0<p\le\infty$ and $0\le\lambda\le 1/p$.

(i) We say that a function $f:\ \mathbb{R}\to\mathbb{C}$ belongs to the (nonperiodic) Morrey space $M_p^\lambda(\mathbb{R})$ if $f\in L_p(B(x,r))$ for all $x\in\mathbb{R}$ and all $r>0$, and the following expression is finite

$$\| f\,|M_p^\lambda(\mathbb{R})\| := \sup_{x\in\mathbb{R}}\sup_{r>0} |B(x,r)|^{-\lambda}\, \|f|L_p(B(x,r))\|\,.$$

(ii) We say that a function $f:\ \mathbb{R}\to\mathbb{C}$, 2π-periodic, belongs to the periodic Morrey space $M_p^\lambda(\mathbb{T})$ if $f\in L_p(B(x,r))$ for all $x\in\mathbb{R}$ and all $r>0$ and the following expression is finite

$$\| f\,|M_p^\lambda(\mathbb{T})\| := \sup_{x\in\mathbb{R}}\sup_{0<r\le\pi} |B(x,r)|^{-\lambda}\, \|f|L_p(B(x,r))\|\,. \tag{1}$$

Remark 1.

(i) Obviously we have

$$M_p^0(\mathbb{R}) = L_p(\mathbb{R}) \qquad \text{and} \qquad M_p^{1/p}(\mathbb{R}) = L_\infty(\mathbb{R})$$

in the sense of equivalent norms. In the periodic case we obtain

$$M_p^0(\mathbb{T}) = L_p(\mathbb{T}) \qquad \text{and} \qquad M_p^{1/p}(\mathbb{T}) = L_\infty(\mathbb{T})$$

in the sense of equivalent norms.

(ii) Let us mention also the trivial embeddings

$$L_\infty(\mathbb{T}) \hookrightarrow M_p^{\lambda_0}(\mathbb{T}) \hookrightarrow M_p^{\lambda_1}(\mathbb{T}) \hookrightarrow L_p(\mathbb{T}) \,, \qquad \lambda_1 \le \lambda_0$$

and

$$L_u(\mathbb{T}) = M_u^0(\mathbb{T}) \hookrightarrow M_{p_1}^{\frac{1}{p_1}-\frac{1}{u}}(\mathbb{T}) \hookrightarrow M_{p_2}^{\frac{1}{p_2}-\frac{1}{u}}(\mathbb{T}) \,, \qquad 0 < p_2 \le p_1 \le u < \infty \,.$$

(iii) Morrey spaces, periodic or not, are not separable if $\lambda > 0$. In some sense these Morrey spaces are relatives of L_∞, and not of L_p, $p < \infty$.

(iv) By periodicity it will be enough to restrict the supremum in (1) to $x \in [-\pi, \pi]$.

Generalized Morrey spaces have been introduced independently by Mizuhara [1] and by Nakai [2]. Here the parameter λ is replaced by a function $\varphi : (0,\infty) \to (0,\infty)$.

Definition 2. Let $0 < p < \infty$ and let $\varphi : (0,\infty) \to (0,\infty)$.

(i) Then the generalized Morrey space $M_p^\varphi(\mathbb{R})$ is the collection of all functions $f : \mathbb{R} \to \mathbb{C}$ such that $f \in L_p(B(x,r))$ for all $x \in \mathbb{R}$ and all $r > 0$ and

$$\|f|M_p^\varphi(\mathbb{R})\| := \sup_{x\in\mathbb{R}} \sup_{0<r<\infty} \varphi(r) \Big(\frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|^p dy\Big)^{\frac{1}{p}} < \infty \,.$$

(ii) Then the generalized periodic Morrey space $M_p^\varphi(\mathbb{T})$ is the collection of all functions $f : \mathbb{R} \to \mathbb{C}$, 2π-periodic, such that $f \in L_p(B(x,r))$ for all $x \in \mathbb{R}$ and all $r > 0$ and

$$\|f|M_p^\varphi(\mathbb{T})\| := \sup_{x\in\mathbb{R}} \sup_{0<r\le\pi} \varphi(r) \Big(\frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|^p dy\Big)^{\frac{1}{p}} < \infty \,.$$

Remark 2. Clearly, if $\varphi(r) := |B(0,r)|^{-\lambda+\frac{1}{p}}$, $r > 0$, then we have coincidence $M_p^\varphi(\mathbb{T}) = M_p^\lambda(\mathbb{T})$, in particular, if $\varphi(r) := |B(0,r)|^{\frac{1}{p}}$, $r > 0$, then $M_p^\varphi(\mathbb{T}) = L_p(\mathbb{T})$.

Of course, we shall need restrictions for φ to develop a reasonable theory. We are mainly interested in smooth peturbations of $|B(0,r)|^{-\lambda+\frac{1}{p}}$. Following Nakai [3] we shall work with the following class of functions.

Definition 3. Let $0 < p < \infty$. Then $\varphi : (0,\infty) \to (0,\infty)$ belongs to the class $\mathcal{G}_p$, if φ there exist positive constants C, C' such that the inequalities

$$\varphi(t_1) \le C\,\varphi(t_2) \qquad \textit{and} \qquad t_2^{-\frac{d}{p}}\,\varphi(t_2) \le C'\,t_1^{-\frac{d}{p}}\,\varphi(t_1)$$

hold for all $0 < t_1 \le t_2 < \infty$.

In the definition of $M_p^\varphi(\mathbb{T})$, we assume that φ is in $\mathcal{G}_p$, that is, there exist some constants $C, C^{'} > 0$ such that the inequalities

$$\varphi(t_1) \le C\varphi(t_2) \quad \text{and} \quad C^{'} t_1^{-\frac{d}{p}}\varphi(t_1) \ge t_2^{-\frac{d}{p}}\varphi(t_2)$$

hold for $0 < t_1 \le t_2 < \infty$.

The nonperiodic and the periodic Hilbert transform are classical objects in harmonic analysis. We give their definitions.

Definition 4. The Hilbert transform of a given function $f \in L_1^{loc}(\mathbb{R})$ is defined as the Cauchy principal value

$$(Tf)(x) := P.V. \int_{-\infty}^{+\infty} \frac{f(t)}{x-t} dt = \lim_{\delta\downarrow 0} \int_{\{t:|t-x|>\delta\}} \frac{f(t)}{x-t} dt$$

at every point $x \in \mathbb{R}$ for which this limit exists.

Definition 5. The periodic Hilbert transform (or conjugate function) of a periodic function $f \in L_1(\mathbb{T})$ is defined as the Cauchy principal value

$$(Hf)(x) := \widetilde{f}(x) = \frac{1}{2\pi} P.V. \int_{-\pi}^{\pi} f(x-u) \cot \frac{u}{2} du =$$

$$= \frac{1}{2\pi} \lim_{\delta \downarrow 0} \int_{\delta < |u| < \pi} f(x-u) \cot \frac{u}{2} du$$

at every point $x \in \mathbb{R}$ for which this limit exists.

We recall the boundedness of the Hilbert transform on generalized periodic Morrey spaces, which have been proved in [4].

Proposition 1 [4, Theorem 2]. Let $1 < p, q, < \infty$ and $\varphi \in G_p$. We assume that there exist some $\varepsilon > 0$ and a constant $C > 0$ such that

$$\frac{t^\varepsilon}{\varphi(t)} \leq C \, \frac{r^\varepsilon}{\varphi(r)} \qquad \text{for all} \quad t \geq r > 0 \, . \tag{2}$$

Then there exists a constant C such that

$$\Big\| \Big(\sum_{j=0}^{\infty} |Hf_j|^q \Big)^{1/q} \Big| M_p^\varphi(\mathbb{T}) \Big\| \leq C \Big\| \sum_{j=0}^{\infty} |f_j|^q \Big)^{1/q} \Big| M_p^\varphi(\mathbb{T}) \Big\|$$

holds for all $f \in M_p^\varphi(\mathbb{T})$.

A standard consequence of the boundedness of the periodic Hilbert transform is an estimate of the operator norm of the partial sum operator of the Fourier series.

$$S[f](x) := \sum_{k=-\infty}^{\infty} c_k e^{ikx}, \qquad x \in \mathbb{R}$$

be the Fourier series $f \in L_1(\mathbb{T})$ with $c_k(f)$. Here $c_k(f)$ is the Fourier coefficient of f given by

$$c_k(f) := \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-ikt} dt.$$

We define

$$S_{N,M}[f](x) := \sum_{k=-N}^{M} c_k(f) e^{ikx}, \qquad N, M \in \mathbb{Z}, \quad N \leq M.$$

The next corollary is generalization of the famous Riesz theorem.

Corollary 1 [4, Theorem 5]. Let $1 < p < \infty$ and a function $\varphi \in G_p$, $\varphi \in \mathcal{G}_p$. We assume that there exist some $\varepsilon > 0$ and a constant $C > 0$ such that

$$\frac{t^\varepsilon}{\varphi(t)} \leq C \, \frac{r^\varepsilon}{\varphi(r)} \qquad \text{for all} \quad t \geq r > 0 \, .$$

Then for all $N, M \in \mathbb{Z}, \quad N \leq M$, we have

$$\|S_{N,M} | M_p^\varphi(\mathbb{T}) \to M_p^\varphi(\mathbb{T})\| \leq \|H | M_p^\varphi(\mathbb{T}) \to M_p^\varphi(\mathbb{T})\|. \tag{3}$$

To reach our goal we need following vector-valued version of Corollary 1.

Theorem 1 [5, Theorem 1]. Let $1 < p < \infty$ and a function $\varphi \in G_p$, $\varphi \in \mathcal{G}_p$. We assume that there exist some $\varepsilon > 0$ and a constant $C > 0$ such that

$$\frac{t^\varepsilon}{\varphi(t)} \leq C \, \frac{r^\varepsilon}{\varphi(r)} \qquad \text{for all} \quad t \geq r > 0 \, .$$

Then for all $(N_j)_j, (M_j)_j$ of complex numbers, satisfying $N_j \leq M_j$ for all j, and all sequences $(f_j)_j \subset M_p^\varphi(\mathbb{T})$ we have

$$\Big\| \Big(\sum_{j=0}^{\infty} |S_{N_j,M_j} f_j | M_p^\varphi(\mathbb{T}) \Big\| \leq C \Big(\|H | M_p^\varphi(\mathbb{T}) \to M_p^\varphi(\mathbb{T})\| \Big) \Big\| \sum_{j=0}^{\infty} |f_j|^q \Big) \frac{1}{q} | M_p^\varphi(\mathbb{T}) \Big\|. \tag{4}$$

Remark 3. In case $\varphi(r) := |B(0,r)|^{\frac{1}{p}}$, $r > 0$, the Theorem 1 has been known before, we refer to [6] and $\varphi(r) := |B(0,r)|^{-\lambda+\frac{1}{p}}$, $r > 0$, we refer to [7].

Looking at the Fourier side Corollary 1 and Theorem 1 can be interpreted as Fourier multipliers assertion with characteristic functions of intervals as multipliers. For later use, we need a variant with smooth multipliers. Smoothness is measured in terms of the Bessel potential spaces $H_2^\kappa(\mathbb{R})$. These classes are defined as follows.

Definition 6. Let $\kappa \geq 0$. Then $H_2^\kappa(\mathbb{R})$ is the collection of all $f \in L_2(\mathbb{R})$ such that

$$\|f|H_2^\kappa(\mathbb{R})\| := \|\mathcal{F}^{-1}[(1+|\xi|^2)^{\kappa/2}\mathcal{F}f(\xi)]|L_2(\mathbb{R})\| < \infty.$$

Theorem 2 [5, Theorem 2]. Let $1 \leq p < \infty$, $1 \leq q \leq \infty$ and $\varphi \in \mathcal{G}_p$. Assume (2) is satisfied. Let $(\Lambda_j)_j$ be a given family of finite and nontrivial intervals. With d_j we denote the length of Λ_j. Let

$$\kappa > \frac{1}{2} + \frac{1}{min(p,q)}.$$

Then there exists a constant C such that the inequality

$$\Big\|\Big(\sum_{j=0}^{\infty}\Big|\sum_{k=-\infty}^{\infty} M_j(k)c_k(f_j)e^{ikx}\Big|^q\Big)^{1/q}|M_p^\varphi(\mathbb{T})\Big\| \leq$$

$$\leq C \sup_{j=0,1,\dots} \|M_j(d_j\cdot)|H_2^\kappa(\mathbb{R})\| \, \|(f_j)_j|M_p^\varphi(\mathbb{T}, l_q)\|$$

holds for all sequences of functions $M_j \in H_2^\kappa(\mathbb{R})$ and all sequences $(f_j)_j$ of trigonometric polynomials such that

$$c_k(f_j) = 0 \qquad \text{if} \quad k \notin \Lambda_j, \qquad j \in \mathbb{N}_0.$$

Now let us to give the definition of generalized periodic Lizorkin-Triebel- Morrey spaces. In fact, for us the scale of the Lizorkin-Triebel-Morrey spaces will be more important one.

Let $\psi \in C_0^\infty(\mathbb{R})$ be a function such that

$$\psi(x) := \begin{cases} 1 & \text{if} \quad |x| \leq 1\,, \\ 0 & \text{if} \quad |x \geq \frac{3}{2}|. \end{cases} \tag{5}$$

Then, with $\phi_0 := \psi$,

$$\phi(x) := \phi_0(x/2) - \phi_0(x) \qquad \text{and} \qquad \phi_j(x) := \phi(2^{-j+1}x), \qquad j \in \mathbb{N}. \tag{6}$$

This implies

$$\sum_{j=0}^{\infty} \phi_j(x) = 1 \quad \text{for all} \quad x \in \mathbb{R}.$$

We shall call $(\phi_j)_{j=0}^\infty$ a smooth dyadic decomposition of unity.

Definition 7. Let $(\phi_j)_j$ be a smooth dyadic decomposition of unity as defined (5), (6). Let $s > 0$, $0 \leq q \leq \infty$, $0 < p < \infty$ and a function $\varphi \in \mathcal{G}_p$. Assume (2). Then $\mathcal{E}^s_{\varphi,p,q}(\mathbb{T})$ is defined to be the set of all $f \in M_p^\varphi(\mathbb{T})$ such that

$$\|f|\mathcal{E}^s_{\varphi,p,q}(\mathbb{T})\| := \Big\|\Big(\sum_{j=0}^{\infty} 2^{jsq}\Big|\sum_{k=-\infty}^{\infty} \phi_j(k)c_k(f)e^{ikx}\Big|^q\Big)^{1/q}\Big|M_p^\varphi(\mathbb{T})\Big\| < \infty.$$

Remark 4. Taking $\varphi(r) := |B(0,r)|^{\frac{1}{p}}$, $r > 0$, we are back in the case of classical periodic Lizorkin-Triebel spaces, i.e., we have

$$\mathcal{E}^s_{\varphi,p,q}(\mathbb{T}) = F^s_{p,q}(\mathbb{T}).$$

We shall call the spaces $\mathcal{E}^s_{\varphi,p,q}(\mathbb{T})$ generalized periodic Lizorkin-Triebel-Morrey spaces. They represent the Lizorkin-Triebel scale built on the generalized Morrey space $M_p^\varphi(\mathbb{T})$. The nonperiodic version of this scale of spaces has been introduced by Tang and Xu in the year 2005 (for Morrey spaces). Lizorkin-Triebel-Morrey spaces, related to generalized Morrey spaces, have been considered recently by Nakamura, Noi and Sawano [7].

Lemma 1 [8, Lemma 4]. Under the restrictions to parameters p, q, s, φ in Definition 7 the periodic Lizorkin-Triebel-Morrey spaces $\mathcal{E}^s_{\varphi,p,q}(\mathbb{T})$ is independent of the chosen smooth dyadic decomposition of unity, i.e., if we change the smooth dyadic decomposition of unity, then this change results in an equivalent quasi-norm.

We turn to an interesting characterization of the spaces $\mathcal{E}^s_{\varphi,p,q}(\mathbb{T})$ by using differences of partial sums of the Fourier series.

We define

$$S_N f(x) = S_{-N,N} f(x) = \sum_{k=-N}^{N} c_k(f) e^{ikx} \quad x \in \mathbb{R}, \quad N \in \mathbb{N}_0.$$

Here $c_k(f)$ is the Fourier coefficient of f given by

$$c_k(f) := \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-ikt} dt.$$

Theorem 3. Let $1 < p, q < \infty, s > 0$ and $\varphi \in \mathcal{G}_p$. Assume (2) is satisfied. A function $f \in M^\varphi_p(\mathbb{T})$ belongs to $\mathcal{E}^s_{\varphi,p,q}(\mathbb{T})$ if and only if

$$\|f|\mathcal{E}^s_{\varphi,p,q}(\mathbb{T})\|^* := \|S_1 f|M^\varphi_p(\mathbb{T})\| + \Big\| \Big(\sum_{j=0}^{\infty} 2^{jsq} |S_{2^{j+1}} f - S_{2^j} f|^q \Big)^{1/q} |M^\varphi_p(\mathbb{T}) \Big\| < \infty.$$

Furthermore the quantities $\|\cdot|\mathcal{E}^s_{\varphi,p,q}\|^*$ and $\|\cdot|\mathcal{E}^s_{\varphi,p,q}\|$ are equivalent on $M^\varphi_p(\mathbb{T})$, there exist two positive constants A,B such that for all $f \in M^\varphi_p(\mathbb{T})$,

$$A\,\|f|\mathcal{E}^s_{\varphi,p,q}\|^* \le \|f|\mathcal{E}^s_{\varphi,p,q}\| \le B\,\|f|\mathcal{E}^s_{\varphi,p,q}\|^*.$$

Proof of Theorem 3. We fix a dyadic decomposition of unity. Let $\psi \in C^\infty_0(\mathbb{R})$ be a function such that

$$\psi(x) := \begin{cases} 1 & \text{if} \quad |x| \le 1\,, \\ 0 & \text{if} \quad |x \ge \frac{3}{2}|. \end{cases}$$

Then, $\phi_0 : \psi(x)$ Then, with $\phi_0 := \psi$,

$$\phi(x) := \phi_0(x/2) - \phi_0(x) \qquad \text{and} \qquad \phi_j(x) := \phi(2^{-j+1}x), \qquad j \in \mathbb{N},$$

we have

$$\sum_{j=0}^{\infty} \phi_j(x) = 1 \qquad \text{for all} \quad x \in \mathbb{R}.$$

It will be convenient to use abbreviation

$$f_j(x) = \sum_{k=-\infty}^{\infty} \phi_j(k) c_k(f) e^{ikx}. \tag{7}$$

We find that

$$S_{2^{j+1}} f - S_{2^j} f = \sum_{2^j < |k| \le 2^{j+1}} c_k(f) e^{ikx} =$$

$$= \sum_{2^j < |k| \le 2^{j+1}} (\phi_{j-1}(k) + \phi_j(k) + \phi_{j+1}(k)) c_k(f) e^{ikx} =$$

$$= (S_{2^{j+1}} f_{j-1} - S_{2^j} f_{j-1}) + (S_{2^{j+1}} f_j - S_{2^j} f_j) + (S_{2^{j+1}} f_{j+1} - S_{2^j} f_{j+1}),$$

where we used abbreviation (7) again. Applying Theorem and generalized Minkowski inequality, we have

$$\Big\| \Big(\sum_{j=0}^{\infty} 2^{jsq} |S_{2^{j+1}} f - S_{2^j} f|^q \Big)^{1/q} |M^\varphi_p(\mathbb{T}) \Big\| \le \sum_{l=-1}^{1} \Big\| \Big(\sum_{j=0}^{\infty} 2^{jsq} |S_{2^{j+1}} f_{j+l} - S_{2^j} f_{j+l}|^q \Big)^{1/q} |M^\varphi_p(\mathbb{T}) \Big\| \le$$

$$\leq C_1 \sum_{l=-1}^{1} \Big\| \Big(\sum_{j=0}^{\infty} 2^{jsq} |f_{j+l}|^q \Big)^{1/q} |M_p^{\varphi}(\mathbb{T}) \Big\| \leq c_2 \| f | \mathcal{E}^s_{\varphi,p,q} \|.$$

The estimate of the term $\|S_1 f | M_p^{\varphi}(\mathbb{T})\|$ can be done in the same manner:

$$\|S_1 f | M_p^{\varphi}(\mathbb{T})\| \leq (\|H|M_p^{\varphi}(\mathbb{T}\| + 1)\|f|M_p^{\varphi}(\mathbb{T}\| = C_3 \|f|M_p^{\varphi}(\mathbb{T})\|.$$

This prove that $A\|f|\mathcal{E}^s_{\varphi,p,q}\|^* \leq \|f|\mathcal{E}^s_{\varphi,p,q}\|$ with an appropriate positive constant A. To prove

$$\|f|\mathcal{E}^s_{\varphi,p,q}\| \leq B\, \|f|\mathcal{E}^s_{\varphi,p,q}\|^*$$

we proceed similarly. Let $j \geq 1$. For brevity, we put

$$g_j := S_{2^{j+1}} f - S_{2^j} f.$$

We start with the identity

$$f_j(x) = \sum_{2^{j-1} < |k| < 3 2^{j-1}} \phi_j(k) c_k(f) e^{ikx}.$$

In the strength of the system's orthonormality $e^{ikx}{}_{k=-\infty}^{+\infty}$ on $[-\pi, \pi]$, we have

$$c_k(g_j) = \frac{1}{2\pi} \int_{-\pi}^{\pi} g_j(x) e^{-ikx} dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} \Big(\sum_{2^j < |\nu| \leq 2^{j+1}} c_\nu(f) e^{i\nu x} \Big) e^{-ikx} dx = c_k(f).$$

Therefore

$$f_j(x) = \sum_{2^{j-1} < |k| < 3 2^{j-1}} \phi_j(k) c_k (g_{j-1} + g_j + g_{j+1}) e^{ikx} =$$

$$= \sum_{l=-1}^{1} \sum_{k=-\infty}^{\infty} \phi_j(k) c_k(g_{j+l}) e^{ikx}.$$

Hence

$$\|f|\mathcal{E}^s_{\varphi,p,q}\| = \Big\| \Big(\sum_{j=0}^{\infty} 2^{jsq} |f_j(x)|^q \Big)^{1/q} \Big| M_p^{\varphi}(\mathbb{T}) \Big\| =$$

$$= \Big\| \Big(\sum_{j=0}^{\infty} 2^{jsq} \Big| \sum_{l=-1}^{1} \sum_{k=-\infty}^{\infty} \phi_j(k) c_k(g_{j+l}) e^{ikx} \Big|^q \Big)^{\frac{1}{q}} \Big| M_p^{\varphi}(\mathbb{T}) \Big\| =$$

$$= \Big\| \Big(\sum_{j=0}^{\infty} \Big| \sum_{l=-1}^{1} \sum_{k=-\infty}^{\infty} \phi_j(k) c_k(2^{js} g_{j+l}) e^{ikx} \Big|^q \Big)^{\frac{1}{q}} \Big|.$$

Then argue by using Theorem 2 instead of Theorem 1 to the last sum, considering

$$M_j(k) = \phi_j(k); \qquad f_j(k) = 2^{js} g_j(x); \qquad c_k(f_j) = c_k(2^{js} g_j); \qquad \Lambda_j = (2^j, 2^{j+1}).$$

Then we get

$$\Big\| \Big(\sum_{j=0}^{\infty} \Big| \sum_{l=-1}^{1} \sum_{k=-\infty}^{\infty} \phi_j(k) c_k(2^{js} g_{j+l}) e^{ikx} \Big|^q \Big)^{\frac{1}{q}} \Big| \leq$$

$$\leq C \cdot \sup_{j=o,1,2...} \|\phi_j(d_j \cdot)|H_2^{\kappa}(\mathbb{R})\| \cdot \|2^{js}(g_j)_j | M_p^{\varphi}(\mathbb{T}, l_q)\| =$$

$$= B \cdot \Big\| \Big(\sum_{j=0}^{\infty} \Big| 2^{js} (g_j)_j \Big|^q \Big)^{\frac{1}{q}} \| M_p^{\varphi}(\mathbb{T}) \Big\| =$$

$$= B \cdot \Big\| \Big(\sum_{j=0}^{\infty} \Big| 2^{jsq} \Big| S_{2^{j+1} f - S_{2^j} f} \Big|^q \Big)^{\frac{1}{q}} \| M_p^{\varphi}(\mathbb{T}) \Big\|.$$

Remark 5. In case $\varphi(r) := |B(0,r)|^{1/p}, r > 0$ this goes back to Lizorkin-Triebel and in case $\varphi(r) := |B(0,r)|^{-\lambda+1/p}$, $r > 0$ this theorem have been proved in [7].

To prove next theorem we add the following.

Corollary 2. [8]. Let $1 < p < \infty$, $1 \le q \le \infty$, $s > 0$ and a function $\varphi \in G_p$. Then for all $f \in \mathcal{N}^s_{\varphi,p,q}(\mathbb{T}) \cup \mathcal{E}^s_{\varphi,p,q}(\mathbb{T})$,

$$\lim_{j\to\infty} \|S_{2^j} f - f|M^{\varphi}_p(\mathbb{T})\| = 0.$$

Now we are going to reach our goal in this paper. The next theorem is basic in this paper. Instead of considering dyadic subsequences of $(S_N)_N$, we switch now to the sequence $(f - S_N)_N$.

Theorem 4. Let $1 < p, q < \infty$, $s > 0$ and a function $\varphi \in \mathcal{G}_p$. A function $f \in M^{\varphi}_p(\mathbb{T})$ belongs to $\mathcal{E}^s_{\varphi,p,q}(\mathbb{T})$ if and only if

$$\|f|\mathcal{E}^s_{\varphi,p,q}\|^{\#} :=$$

$$:= \|S_1 f|M^{\varphi}_p(\mathbb{T})\| + \left\|\Big(\sum_{N=1}^{\infty} N^{(s-1/q)q}|f - S_N f|^q\Big)^{1/q}|M^{\varphi}_p(\mathbb{T})\right\| < \infty. \tag{8}$$

Furthermore the quantities $\|\cdot|\mathcal{E}^s_{\varphi,p,q}\|^{\#}$ and $\|\cdot|\mathcal{E}^s_{\varphi,p,q}\|$ are equivalent on $M^{\varphi}_p(\mathbb{T}^d)$, there exist two positive constants A,B such that for all $f \in M^{\varphi}_p(\mathbb{T})$,

$$A\|f|\mathcal{E}^s_{\varphi,p,q}\|^{\#} \le \|f|\mathcal{E}^s_{\varphi,p,q}\| \le B\|f|\mathcal{E}^s_{\varphi,p,q}\|^{\#}.$$

Proof of Theorem 4. By Theorem 3, it will be sufficient to compare $\|\cdot|\mathcal{E}^s_{\varphi,p,q}\|^{\#}$ and $\|\cdot|\mathcal{E}^s_{\varphi,p,q}\|^{*}$.

Step 1. We shall prove that

$$\|f|\mathcal{E}^s_{\varphi,p,q}\|^{\#} \le c_1\|f|\mathcal{E}^s_{\varphi,p,q}\|^{*} \tag{9}$$

with some constant C_1 independent of f. First, we split the sum $\sum_{N=1}^{\infty}$ into dyadic blocks. More exactly, we set

$$\sum_{N=1}^{\infty} N^{(s-1/q)q}|f - S_N f|^q = \sum_{j=0}^{\infty}\sum_{N=2^j}^{2^{j+1}-1} N^{(s-1/q)q}|f - S_N f|^q \le$$

$$\le C_2\sum_{j=0}^{\infty} 2^{j(s-1/q)q}\sum_{N=2^j}^{2^{j+1}-1}|f - S_N f|^q,$$

where C_2 is equal to 1 if $s - 1/q \le 0$ and equal to $2^{(s-1/q)q}$ otherwise. Next, we use the identities

$$f - S_N f = f - S_{2^{j+1}} f + S_{2^{j+1}} f - S_N f$$

and

$$S_{2^{j+1}} f - S_N f = S_{N,2^{j+1}}\Big(S_{2^{j+1}} f - S_{2^j} f\Big).$$

Inserting these identities in the previous inequality and applying Theorem 1 yields

$$\left\|\Big(\sum_{N=1}^{\infty} N^{(s-1/q)q}|f - S_N f|^q\Big)^{1/q}|M^{\varphi}_p(\mathbb{T}^d)\right\| \le C_2\left\|\Big(\sum_{j=0}^{\infty} 2^{j(s-1/q)q}2^j|f - S_{2^{j+1}} f|^q\Big)^{1/q}|M^{\varphi}_p(\mathbb{T}^d)\right\| +$$

$$+C_2\left\|\Big(\sum_{j=0}^{\infty} 2^{j(s-1/q)q}\sum_{N=2^j}^{2^{j+1}-1}|S_{N,2^{j+1}}\Big(S_{2^{j+1}} f - S_{2^j} f|^q\Big)^{1/q}|M^{\varphi}_p(\mathbb{T}^d)\right\| +$$

$$+C_3\left\|\Big(\sum_{j=0}^{\infty} 2^{j(s-1/q)q}\sum_{N=2^j}^{2^{j+1}-1}|S_{2^{j+1}} f - S_{2^j} f|^q\Big)^{1/q}|M^{\varphi}_p(\mathbb{T}^d)\right\|.$$

Since

$$\left\|\Big(\sum_{j=0}^{\infty} 2^{j(s-1/q)q}\sum_{N=2^j}^{2^{j+1}-1}|S_{2^{j+1}} f - S_{2^j} f|^q\Big)^{1/q}|M^{\varphi}_p(\mathbb{T}^d)\right\| \le \left\|\Big(\sum_{j=0}^{\infty} 2^{jsq}|S_{2^{j+1}} f - S_{2^j} f|^q\Big)^{1/q}|M^{\varphi}_p(\mathbb{T}^d)\right\|,$$

it remains to estimate

$$\Big\|\Big(\sum_{j=0}^{\infty}2^{jsq}|f-S_{2^{j+1}}f|^q\Big)^{1/q}|M_p^{\varphi}(\mathbb{T}^d)\Big\|.$$

Obviously, as a consequence of Corollary 2, we have

$$f-S_{2^{j+1}}f=\lim_{M\to\infty}S_{2^M}f-S_{2^{j+1}}f=\lim_{M\to\infty}\sum_{l=j+1}^{M-1}S_{2^{l+1}}f-S_{2^l}f=$$

$$=\sum_{l=j+1}^{\infty}S_{2^{l+1}}f-S_{2^l}f=\sum_{l=1}^{\infty}S_{2^{l+j+1}}f-S_{2^{l+j}}f.$$

Hence it follows that

$$\Big\|\Big(\sum_{j=0}^{\infty}2^{jsq}|f-S_{2^{j+1}}f|^q\Big)^{1/q}|M_p^{\varphi}(\mathbb{T}^d)\Big\|\le$$

$$\le\sum_{l=1}^{\infty}\Big\|\Big(\sum_{j=0}^{\infty}2^{jsq}|S_{2^{l+j+1}}f-S_{2^{j+l}}f|^q\Big)^{1/q}|M_p^{\varphi}(\mathbb{T}^d)\Big\|\le$$

$$\le\sum_{l=1}^{\infty}2^{-ls}\Big\|\Big(\sum_{j=0}^{\infty}2^{(j+l)sq}|S_{2^{l+j+1}}f-S_{2^{l+j}}f|^q\Big)^{1/q}|M_p^{\varphi}(\mathbb{T}^d)\Big\|\le\|f|\mathcal{E}^s_{\varphi,p,q}\|^*\Big(\sum_{l=1}^{\infty}2^{-ls}\Big).$$

Since $s>0$, the geometric series is convergent. This completes the proof of (9).

Step 2. We shall prove that

$$\|f|\mathcal{E}^s_{\varphi,p,q}\|^*\le c_4\|f|\mathcal{E}^s_{\varphi,p,q}\|^{\#} \tag{10}$$

with some constant c_4 independent of f. This time we use the identity

$$S_{2^{j+1}}f-S_{2^j}f=\sum_{2^j<\|k\|_\infty\le 2^{j+1}}c_k(f-S_Nf)e^{ikx},\quad N=2^{j-1},...,2^j-1,$$

which implies that

$$|S_{2^{j+1}}f-S_{2^j}f|^q=2^{-(j-1)}\sum_{N=2^{j-1}}^{2^j-1}\Big|\sum_{2^j<\|k\|_\infty\le 2^{j+1}}c_k(f-S_Nf)e^{ikx}\Big|^q.$$

Hence it follows that

$$\Big\|\Big(\sum_{j=0}^{\infty}2^{jsq}|S_{2^{j+1}}f-S_{2^j}f|^q\Big)^{1/q}\Big|M_p^{\varphi}(\mathbb{T}^d)\|=$$

$$=\Big\|\Big(\sum_{j=0}^{\infty}2^{jsq}2^{-(j-1)}\sum_{N=2^j}^{2^j-1}\Big|\sum_{2^j<|k|\le 2^{j+1}}c_k(f-S_Nf)e^{ikx}\Big|^q\Big)^{1/q}|M_p^{\varphi}(\mathbb{T}^d)\Big\|\le$$

$$\le C_5\Big\|\Big(\sum_{j=0}^{\infty}2^{j(s-1/q)q}\sum_{N=2^j}^{2^j-1}|f-S_Nf|^q\Big)^{1/q}|M_p^{\varphi}(\mathbb{T}^d)\Big\|,$$

where we used Theorem 1 in the last step. Since

$$2^{j(s-1/q)q}\le C_6N^{(s-1/q)q},\quad 2^{j-1}\le N\le 2^j,$$

with some constant C_6 independent of j, this yields (10).

Remark 6. In case $\varphi(r):=|B(0,r)|^{1/p},r>0$ we refer [6] and in case $\varphi(r):=|B(0,r)|^{-\lambda+1/p},r>0$ this theorem have been proved in [9].

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А.Н. Адилханов, Ж.Ж. Байтуякова, Д.Т. Матин

Жалпыланған периодты Морри кеңістігінде Фурье қатарларымен күшті қосындылау

Соңғы жылдары Морри типті кеңістіктерді зерттеу үлкен қызығушылық тудырды. G.Di Fazioand, M. Ragusa жұмыстарында және T. Mizuhara кітабында Морри типтес кеңістіктердің дербес туындылы теңдеулердің жиі қолданулылары көрсетілген. T. Mizuhara және E. Nakai ұсынған жалпыланған Морри кеңістіктерінде параметр және функция енгізілген. Осы жұмыста Лизоркина-Трибель кеңістігі Морри кеңістіктегі шкалада жасалған. Алдымен, Морри мен жалпыланған Морри кеңістіктерінің анықтамасы берілді. Әрі қарай Гильберт түрлендіруінің шенелгендігі көрсетілді. Бұл біздің ең басты құралымыз. Периодты емес Морри кеңістігіндегі Гильберт түрлендіруінің шенелгендігін қолдана отырып, периодты Морри кеңістігіне өткіземіз. Гильберт түрлендіруінің шенелгендігі Фурье қатарының дербес қосындысының оператор нормасының бірқалыпты бағалауын білдіреді. Векторлы түрдегі Фурье мультипликаторы зерттелді. Осының негізінде белгілі Рисс теоремасының векторлы түрі алынады. Мұнда біз Лизоркин-Трибель кеңістіктерінің жалпыланған Морри кеңістігіндегісін қарастырамыз. Жұмыстың соңында Фурье қатарының дербес қосындысының айырмасын қолдана отырып, жалпыланған Трибель-Лизоркин-Морри кеңістігінде $\mathcal{E}^s_{\varphi,p,q}(T)$ қызықты сипаттамасы алынған және Морри кеңістігінде Фурье қатарының аппроксимациялауы көрсетілген.

Кілт сөздер: Морри кеңістігі, жалпыланған Морри кеңістігі, қатаң қосындылау, Рисс теоремасының векторлы түрі.

А.Н. Адилханов, Ж.Ж. Байтуякова, Д.Т. Матин

Сильная суммируемость рядов Фурье в обобщенных периодических пространствах Морри

В последние годы большое внимание уделялось изучению пространств типа Морри. В работах G.Di Fazioand, M. Ragusa и в книге T. Mizuhara дано множество приложений уравнений в частных производных в пространствах Морри и пространствах Лизоркина-Трибеля. Разработана теория обобщенных пространств Трибеля-Лизоркина-Морри. Обобщенные пространства Морри, предложенные T. Mizuhara и E.Nakai, снабжены параметром и функцией. Сначала дадим определение пространств Морри и обобщенных пространств Морри. Затем напомним ограниченность периодического преобразования Гильберта. Это будет нашим основным инструментом для всех, что следует. Более или менее элементарным образом мы переносим известные утверждения ограниченности для преобразования Гильберта на пространства Морри, определенные на $\mathbb{R}$ на периодических пространствах Морри. Ограниченность преобразования Гильберта подразумевает равномерные оценки операторных норм частичной суммы ряда Фурье. Затем мы изучаем векторнозначную теорему Фурье-множителя для гладких мультипликаторов. Впоследствии мы изучаем векторнозначную версию знаменитой теоремы Рисса. Здесь мы концентрируемся на представлениях Лизоркина. В настоящей работе изучены пространства Лизоркина-Трибеля на шкале обобщенного периодического пространства Морри. Наконец, мы получили интересную характеристику пространства $\mathcal{E}^s_{\varphi,p,q}(T)$, используя разности частичных сумм ряда Фурье и следствие для сильной аппроксимации рядов Фурье в пространствах Морри.

Ключевые слова: пространства Морри, обобщенные пространства Морри, сильная суммируемость, векторно-значная версия теоремы Рисса.

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Waves of elastic stresses in the doubly-connected domain

In this paper, we consider a nonstationary mixed problem of impact of a rigid plate on the face surface of the base body containing inside itself a central foreign inclusion. Formulated in terms of stresses and velocities of displacements, the mixed problem is modeled numerically by means of an explicit difference scheme of a through count, based on the method of the spatial characteristics. The results of a change in the wave fields in a doubly connected domain are presented in the form of graphs. The analysis of numerical information made it possible to investigate the dynamic effects of stress concentration in the neighborhood of the contact of a doubly-connected domain, as well as near the corner points of a foreign inclusion.

Keywords: elastic, wave process, stress, speed, foreign inclusion, numerical solution, boundary condition.

The leading factor determining the efficiency of modern engineering structures is their layered heterogeneous structure. In connection with this, the study of dynamic wave processes in deformable multiply connected media in order to determine the nature of possible damage has, in addition to purely scientific interest, an important applied value. In general, the number of works devoted to dynamic problems, taking into account a number of weakening factors, is very small, they do not consider all aspects of their performance under the conditions of nonstationary external loads [1–4]. However, the interest in these problems, primarily due to the importance of solving complex practical problems, is great, and further improvement of numerical methods in various modifications with the use of increasingly sophisticated computer technology should lead to a substantial development of this direction.

Statement of the problem. A strip with a rectangular cross-section of finite sizes, consisting of a main body and a central rectangular foreign inclusion, occupies a region in the Cartesian coordinate system $x_1 0 x_2$ (Fig. 1). The physical and mechanical properties of the body ($i = 1$) and the foreign inclusion ($i = 2$) are characterized by the density of the material ρ_i, the velocities of the longitudinal and transverse elastic waves a_i, b_i.

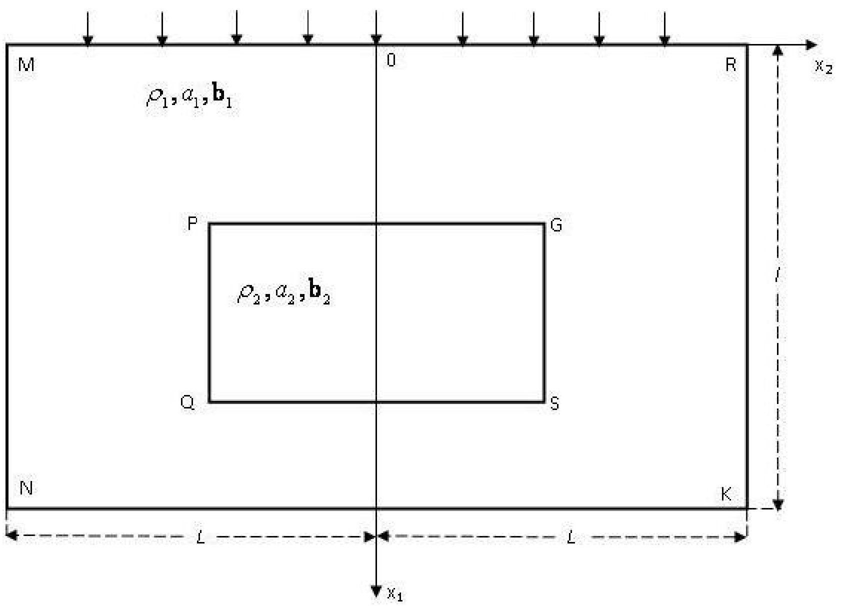

Figure 1. Study area

At the initial time $t = 0$, an absolute rigid body with a displacement velocity $v_1^{(1)} = f(t)$ and $v_2^{(1)} = 0$ strikes the outer boundary $x_1 = 0$, $|x_2| \leq l$ of the principal body in a static equilibrium state of the doubly-connected domain. The problem is to determine within the doubly-connected domain $0 \leq x_1 \leq \ell$, $|x_2| \leq L$ the stress and displacement velocity fields caused by fronts of incident and repeatedly reflected, refractive and diffracted elastic waves at time moments $t > 0$.

A mathematical model of the wave process at interior points of a doubly-connected domain is a linear system of dynamical equations of hyperbolic type containing as unknowns the dimensionless stresses $p^{(i)}, q^{(i)}, \tau^{(i)}$, speed of displacements v_1, v_2 [5]:

$$\rho_{(i)} \cdot v_{1,t}^{(i)} - p_{,1}^{(i)} - q_{,1}^{(i)} - \tau_{,2}^{(i)} = 0; \qquad \rho_{(i)} \cdot v_{2,t}^{(i)} - p_{,2}^{(i)} + q_{,2}^{(i)} - \tau_{,1}^{(i)} = 0;$$

$$\begin{aligned} \frac{1}{\rho_i \cdot (a_i^2 - b_i^2)} \cdot p_{,t}^{(i)} - v_{1,1}^{(i)} - v_{2,2}^{(i)} = 0; \\ \frac{1}{\rho_i \cdot b_i^2} \cdot q_{,t}^{(i)} - v_{1,1}^{(i)} + v_{2,2}^{(i)} = 0; \\ \frac{1}{\rho_i \cdot b_i^2} \cdot \tau_{,t}^{(i)} - v_{1,2}^{(i)} - v_{2,1}^{(i)} = 0, \end{aligned} \tag{1}$$

where

$$a_i = \frac{a_i^0}{a_m^0}, \quad b_i = \frac{b_i^0}{a_m^0}, \quad \rho_i = \frac{\rho_i^0}{\rho_m^0}, \quad (i = 1, 2).$$

Here $v_1^{(i)}(x_1, x_2, t)$, $v_2^{(i)}(x_1, x_2, t)$ are the components of the displacement velocity vector in the direction of the coordinate axes x_1 and x_2. The transition to dimensionless variables was carried out according to the formulas [5]:

$$t = \frac{t^0 \cdot a_m^0}{b^0}, \quad x_k = \frac{x_k^0}{b^0}, \quad v_k = \frac{v_k^0}{a_m^0} \quad (k = 1, 2);$$

$$p = \frac{\sigma_{11}^0 + \sigma_{22}^0}{2 \cdot \rho_m^0 \cdot a_m^{02}}, \quad q = \frac{\sigma_{11}^0 - \sigma_{22}^0}{2 \cdot \rho_m^0 \cdot a_m^{02}}, \quad \tau = \frac{\sigma_{12}^0}{\rho_m^0 \cdot a_m^{02}}. \tag{2}$$

The upper index «0» is given to the dimensional values; b^0 is the characteristic size; the index $i (i = 1, 2)$ corresponds to the main body if $i = 1$, and to a foreign inclusion if $i = 2$; the index m refers to a material in which the velocity of propagation of longitudinal elastic waves is a maximum ($a_m^0 = \max_i a_i^0$); $\sigma_{11}^0, \sigma_{22}^0, \sigma_{12}^0$ are the components of the stress tensor. We assume that the velocity of propagation of longitudinal elastic waves in the main body is the greatest.

To determine the wave field in a doubly-connected domain caused by a dynamic action on the face frontier $x_1 = 0$, $|x_2| \leq L$, of the main body, it is necessary to integrate for $t > 0$ the hyperbolic system of differential equations (1) with zero initial data

$$v_1^{(i)} = v_2^{(i)} = p^{(i)} = q^{(i)} = \tau^{(i)} = 0 \quad (i = 1, 2) \tag{3}$$

and the following boundary conditions for $t \geq 0$:

$$v_1^{(1)} = f(t), \quad v_2^{(1)} = 0 \qquad \text{for} \quad x_1 = 0, \quad |x_2| \leq L; \tag{4}$$

$$p^{(1)} - q^{(1)} = 0, \quad \tau^{(1)} = 0 \qquad \text{for} \quad x_2 = L, \quad 0 \leq x_1| \leq \ell; \tag{5}$$

$$v_1^{(1)} = 0, \quad v_2^{(1)} = 0 \qquad \text{for} \quad x_1 = \ell, \quad |x_2| \leq L, \tag{6}$$

$$p^{(1)} + q^{(1)} = p^{(2)} + q^{(2)}, \quad \tau^{(1)} = \tau^{(2)}, \quad v_1^{(1)} = v_1^{(2)}, \quad v_2^{(1)} = v_2^{(2)} \tag{7}$$

on contact borders PG and QS and

$$p^{(1)} - q^{(1)} = p^{(2)} - q^{(2)}, \quad \tau^{(1)} = \tau^{(2)}, \quad v_1^{(1)} = v_1^{(2)}, \quad v_2^{(1)} = v_2^{(2)} \tag{8}$$

on contact borders PQ and GS.

Here $f(t)$ is a given function that monotonically increases to the maximum value $f(t_0)$, and then decreases monotonically. The zero initial conditions (3) mean that a doubly connected region is at rest up to the time $t \leq 0$. The boundary condition (4) corresponds to the specification of the normal component of the particles' velocity and to the absence of a tangential component of the particles' velocity on the boundary $x_1 = 0$, $|x_2| \leq L$ of the base body for all instants of time. The boundary conditions (5) mean that the lateral surfaces $|x_2| = L$ of the base body are free from stresses for all instants of time. The boundary condition (6) corresponds to the condition of rigid fixation of the points of the surface $|x_2| = \ell$ of the base body.

The boundary conditions (7) and (8) at the contact boundaries of the main body and the foreign inclusion correspond to the usual conditions of rigid contact. The conditions connecting the stresses at the contact boundaries are the formulation of Newton's third law, and the remaining ones that connect the particle velocities ensure the continuity of the doubly-connected domain.

At the accepted loading in the body it is appeared a complex process of propagation of longitudinal waves in the directions of the axes x_1, x_2 and propagation of transverse waves. After a while (depending on the size and speed of propagation of disturbances) they begin to interfere. To determine the stresses and velocities of particles in a doubly connected region, it is necessary to integrate the hyperbolic system of differential equations (1) for given zero and boundary conditions (3)–(8).

The solution of the system of equations (1) for initial (3) and boundary (4)–(8) conditions is found by the method of spatial characteristics at the nodal points, into which the entire doubly-connected region is divided [6]. The peculiarity of the considered body is that on the contact boundaries of the doubly connected domain, as well as at the inner corner points (P, G, Q, S) of foreign inclusion, the «smoothness» of the functions «usual» for dynamic problems is violated. In general, as we know, there were no methods for solving the tasks with such singularities. In addition to the known relations [6], the calculated relations on the contact boundaries of the doubly-connected domain and also at the internal corner points (P, G, Q, S) of the foreign inclusion are obtained [5].

The developed technique was numerically realized for a rectangular strip $(0 \leq x_1 \leq 14 \cdot h)$, $|x_2| \leq 12 \cdot h$ with central foreign inclusion $(4 \cdot h \leq x_1 \leq 10 \cdot h)$, $|x_2| \leq 4 \cdot h$ for hard coupling of dissimilar materials. The material of the main body is steel ($i = 1$, $p_1^0 = 7.9 \cdot 10^3$ kg / m ^{3}, $a_1^0 = 5817$ m/sec, $b_1^0 = 3109$ m/sec), and the foreign inclusion - copper ($i = 2$, $p_2^0 = 8.9 \cdot 10^3$ kg/m ^{3}, $a_2^0 = 4557$ m/sec, $b_2^0 = 2131$ m/s). The program, written in the Fortran language, allows us to calculate a sufficiently wide class of dynamic problems describing unsteady wave processes in multiply connected isotropic media. The wavefield parameters were calculated for a band with a central foreign inclusion at the following values of the initial data: $\ell = 0.7, L = 0.6, h = \triangle x_1 = \triangle x_2 = 0.05$, $k = \triangle t = 0.025, f(t) = t \cdot e^t$ for $t \geq 0$ and for $f(t) = 0$. The time step k is chosen in accordance with the necessary stability conditions

$$\left(\frac{k}{h}\right)^2 \leq min\left\{\frac{\gamma^2}{\gamma^2+1}, \frac{\gamma^2}{(2\gamma^2-1)}\right\}$$

of the explicit finite-difference calculation scheme used. For comparison, the calculation was carried out for a single-connected steel strip without foreign inclusion with the same boundary conditions.

The results of calculations are presented in the form of graphs of the variation of various functions as a function of time at fixed points and the distribution of these same functions wih respect to the coordinate for certain fixed time instants. An analysis of the obtained results reveals certain features of the process of propagation of perturbations in a doubly connected medium. Because of the symmetry of the location of the foreign inclusion and the nature of the loading, the normal stresses $p^{(i)}, q^{(i)}$ and the longitudinal velocity $v_1^{(i)}$ are symmetric, and the tangent stress $\tau^{(i)}$ and the transverse velocity $v_2^{(i)}$ are antisymmetric with respect to the axis $x_2 = 0$. In this connection, Figures 2–4 show the results of calculations only for positive values of $x_2, (x_2 \geq 0)$.

The calculated oscillograms of the transverse $v_2^{(i)}$ particle velocity in the time interval $t \in [0, 100 \cdot k]$ at nine fixed observation points $1(x_1 = 2 \cdot h,\ x_2 = 2 \cdot h)$, $2(x_1 = 2 \cdot h,\ x_2 = 10 \cdot h)$, $3(x_1 = 7 \cdot h,\ x_2 = 2 \cdot h)$, $4(x_1 = 7 \cdot h,\ x_2 = 10 \cdot h)$, $5(x_1 = 12 \cdot h,\ x_2 = 2 \cdot h)$, $6(x_1 = 12 \cdot h,\ x_2 = 10 \cdot h)$, $7(x_1 = 0,\ x_2 = 0)$, $8(x_1 = 7 \cdot h,\ x_2 = 4 \cdot h)$, $9(x_1 = 4 \cdot h,\ x_2 = 2 \cdot h)$ are represented by the curves in Figure 2. The solid curves with round points indicate the velocities displacements arising in a simply connected strip without foreign inclusion with the same boundary conditions. The appearance time of the transverse component of the velocity $v_2^{(i)}$ and its magnitude are completely determined by the transverse-wave fronts. The transverse component of the velocity $v_2^{(i)}$ at point 1 is small. Therefore, at the point under consideration, near the symmetry axis $x_2 = 0$, we can assume that the condition for a quasi-one-dimensional motion is satisfied.

The displacement speed $v_2^{(i)}$, generated by the diffraction waves from the angle $R(x_1 = 0, x_2 = 0.6)$, appears at point 2 before $(t \approx 5 \cdot k)$ than at the point 1 $(t \approx 24 \cdot k)$. At $t \approx 13 \cdot k$ point 4 is already covered by additional transverse motion due to arrival of diffracted waves emanating from the angle R and from the free surface $RK(x_2 = 0.6)$ of the base body. From the form of the velocity of displacement $v_2^{(i)}$ at the point $3(t \approx 15 \cdot k)$ (for the simply connected (curve 3 – • – –) and doubly connected (curve 3 - -) regions), the effect of the diffraction of the angle G of a foreign inclusion is well traced. The appearance of the transverse velocity $v_2^{(i)}$ at point 5 for $(t \approx 23 \cdot k)$ is due to the diffraction of transverse waves from the angle S of foreign inclusion.

At a distance of $10 \cdot h$ from the axis $x_2 = 0$ (points 2, 4, 6), the impact of the diffraction fronts is more significant than their influence at points 1, 3, 5 located near the axis of symmetry. The motion polarized in the transverse direction has less propagation velocity than the velocity of the displacement of the longitudinal wave. The diagrams for the transverse velocity of displacements $v_2^{(i)}$ are characterized by the presence of a set of local extrema, which is caused by the interaction of multiply reflected, refracted and diffracted waves. And this, in turn, determines the pulsating nature of the movement in the transverse direction.

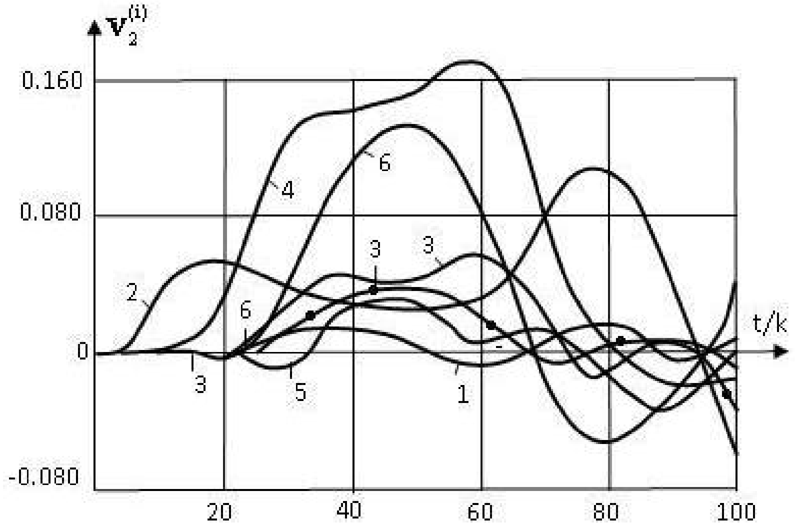

Figure 2. Oscillogram of transverse velocities $v_2^{(i)}$
of particles at nine fixed observation points

Comparison of kinematic parameters of motion at six points of a doubly connected medium with corresponding values in a simply connected medium shows that the influence of the material of foreign inclusion with acoustic stiffness, which is twice less than the stiffness of the material of the main body, significantly affects the magnitude and character of the change in the velocity vector $v_2^{(i)}$ of particles in time at points 1, 3 , 5. As well as the presence of foreign inclusion significantly affects the early appearance of the transverse component of the displacement velocity $v_2^{(i)}$ at points 3 and 5.

Figure 3 shows the isolines of normal stresses $p^{(i)} + q^{(i)} = const$ in the plane $x_1/h \cdot x_2/h$ for the time $(t \approx 50 \cdot k)$. Sequential action of plane wave fronts, their interaction with reflected, refracted and diffracted waves leads to a dynamic concentration of compressive stresses near the corner points (GS) of foreign inclusion in which they reach local maxima. At this time, a relatively high concentration is formed around the corner point G. The degree of stress concentration around the corner points varies with time. The maximum concentration $(t \approx 50 \cdot k)$ is observed near the corner points R, K of the strip in the vicinity of which the stress field has the largest gradients.

In conditions of the existence of internal points with maximum stresses, it becomes important to ensure the compatibility of the deformation of the materials of the foreign inclusion and the main body.

The distribution of the tangential stress $\tau^{(i)}$ in x_2/h in the sections $x_1 = 2 \cdot h(1)$, $x_1 = 4 \cdot h(2)$, $x_1 = 7 \cdot h(3)$, $x_1 = 10 \cdot h(4)$, $x_1 = 12 \cdot h(5)$, is shown in Figure 3 for the time moment $(t \approx 50 \cdot k)$. The maximum values of the stresses in sections 1 and 5 arise in the vicinity.

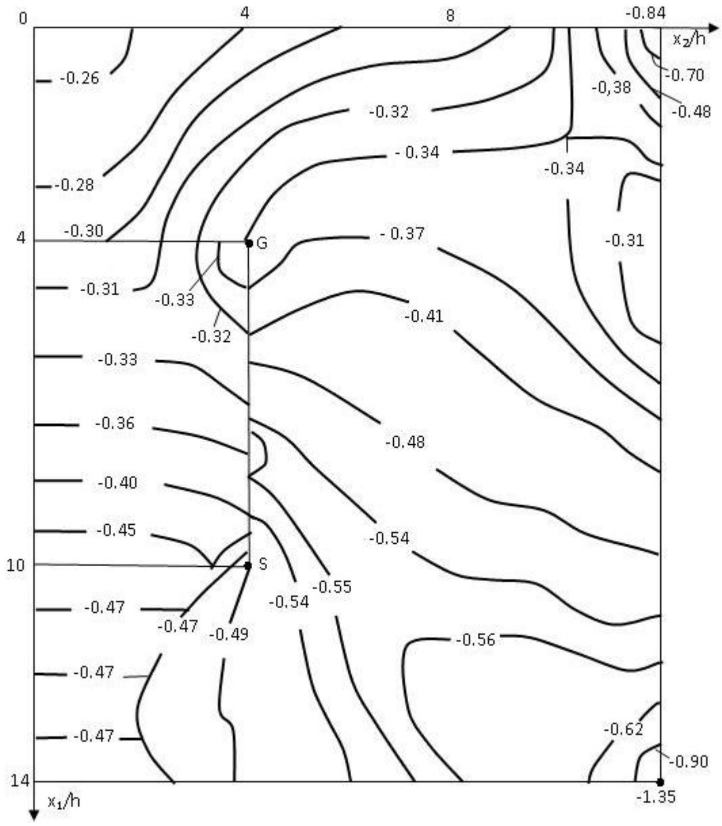

Figure 3. Isolines of normal stresses $p^{(i)} + q^{(i)} = const$ at the time $t = 50 \cdot k$

The maximum values of the stresses in sections 1 and 5 arise in the vicinity of free surface RK of the band. In the neighborhood of the corner points G, S of a foreign inclusion located in sections 2, 4, significant changes in the behavior of tangential stresses are noted. The presence of sections of the sharp changes in $\tau^{(i)}$ is due to the piecewise heterogeneity of the material properties and the additional wave diffraction caused by them. At the moment considered, the effect of diffraction waves emanating from the corner points of a foreign inclusion on the nature of the change of the tangent stresses $\tau^{(i)}$ is weaker in the remaining sections along the x_2/h axis, but over time it becomes more noticeable (Fig. 4).

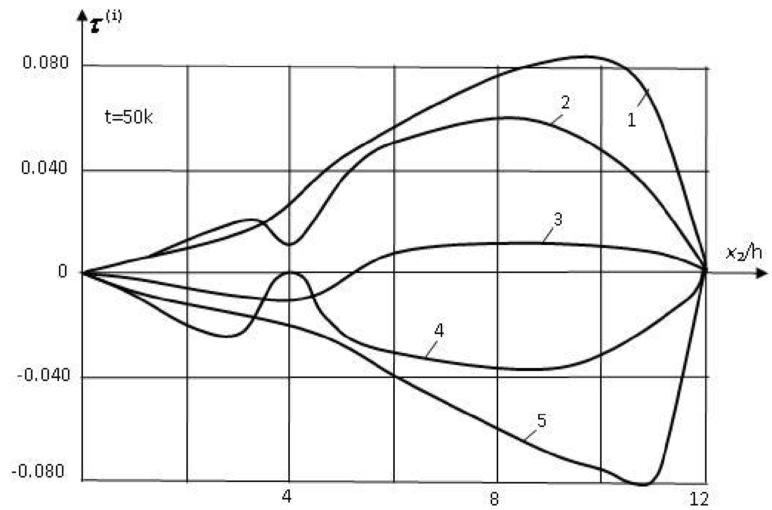

Figure 4. Change of tangent stress $\tau^{(i)}$ with respect to x_2/h in cross-sections $x_1 = 2h(1)$, $x_1 = 4h(2)$, $x_1 = 7h(3)$, $x_1 = 10h(4)$, $x_1 = 12h(5)$

As a result of the studies carried out, it can be concluded that the developed calculation technique for nonstationary dynamic problems with a centrally located foreign inclusion of a rectangular shape quite correctly conveys the main regularities and features of the ongoing wave processes and makes it possible to study the stress-strain states in simply connected and multiply connected media with a complex system of inhomogeneities of the considered types.

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Екібайламды облыстағы серпімді кернеулік толқындар

Мақалада орталық ішкі бөлігі басқа бөгде материалмен толтырылған негізгі дененің сыртқы бет жағынан қатты штамппен ұрған кездегі стационар емес аралас есеп қарастырылған. Кернеулер мен жылдамдықтар терминінде қойылған аралас есеп айқын айырымдық схема, атап айтқанда, сандық кеңістіктік сипаттамалар әдісімен шешілген. Екібайламды облыста пайда болатын толқындық өрістің өзгеру процесі графиктер түрінде берілген. Алынған сандық нәтижелерді талдау барысында екібайламды облыстың түйіскен жанасу нүктелері мен бөгде дененің бұрыштық нүктелерінің маңайында кернеулер концентрациясының динамикалық эффектісін зерттеуге мүмкіндік берді.

Кілт сөздер: серпімділік, толқындық үрдіс, кернеулік, бөгде дені, сандық шешім.

Н.Аширбаев, Ж.Аширбаева, Т.Султанбек, М.Шоманбаева

Волны упругих напряжений в двухсвязной области

В работе рассмотрена нестационарная смешанная задача об ударе жесткого штампа по лицевой поверхности основного тела, содержащего внутри себя центральное инородное включение. Сформулированная в терминах напряжений и скоростей перемещений смешанная задача моделируется численно с помощью явной разностной схемы сквозного счета, основанной на методе пространственных характеристик. В виде графиков представлены результаты изменения волновых полей в двухсвязной области. Анализ числовой информации позволил исследовать динамические эффекты концентрации напряжений в окрестности контакта двухсвязной области, а также вблизи угловых точек инородного включения.

Ключевые слова: упругость, волновой процесс, напряжение, скорость, инородное включение, численное решение, граничное условие.

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On the unique solvability of a family of multipoint-integral boundary value problems for a third order differential equation

A family multipoint-integral boundary value problems for a third order differential equation with variable coefficients is considered. The questions of a existence unique solution of the considered problem and ways of its construction are investigated. The family multipoint-integral boundary value problems for the differential equation of third order with variable coefficients is reduced to a family multipoint-integral boundary value problems for a system of three differential equations by introducing new functions. For solve of resulting family of multipoint-integral boundary value problems is applied a parametrization method. An algorithms of finding the approximate solution to the family multipoint-integral boundary value problems for the system of three differential equations are proposed and their convergence is proved. The conditions of the unique solvability of the family multipoint-integral boundary value problems for the system of three differential equations are obtained in the terms of initial data. The results also formulated relative to the original of the family multipoint-integral boundary value problems for the differential equation of third order with variable coefficients.

Keywords: multipoint-integral boundary value problem, third order differential equations, parameter, family of multipoint-integral boundary value problems, system of differential equations, algorithm, unique solvability.

Introduction

Mathematical modeling of various processes in physics, chemistry, biology, technology, ecology, economics and others are leaded to multipoint-integral boundary value problems for differential equations of higher orders with variable coefficients and parameters [1–8]. The problems of solvability of multipoint-integral boundary value problems remain important for applications because they are directly connected with the theory of splines and interpolations and used in the theory of multi-support beams. Despite the presence of numerous works, general statements of multipoint-integral problems for ordinary differential equations with parameters remain poorly studied up to now. The method of Green functions proves to be the main method for the investigation and solution of family multipoint-integral boundary value problems. This method reflects the specific features of the analyzed boundary value problems. However, the problem of construction of the Green function is quite complicated due to the complex nature of the investigated object and the absence of the required information about its properties.

One of possible ways of overcoming these difficulties is connected with the development of constructive methods aimed at the investigation and solving of family multipoint-integral boundary value problems for higher order differential equations without using the fundamental matrix and the Green function. Thus, in [9], a parametrization method was proposed for the investigation and solving of two-point boundary value problems for ordinary differential equations. Parallel with construction of the coefficient criteria for the unique solvability of the investigated problem, parametrization method enables one to propose algorithms for finding the solution of this problem. In [10, 11], the parametrization method was applied to multipoint boundary value problem for ordinary differential equations. A family of multipoint boundary value problems for system of differential equations and multipoint nonlocal problem for system of hyperbolic equations were considered in [12, 13].

In the present paper we study of a questions the existence and uniqueness of solutions to family of multipoint-integral boundary value problems for the third-order differential equations and the methods of finding its approximate solutions. For these purposes, we are applied parametrization method [9] for solve of this problem.

We consider the family multipoint-integral boundary value problems for the third-order differential equation

$$\frac{\partial^3 z}{\partial t^3} = A_1(t,x)\frac{\partial^2 z}{\partial t^2} + A_2(t,x)\frac{\partial z}{\partial t} + A_3(t,x)z + f(t,x), \qquad t \in (0,T), \quad x \in [0,\omega]; \tag{1}$$

$$\sum_{i=0}^{m}\Big\{\alpha_{i1}(x)\frac{\partial^2 z(t_i,x)}{\partial t^2}+\beta_{i1}(x)\frac{\partial z(t_i,x)}{\partial t}+\gamma_{i1}(x)z(t_i,x)\Big\}+$$
$$+\int_0^T\Big\{K_{11}(\tau,x)\frac{\partial^2 z(\tau,x)}{\partial \tau^2}+K_{12}(\tau,x)\frac{\partial z(\tau,x)}{\partial \tau}+K_{13}(\tau,x)z(\tau,x)\Big\}d\tau=d_1(x), \qquad x\in[0,\omega]; \qquad (2)$$

$$\sum_{i=0}^{m}\Big\{\alpha_{i2}(x)\frac{\partial^2 z(t_i,x)}{\partial t^2}+\beta_{i2}(x)\frac{\partial z(t_i,x)}{\partial t}+\gamma_{i2}(x)z(t_i,x)\Big\}+$$
$$+\int_0^T\Big\{K_{21}(\tau,x)\frac{\partial^2 z(\tau,x)}{\partial \tau^2}+K_{22}(\tau,x)\frac{\partial z(\tau)}{\partial \tau}+K_{23}(\tau,x)z(\tau,x)\Big\}d\tau=d_2(x); \qquad (3)$$

$$\sum_{i=0}^{m}\Big\{\alpha_{i3}(x)\frac{\partial^2 z(t_i,x)}{\partial t^2}+\beta_{i3}(x)\frac{\partial z(t_i,x)}{\partial t}+\gamma_{i3}(x)z(t_i,x)\Big\}+$$
$$+\int_0^T\Big\{K_{31}(\tau,x)\frac{\partial^2 z(\tau,x}{\partial \tau^2}+K_{32}(\tau,x)\frac{\partial z(\tau,x)}{\partial \tau}+K_{33}(\tau,x)z(\tau,x)\Big\}d\tau=d_3(x). \qquad (4)$$

Here $z(t,x)$ is unknown function, the functions $A_k(t,x)$, $f(t,x)$ are continuous on $\Omega=[0,T]\times[0,\omega]$, $k=1,2,3$, $\alpha_{ij}(x)$, $\beta_{ij}(x)$, $\gamma_{ij}(x)$, $d_j(x)$ are continuous functions on $[0,\omega]$, the functions $K_j(t,x)$ are continuous on Ω, $i=\overline{0,m}$, $j=1,2,3$, $0=t_0<t_1<t_2<...<t_{m-1}<t_m=T$.

Let $C(\Omega,R)$ be a space of continuous functions $z:\Omega\to R$ on Ω with norm $||u||_0=\max\limits_{(t,x)\in\Omega}|z(t,x)|$. The function $z(t,x)\in C(\Omega,R)$, that has partial derivatives $\frac{\partial z(t,x)}{\partial t}\in C(\Omega,R)$, $\frac{\partial^2 z(t,x)}{\partial t^2}\in C(\Omega,R)$, $\frac{\partial^3 z(t,x)}{\partial t^3}\in C(\Omega,R)$ is called a solution to family of problems (1)–(4) if it satisfies third-order differential equation (1) for all $(t,x)\in\Omega$ and meets the boundary conditions (2), (3) and (4) for all $x\in[0,\omega]$.

For fixed $x\in[0,\omega]$ problem (1)–(4) is a linear multipoint-integral boundary value problem for the third order ordinary differential equations. Suppose a variable x is changed on $[0,\omega]$; then we obtain a family of multipoint-integral boundary value problems for ordinary differential equations.

We will investigate the existence of a unique solution to the family of multipoint-integral boundary value problems for the third-order differential equation (1)–(4). We use parametrization method for solve of the family problems (1)–(4) and construct of its approximate solutions. An algorithms of finding the approximate solution to the family of multipoint-integral boundary value problems for the system of three differential equations are constructed and their convergence is proved. The conditions of the unique solvability of the family of multipoint-integral boundary value problems for the system of three differential equations are established in the terms of initial data. The results also formulated relative to the original of the family multipoint-integral boundary value problems for the differential equation of third order with variable coefficients. The obtained results are applied to a family of multi-point boundary value problems for the third order ordinary differential equation. The efficiency of the proposed approach for solve of the multi-point boundary value problems for the third order differential equations that arise in applications. The results can also be used in the study and solve of a nonlinear multipoint-integral boundary value problems for the third order differential equations. Some types of problems (1)–(4) were studied in [1-8]. For fixed x and $K_{ij}(t,x)=0$, $i=\overline{1,3}$, $j=\overline{1,3}$, the problem (1)–(4) were considered in [14]. At fixed x the problem (1)–(4) were investigated in [15].

1 Scheme of the method

We introduce the following notations

$$A(t,x)=\begin{pmatrix}0&1&0\\0&0&1\\A_3(t,x)&A_2(t,x)&A_1(t,x)\end{pmatrix},\quad F(t,x)=\begin{pmatrix}0\\0\\f(t,x)\end{pmatrix},\quad d(x)=\begin{pmatrix}d_1(x)\\d_2(x)\\d_3(x)\end{pmatrix};$$

$$M_i(x)=\begin{pmatrix}\gamma_{i1}(x)&\beta_{i1}(x)&\alpha_{i1}(x)\\\gamma_{i2}(x)&\beta_{i2}(x)&\alpha_{i2}(x)\\\gamma_{i3}(x)&\beta_{i3}(x)&\alpha_{i3}(x)\end{pmatrix},\qquad K(t,x)=\begin{pmatrix}K_{13}(t,x)&K_{12}(t,x)&K_{11}(t,x)\\K_{23}(t,x)&K_{22}(t,x)&K_{21}(t,x)\\K_{33}(t,x)&K_{32}(t,x)&K_{31}(t,x)\end{pmatrix}.$$

I is identity matrix of dimension 3.

Problem (1)–(4) can be write in the vector-matrix form

$$\frac{\partial u}{\partial t} = A(t,x)u + F(t,x); \tag{5}$$

$$\sum_{i=0}^{m} M_i(x)u(t_i,x) + \int_0^T K(\tau,x)u(\tau,x)d\tau = d(x), \tag{6}$$

where $u(t,x) = (u_1(t,x), u_2(t,x), u_3(t,x))'$, $u_1(t,x) = z(t,x)$, $u_2(t,x) = \frac{\partial z(t,x)}{\partial t}$, $u_3(t,x) = \frac{\partial^2 z(t,x)}{\partial t^2}$.

A continuously differentiable function $u : \Omega \to R^3$ is called a solution of the family multipoint-integral boundary value problems (5), (6) if it satisfies system (5) for all $(t,x) \in \Omega$ and condition (6) for all $x \in [0,\omega]$. By $\mu(x)$, we denote the value of the function $u(t,x)$ for $t = t_0$.

We perform the change $u(t,x) = \widetilde{u}(t,x) + \mu(x)$ in the problem (5), (6).

Then problem (5), (6) is reduced to the following equivalent problem with unknown functional parameter $\mu(x)$:

$$\frac{\partial \widetilde{u}}{\partial t} = A(t,x)\widetilde{u} + A(t,x)\mu(x) + F(t,x); \tag{7}$$

$$\widetilde{u}(t_0,x) = 0, \qquad x \in [0,\omega]; \tag{8}$$

$$\sum_{i=0}^{m} M_i(x)\mu(x) + \sum_{i=1}^{m} M_i(x)\widetilde{u}(t_i,x) + \int_0^T K(\tau,x)\widetilde{u}(\tau,x)d\tau + \int_0^T K(\tau,x)d\tau\mu(x) = d(x), \quad x \in [0,\omega]. \tag{9}$$

A pair $(\widetilde{u}(t,x), \mu(x))$ is called a solution to family of problems with parameter (7)–(9) if the function $\widetilde{u}(t,x)$ is continuous and continuously differentiable by t on Ω, satisfies of the system (7) for all $(t,x) \in \Omega$, initial condition (8) and multipoint-integral condition (8) for all $x \in [0,\omega]$.

Problems (5)–(6) and (7)–(9) are equivalent. If a vector function $u(t,x)$ is a solution to family of multipoint-integral problems (5), (6), then a pair $(\widetilde{u}(t,x), \mu(x))$, where $\widetilde{u}(t,x) = u(t,x) - u(t_0,x)$, $\mu(x) = u(t_0,x)$, be a solution to family of problems with functional parameter (7)–(9). And conversely, if a pair $(\widetilde{u}^*(t,x), \mu^*(x))$ is a solution to family of problems with functional parameter (7)–(9), then a vector function $u^*(t,x) = \widetilde{u}^*(t,x) + \mu^*(x)$ be a solution to original family of multipoint-integral problems (5), (6). At fixed $\mu(x)$ the problem (7), (8) is a family of Cauchy problems for system of three differential equations and the relation (9) is connected a values of function $\widetilde{u}(t,x)$ with unknown parameter $\mu(x)$.

A solution of the family of Cauchy problems (7), (8) is equivalent to a family of Volterra integral equations second kind

$$\widetilde{u}(t,x) = \int_0^t A(\tau,x)\widetilde{u}(\tau,x)d\tau + \int_0^t A(\tau,x)d\tau\mu(x) + \int_0^t F(\tau,x)d\tau, \qquad (t,x) \in \Omega. \tag{10}$$

Substituting the right-hand side of the integral equation (10) instead of the function $\widetilde{u}(\tau,x)$ at $t = \tau$ and repeating the process νth time ($\nu = 1,2,3,...$), we get

$$\widetilde{u}(t,x) = D_\nu(t,x)\mu(x) + G_\nu(t,x,\widetilde{u}) + \widetilde{F}_\nu(t,x), \qquad (t,x) \in \Omega, \tag{11}$$

where

$$D_\nu(t,x) = \int_0^t A(\tau,x)d\tau + \int_0^t A(\tau,x)\int_0^\tau A(\tau_1,x)d\tau_1 d\tau + ... +$$

$$+ \int_0^t A(\tau,x)\int_0^\tau A(\tau_1,x) ... \int_0^{\tau_{\nu-1}} A(\tau_\nu,x)d\tau_\nu d\tau_{\nu-1} ... d\tau_1 d\tau;$$

$$G_\nu(t,x,\widetilde{u}) = \int_0^t A(\tau,x)\int_0^\tau A(\tau_1,x) ... \int_0^{\tau_{\nu-1}} A(\tau_\nu,x)\widetilde{u}(\tau_\nu,x)d\tau_\nu d\tau_{\nu-1} ... d\tau_1 d\tau;$$

$$\widetilde{F}_\nu(t,x) = \int\limits_0^t F(\tau,x)d\tau + \int\limits_0^t A(\tau,x)\int\limits_0^\tau F(\tau_1,x)d\tau_1 d\tau + ...+$$

$$+\int\limits_0^t A(\tau,x)\int\limits_0^\tau A(\tau_1,x)...\int\limits_0^{\tau_{\nu-1}} F(\tau_\nu,x)d\tau_\nu d\tau_{\nu-1}...d\tau_1 d\tau.$$

From the representation (11) we determine the values of function $\widetilde{u}(t,x)$ for $t=t_i$, $i=\overline{1,m}$, $t=\tau$ and substitute them into the appropriate expression (9). Then, we obtain

$$\Big[M_0(x)+\sum_{i=1}^m M_i(x)[I+D_\nu(t_i,x)]+\int\limits_0^T K(\tau,x)[I+D_\nu(\tau,x)]d\tau\Big]\mu(x) = d(x)-\sum_{i=1}^m M_i(x)\widetilde{F}_\nu(t_i,x)-$$

$$-\int\limits_0^T K(\tau,x)\widetilde{F}_\nu(\tau,x)d\tau - \sum_{i=1}^m M_i(x)G_\nu(t_i,x,\widetilde{u}) - \int\limits_0^T K(\tau,x)G_\nu(\tau,x,\widetilde{u})d\tau. \quad (12)$$

The relation (12) is a linear system of three functional equations with respect to parameter $\mu(x)$.

Introduce the notation $Q_\nu(T,x) = M_0(x)+\sum\limits_{i=1}^m M_i(x)[I+D_\nu(t_i,x)]+\int\limits_0^T K(\tau,x)[I+D_\nu(\tau,x)]d\tau$. If for some $\nu\in\mathbb{N}$ the (3×3) matrix $Q_\nu(T,x)$ is invertible for all $x\in[0,\omega]$, then at fixed values $\widetilde{u}$ the functional parameter $\mu(x)$ is uniquely determined from system (12). So, for finding a solution to family of problems (7)–(9) we have a closed system of equations (10) and (12).

2 Algorithm and main result

If the function $\widetilde{u}(t,x)$ is known, then parameter $\mu(x)$ can be found from the system of functional equations (12). Conversely, if parameter $\mu(x)$ is known, then function $\widetilde{u}(t,x)$ can be found from the family of Cauchy problem for system of differential equations (7), (8). Since neither $\widetilde{u}(t,x)$ nor $\mu(x)$ are known, we use the iterative method and find the solution of family problems with parameter (7)–(9) in the form of a pair $(\widetilde{u}^*(t,x),\mu^*(x))$ as the limit of a sequence $(\widetilde{u}^{(k)}(t,x),\mu^{(k)}(x))$, $k=0,1,2,...$, determined according to the following algorithm:

Step 0. Assume that, for chosen $\nu\in\mathbb{N}$ the matrix $Q_\nu(T,x): R^3\to R^3$ is invertible for all $x\in[0,\omega]$. We use the initial condition (8). We determine the initial approximation in the parameter $\mu^{(0)}(x)$ from the system of functional equations $Q_\nu(T,x)\mu(x) = d(x)-\sum\limits_{i=1}^m M_i(x)\widetilde{F}_\nu(t_i,x)$ for all $x\in[0,\omega]$.

We solve the family of Cauchy problems (7), (8) for $\mu(x)=\mu^{(0)}(x)$ and find of function $\widetilde{u}^{(0)}(t,x)$ for all $(t,x)\in\Omega$.

Step 1. Substituting the obtained function $\widetilde{u}^{(0)}(t,x)$ for $\widetilde{u}(t,x)$, from the system of functional equations (12), we obtain $\mu^{(1)}(x)$ for all $x\in[0,\omega]$. Further, we solve the family of Cauchy problems (7), (8) for $\mu(x)=\mu^{(1)}(x)$ and find of function $\widetilde{u}^{(1)}(t,x)$ for all $(t,x)\in\Omega$.

And so on.

Step k. Substituting the obtained function $\widetilde{u}^{(k-1)}(t,x)$ for $\widetilde{u}(t,x)$, from the system of functional equations (12), we get $\mu^{(k)}(x)$ for all $x\in[0,\omega]$. Solving the family of Cauchy problems (7), (8) for $\mu(x)=\mu^{(1)}(x)$, we find $\widetilde{u}^{(k)}(t,x)$ for all $(t,x)\in\Omega$, $k=0,1,2,...$.

Introduce a notations

$$a(x) = \max_{t\in[0,T]}||A(t,x)|| = \max\Big(1, \max_{t\in[0,T]}\{|A_1(t,x)|+|A_2(t,x)|+|A_3(t,x)|\}\Big);$$

$$\kappa(x) = \max_{t\in[0,T]}||K(t,x)|| = \max_{t\in[0,T]}\max_{i=\overline{1,3}}\{|K_{i1}(t,x)|+|K_{i2}(t,x)|+|K_{i3}(t,x)|\}.$$

The following theorem establishes sufficient conditions for the applicability and convergence of the algorithm proposed above, which also guarantee the unique solvability of problem (5), (6).

Theorem 1. Let for some $\nu\in\mathbb{N}$ the matrix $Q_\nu(T,x): R^3\to R^3$ is invertible for all $x\in[0,\omega]$ and let the following inequalities be true:

a) $||[Q_\nu(T,x)]^{-1}||\le\eta_\nu(T,x)$, where $\eta_\nu(T,x)$ is a positive continuous on $x\in[0,\omega]$ function;

b) $q_\nu(T,x) = \eta_\nu(T,x)\cdot\Big(\sum\limits_{i=1}^m ||M_i(x)||+\kappa(x)T\Big)\max\limits_{i=\overline{1,m}}\Big[e^{a(x)t_i}-1-\sum\limits_{j=1}^\nu\frac{[a(x)t_i]^j}{j!}\Big]\le\chi<1$, where χ is a constant.

Then family of multipoint-integral boundary value problems (5), (6) has a unique solution.

The proof of theorem 1 is similar of proof theorem 1.

By using the parametrization method, we split the procedure of determination of unknown functions into two part:

1) determination of the unknown function $\widetilde{u}(t,x)$ from the family of Cauchy problems for system of three partial differential equations (7), (8);

2) determination of the introduced parameter $\mu(x)$ from the system of functional equations (12).

Taking into account the notations and the equivalent transition to problem (5), (6), we have.

Theorem 2. Let for some $\nu \in \mathbb{N}$ the matrix $Q_\nu(T,x) : R^3 \to R^3$ is invertible for all $x \in [0,\omega]$ and let the inequalities a), b) of Theorem 1 be true.

Then family of multipoint-integral boundary value problems for the third-order differential equation (1)–(4) has a unique solution.

Example. We consider the family of boundary value problems [8]:

$$\frac{\partial^3 z}{\partial t^3} = p(t,x)z + f(t,x) + r, \qquad t \in (a,b), \quad x \in [0,1]; \tag{13}$$

$$z(a,x) = \alpha, \qquad x \in [0,1]; \tag{14}$$

$$\frac{\partial z(a,x)}{\partial t} = \beta_1, \qquad x \in [0,1]; \tag{15}$$

$$\frac{\partial z(b,x)}{\partial t} = \beta_2, \qquad x \in [0,1]. \tag{16}$$

Assume that the functions $f(t,x)$ and $p(t,x)$ are given, and $p(t,x) = 0$ for $t \in [a,c) \cup (d,b]$, $\;a < c < d < b$, and $x \in [0,1]$, the parameter r, α, β_1, and β_2 are constants.

For this problem

$$A(t,x) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & p(t,x) \end{pmatrix}, \qquad F(t,x) = \begin{pmatrix} 0 \\ 0 \\ f(t,x) + r \end{pmatrix};$$

$$M_0(x) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad M_1(x) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad K(t,x) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \tilde{d}(x) = \begin{pmatrix} \alpha \\ \beta_1 \\ \beta_2 \end{pmatrix};$$

$$\widetilde{D}_\nu(a,t,x) = \int_a^t A(\tau,x)d\tau + \int_a^t A(\tau,x)\int_a^\tau A(\tau_1,x)d\tau_1 d\tau + ...+$$

$$+\int_a^t A(\tau,x)\int_a^\tau A(\tau_1,x)...\int_a^{\tau_{\nu-1}} A(\tau_\nu,x)d\tau_\nu d\tau_{\nu-1}...d\tau_1 d\tau, \quad \nu = 1,2,...,$$

$$\delta(x) = \max\Big(1, \max_{t\in[a,b]} |p(t,x)|\Big).$$

Theorem 3. Let for some $\nu \in \mathbb{N}$ the (3×3) matrix $Q_\nu(a,b,x) = M_0(x) + M_1(x)[I + \widetilde{D}_\nu(a,b,x)$ is invertible for all $x \in [0,1]$ and let the following inequalities be true:

a) $||[\widetilde{Q}_\nu(a,b,x)]^{-1}|| \le \widetilde{\eta}_\nu(a,b,x)$, where $\widetilde{\eta}_\nu(a,b,x)$ is a positive continuous on $[0,1]$ function;

b) $\widetilde{q}_\nu(a,b,x) = \widetilde{\eta}_\nu(a,b,x) \cdot \left[e^{\delta(x)(b-a)} - 1 - \sum\limits_{j=1}^{\nu} \frac{[\delta(x)(b-a)]^j}{j!}\right] \le \chi < 1.$

Then family of two-point boundary value problems for the third-order differential equation (13)–(16) has a unique solution.

Note that in the repeated integrals of the $\widetilde{D}_\nu(a,t,x)$ the element of the matrix $A(t,x)$ is function $p(t,x)$ and it will be calculated on the interval $[c,d] \times [0,1]$.

Let $p(t,x) = 1$ for $t \in [c,d] \times [0,1]$, $p(t,x) = 0$ for $t \in [a,c) \cup (d,b]$, $x \in [0,1]$.

Then the matrix $Q_\nu(a,b,x)$ independent on x. In this case, the conditions of Theorem 3 will be formulated only in the terms of numbers a, b, c, d.

We have

Theorem 4. Let the (3×3) *matrix* $Q_1(a,b)=\begin{pmatrix} 1 & b-a & 0\\ 0 & 1+b-a & 0\\ 0 & 1 & d-c\end{pmatrix}$ *is invertible and let the following inequalities be true:*

a) $\|[\widetilde{Q}_1(a,b)]^{-1}\|\leq \max\Big(\frac{1}{d-c},1\Big)+\max\Big(b-a,1,\frac{1}{d-c}\Big)\frac{1}{1+b-a}$*;*

b) $\widetilde{q}_\nu(a,b)=\Big[\max\Big(\frac{1}{d-c},1\Big)+\max\Big(b-a,1,\frac{1}{d-c}\Big)\frac{1}{1+b-a}\Big]\cdot\Big[e^{(b-a)}-1-(b-a)\Big]<1.$

Then two-point boundary value problem for the third-order differential equation (13)–(16) has a unique solution.

So, the family of multipoint-integral boundary value problems for third order differential equation (1)–(4) is reduced to an equivalent family of multipoint-integral boundary value problems for system of differential equation first order. For solve of the family multipoint-integral boundary value problems for system of differential equations results of articles [10-13] are used. Algorithms of finding solutions to the family of multipoint-integral boundary value problems for differential equations are constructed and their convergence is proved. The conditions of the unique solvability to the family of multipoint-integral boundary value problems for third order differential equations are established.

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Үшінші ретті дифференциалдық теңдеу үшін көпнүктелі-интегралдық шеттік есептер әулетінің бірмәнді шешілімділігі туралы

Мақалада коэффициенттері айнымалы үшінші ретті дифференциалдық теңдеу үшін көпнүктелі-интегралдық шеттік есептер әулеті қарастырылды. Бұл есептің жалғыз шешімінің бар болуы мәселелері мен оны табу жолдары зерттелді. Жаңа функциялар енгізу әдісі арқылы коэффициенттері айнымалы үшінші ретті дифференциалдық теңдеу үшін көпнүктелі-интегралдық шеттік есептер әулеті үш дифференциалдық теңдеуден тұратын жүйе үшін көпнүктелі-интегралдық шеттік есептер әулетіне келтіріледі. Нәтижесінде алынған көпнүктелі-интегралдық шеттік есептер әулетін шешу үшін параметрлеу әдісі қолданылды. Үш дифференциалдық теңдеуден тұратын жүйе үшін көпнүктелі-интегралдық шеттік есептер әулетінің жуық шешімдерін табу алгоритмдері ұсынылған және оның жинақтылығы дәлелденген. Үш дифференциалдық теңдеуден тұратын жүйе үшін көпнүктелі-интегралдық шеттік есептер әулетінің бірмәнді шешілімділігінің шарттары бастапқы берілімдер терминінде алынған. Нәтижелер сәйкесінше бастапқы коэффициенттері айнымалы үшінші ретті дифференциалдық теңдеу үшін көпнүктелі-интегралдық шеттік есептер әулеті үшін де тұжырымдалған.

Кілт сөздер: көпнүктелі-интегралдық шеттік есеп, үшінші ретті дифференциалдық теңдеулер, параметр, көпнүктелі-интегралдық шеттік есептер әулеті, дифференциалдық теңдеулер жүйесі, алгоритм, бірмәнді шешімділік.

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Об однозначной разрешимости семейства многоточечно-интегральных краевых задач для дифференциального уравнения третьего порядка

В статье рассмотрено семейство многоточечно-интегральных краевых задач для дифференциального уравнения третьего порядка с переменными коэффициентами. Исследованы вопросы существования единственного решения рассматриваемой задачи и способы его построения. Методом введения новых функций семейство многоточечно-интегральных краевых задач для дифференциального уравнения третьего порядка с переменными коэффициентами сводится к семейству многоточечно-интегральных краевых задач для системы трех дифференциальных уравнений. Для решения полученного семейства многоточечно-интегральных краевых задач применяется метод параметризации. Предложены алгоритмы нахождения приближенного решения семейства многоточечно-интегральных краевых задач для системы трех дифференциальных уравнений, и доказана их сходимость. Получены условия однозначной разрешимости семейства многоточечно-интегральных краевых задач для системы трех дифференциальных уравнений в терминах исходных данных. Результаты также сформулированы относительно исходного семейства многоточечно-интегральных краевых задач для дифференциального уравнения третьего порядка с переменными коэффициентами.

Ключевые слова: многоточечно-интегральная краевая задача, дифференциальное уравнение третьего порядка, параметр, семейство многоточечно-интегральных краевых задач, алгоритм, однозначная разрешимость.

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The first boundary value problem with deviation from the characteristics for a second-order parabolic-hyperbolic equation

We pose and investigate the first boundary value problem using a model second order equation of parabolic-hyperbolic type with A.M. Nakhushev's conditions violated relative to coefficients. Despite these conditions are violated an a priori estimate similar to the a priori estimate obtained by A.M. Nakhushev takes place for solving the first boundary-value problem under study.

Keywords: equation of mixed parabolic-hyperbolic type, the first boundary value problem, a priori estimate of the solution.

Introduction

Consider the equation

$$f = Lu \equiv \begin{cases} u_{yy} - k(y)u_{xx}, & y < 0, \\ u_{yy} + u_x, & y > 0, \end{cases} \tag{1}$$

where $k(y) \geq k_1 > 0$, $f = f(x, y)$ are given functions, and $u = u(x, y)$ is an unknown one. For $y < 0$, equation (1) coincides with the Chaplygin equation [1; 21], and for $y > 0$, it is a parabolic equation backward in time (with x standing for a time variable). Thus, equation (1) is a second-order parabolic-hyperbolic equation with non-characteristic line of degeneracy [2].

A great number of scientific researches are devoted to the study of boundary-value problems for second-order parabolic-hyperbolic equations with non-characteristic line of degeneracy. For example, in [3], by the spectral method, a priori estimates in the L_p- and C-classes for solution of the Tricomi problem for an equation of the form (1), are obtained. In [4–8], boundary value problems with deviation from the characteristics for parabolic-hyperbolic equations are studied.

In [9], a method enabling one to formulate well-posed boundary value problems for a class of linear parabolic-hyperbolic equations of the form

$$Lu \equiv u_{yy} - k(x, y)u_{xx} + a(x, y)u_x + b(x, y)u_y + c(x, y)u = f(x, y), \tag{2}$$

in a bounded domain Ω with piecewise smooth boundary Σ, is given. It is assumed that the coefficients of (2) are continuous and satisfy the Nakhushev conditions, namely:

$$k_x(x, y),\ a_x(x, y),\ c_x(x, y) \in C\left(\overline{\Omega}\right), \quad f(x, y) \in L_2\left(\Omega\right), \tag{3}$$

$$k(x, y) \geq 0, \qquad \forall (x, y) \in \Omega, \tag{4}$$

$$a(x, y) > 0, \quad 2a(x, y) + k_x(x, y) \geq 0, \qquad \forall (x, y) \in \Omega_0, \tag{5}$$

where Ω_0 is the domain of parabolicity of (2). In particular, all parts of Σ which should be free from boundary data, are specified. Moreover, an a priori estimate for such a problem for (2) is obtained. The latter implies the uniqueness of a regular solution and the existance of a weak one. In [10], this problem is called the first boundary value problem for parabolic-hyperbolic equations.

In what follows, by Ω we denote the union of Ω_1, Ω_2 and J_r, i.e. $\Omega = \Omega_1 \cup \Omega_2 \cup J_r$, where

$$\Omega_1 = \{(x, y) : 0 < x < r,\ 0 < y < \varphi(x)\};$$

$$\Omega_2 = \{(x, y) : 0 < x \leq l,\ \gamma_1(x) < y < 0\} \cup \{(x, y) : l \leq x < r,\ \gamma_2(x) < y < 0\}$$

and

$$J_r = \{(x,y) : 0 < x < r,\ y = 0\}.$$

It is assumed that

$$\varphi(x) \in C^1[0,r], \quad \gamma_1(x) \in C^1[0,l], \quad \gamma_2(x) \in C^1[l,r]$$

and

$$\varphi(x) > 0, \quad \gamma_1'(x) < 0, \quad \gamma_2'(x) > 0, \qquad \gamma_1(0) = \gamma_2(r) = 0, \quad \gamma_1(l) = \gamma_2(l).$$

We will use the following notations:

$$A = A(0,0); \quad B = B(r,0); \quad C = C(l,\gamma_1(l)); \quad A_0 = A_0(0,\varphi(0)); \quad B_0 = B_0(r,\varphi(r));$$

$$\sigma_0 = \{(x,y) : 0 < x < r,\ y = \varphi(x)\}; \quad \sigma_1 = \{(x,y) : 0 < x < l,\ y = \gamma_1(x)\}$$

and

$$\sigma_2 = \{(x,y) : l < x < r,\ y = \gamma_2(x)\}.$$

We also require $f(x,y)$ and $k(y)$ to be continuous, i.e.

$$f \in C\left(\bar{\Omega}_i\right), \quad i = 1,2; \qquad k(y) \in C\left[\gamma_1(l), 0\right].$$

As it is shown in [9], the well-posedness of the first boundary-value problem for equation (1) strongly depends on the mutual location of curves σ_1 and σ_2, and the characteristics

$$AC_1 : x + \int\limits_0^y \sqrt{k(t)}dt = 0 \qquad \text{and} \qquad C_1B : x - \int\limits_0^y \sqrt{k(t)}dt = r.$$

Here, C_1 is the point of intersection of the characteristics passing through the points A and B.

In this paper, we consider equation (1) with coefficients not satisfying the Nakhushev conditions. We state and solve the first boundary value problem in the case when the curves σ_1 and σ_2 lie entirely in the characteristic triangle ABC_1. We also prove an analogue of the a priori estimate obtained in [9].

Formulation of the problem

By substitution [9]:

$$u(x,y) = \upsilon(x,y)\, \exp\left(\mu\, x\right), \tag{6}$$

the given operator Lu associated with the operator $L_\mu \upsilon$ according to the formula

$$L_\mu \upsilon = \begin{cases} \upsilon_{yy} - k(y)\, \upsilon_{xx} - 2\mu\, k(y)\, \upsilon_x - \mu^2\, k(y)\, \upsilon, & y < 0; \\ \upsilon_{yy} + \upsilon_x + \mu \upsilon, & y > 0, \end{cases}$$

where μ is some negative number. The operators Lu and $L_\mu \upsilon$ when replacing (6) will be related by

$$Lu = \exp\left(\mu x\right) L_\mu \upsilon.$$

The Regular solution of the equation

$$L_\mu \upsilon = f_\mu, \ f_\mu = \exp\left(-\mu x\right) f \tag{7}$$

in the domain Ω we call any function $\upsilon = \upsilon(x,y)$ for the class $C\left(\bar{\Omega}\right) \cap C^1\left(\Omega\right) \cap C^2\left(\Omega_1\right) \cap C_y^2\left(\Omega_2\right)$, which turns the equation into an identity upon substitution.

Problem 1. Find in the domain Ω the regular solution $\upsilon = \upsilon(x,y)$ to the equation (7) of $\upsilon(x,y) \in C^1\left(\bar{\Omega}\right) \cap C^2\left(\Omega_i\right)$, $i = \overline{1,2}$ satisfying the boundary conditions

$$\upsilon = 0 \qquad \forall\ (x,y) \in BB_0 \cup \sigma_0 \cup \sigma_2; \tag{8}$$

$$\upsilon_y = 0 \qquad \forall\ (x,y) \in \sigma_1. \tag{9}$$

Theorem to get a priori estimation of the solution to problem 1

Further, we use the following notation:

$$(u,\, \upsilon)_0 = \int\limits_{\Omega} u\,\upsilon d\Omega; \quad \|u\|_0^2 = \int\limits_{\Omega} u^2 d\Omega;$$

$$\|u\|_1^2 = \int\limits_{\Omega} \left[u^2 + u_x^2 + u_y^2\right] d\Omega.$$

The following Theorem holds true.

Theorem 1. Assume the curves σ_0, σ_1 and σ_2, restricting the domain Ω are such that they possess the following properties:

$$\sigma_0 \colon y = \varphi(x) \in C^1\left[0,\, r\right], \text{ and besides } \varphi'(x) \le 0 \;\; \forall\, x \in [0, r]; \tag{10}$$

$$\sigma_1 \,:\, y = \gamma_1(x) \in C^1\left[0,\, l\right];\; \sigma_2 = CB \,:\, y = \gamma_2(x) \in C^1\left[l,\, r\right]; \tag{11}$$

$$-1 < \sqrt{k(y)}\,\gamma_1'(x) < 0 \qquad \forall\; (x,y) \in AC; \tag{12}$$

$$0 < \sqrt{k(y)}\,\gamma_2'(x) \le 1 \qquad \forall\; (x,y) \in CB. \tag{13}$$

Then to solve $\upsilon = \upsilon(x,y)$ of problem 1 we have the energy inequality

$$\|\upsilon\|_1 \le M_1 \left\|f_\mu\right\|_0, \tag{14}$$

where the function $\upsilon = \upsilon(x,y)$ is associated with the solution $u = u(x,y)$ of original equation (1) according to the formula (6), and M_1 is the positive constant that does not depend on the sought function $\upsilon(x,y)$.

Proof. For the operator $L_\mu \upsilon$ as $y < 0$ the equality

$$2\,\delta(x)\,\upsilon_x\,L_\mu\upsilon = 2\,\delta(x)\,\upsilon_x\,\left[\upsilon_{yy} - k(y)\upsilon_{xx} - 2\mu\,k(y)\,\upsilon_x - \mu^2 k(y)\upsilon\right] =$$

$$= \frac{\partial}{\partial y}\left[2\,\delta(x)\,\upsilon_x\,\upsilon_y\right] - \frac{\partial}{\partial x}\left[\delta(x)\,\left(k(y)\upsilon_x^2 + \upsilon_y^2 + \mu^2 k(y)\upsilon^2\right)\right] +$$

$$+\delta(x)\left[\alpha\left(k(y)\upsilon_x^2 + \upsilon_y^2\right) - 4\mu\,k(y)\upsilon_x^2 + \alpha\mu^2 k(y)\upsilon^2\right],$$

holds true while as $y > 0$ we have the equality

$$2\delta(x)\,L_\mu\upsilon = 2\,\delta(x)\,\left[\upsilon_{yy} + \,\upsilon_x + \mu\upsilon\right] = \frac{\partial}{\partial x}\left[\delta(x)\,\left(\mu\upsilon^2 - \upsilon_y^2\right)\right] + \frac{\partial}{\partial y}\left[2\delta(x)\upsilon_x\upsilon_y\right] +$$

$$+\delta(x)\left[2\upsilon_x^2 + \alpha\,\upsilon_y^2 - \mu\alpha\upsilon^2\right],$$

where $\delta(x) = \exp\left(\alpha x\right)$, $\alpha > 0$. With the above equalities, it is easy to verify that for any function $\upsilon(x,y) \in C^1\left(\bar{\Omega}\right) \cap C^2\left(\Omega_i\right)$, $i = \overline{1,2}$ holds

$$2\left(\delta(x)\upsilon_x,\, L_\mu\upsilon\right)_0 = \int\limits_{\Omega} 2\delta(x)\,\upsilon_x\,L_\mu\upsilon d\Omega = \int\limits_{\Omega_1} 2\delta(x)\,\upsilon_x\,L_\mu\upsilon d\Omega_1 + \int\limits_{\Omega_2} 2\delta(x)\,\upsilon_x\,L_\mu\upsilon d\Omega_2 =$$

$$= \int\limits_{\Omega_1} 2\,\delta(x)\,\upsilon_x\,\left[\upsilon_{yy} - k(y)\upsilon_{xx} - 2\mu\,k(y)\,\upsilon_x - \mu^2 k(y)\upsilon\right] d\Omega_1 +$$

$$+ \int\limits_{\Omega_2} 2\delta(x)\,\upsilon_x\,\left[\upsilon_{yy} + \upsilon_x + \mu\upsilon\right] d\Omega_2 =$$

$$= \int\limits_{\Omega_1} \frac{\partial}{\partial y}\left[2\,\delta(x)\,\upsilon_x\,\upsilon_y\right] - \frac{\partial}{\partial x}\left[\delta(x)\left(k(y)\upsilon_x^2 + \upsilon_y^2 + \mu^2 k(y)\upsilon^2\right)\right] d\Omega_1 +$$

$$+\int\limits_{\Omega_2} \frac{\partial}{\partial x}\left[\delta(x)\,\left(\mu\upsilon^2-\upsilon_y^2\right)\right]+\frac{\partial}{\partial y}\left[2\delta(x)\upsilon_x\upsilon_y\right]d\Omega_2+$$

$$+\int\limits_{\Omega_1} \delta(x)\left[\alpha\left(k(y)\upsilon_x^2+\upsilon_y^2\right)-4\mu\, k(y)\upsilon_x^2+\alpha\mu^2k(y)\upsilon^2\right]d\Omega_1+$$

$$+\int\limits_{\Omega_2} \delta(x)\left[2\upsilon_x^2+\alpha\,\upsilon_y^2-\mu\alpha\upsilon^2\right]d\Omega_2.$$

Applying the Green formula to the last equation, we obtain

$$2\left(\delta(x)\upsilon_x,\, L_\mu\upsilon\right)_0=-\int\limits_{\Gamma_1}\left[2\,\delta(x)\,\upsilon_x\,\upsilon_y\right]dx+\left[\delta(x)\left(k(y)\upsilon_x^2+\upsilon_y^2+\mu^2k(y)\upsilon^2\right)\right]dy+$$

$$+\int\limits_{\Gamma_2}\left[\delta(x)\left(\mu\upsilon^2-\upsilon_y^2\right)\right]dy-\left[2\delta(x)\upsilon_x\upsilon_y\right]dx+$$

$$+\int\limits_{\Omega_1}\delta(x)\left[(\alpha-4\mu)\,k(y)\upsilon_x^2\;+\alpha\,\upsilon_y^2+\alpha\mu^2k(y)\upsilon^2\right]d\Omega_1+$$

$$+\int\limits_{\Omega_2}\delta(x)\left[2\upsilon_x^2+\alpha\,\upsilon_y^2-\mu\alpha\upsilon^2\right]d\Omega_2=$$

$$=\int\limits_{A}^{A_0}\delta(x)\left[\upsilon_y^2\,(0,\,y)-\mu\upsilon^2\,(0,\,y)\right]dy+\int\limits_{B}^{B_0}\delta(x)\left[\mu\upsilon^2\,(r,y)-\upsilon_y^2\,(r,y)\right]dy+$$

$$+\int\limits_{A_0}^{B_0}\delta(x)\left[2\upsilon_x\upsilon_y dx+\left(\upsilon_y^2-\mu\,\upsilon^2\right)dy\right]-$$

$$-\int\limits_{A}^{C}\delta(x)\left[2\upsilon_x\,\upsilon_y dx+\left(k(y)\upsilon_x^2+\upsilon_y^2+\mu^2k(y)\upsilon^2\right)dy\right]-$$

$$-\int\limits_{C}^{B}\delta(x)\left[2\upsilon_x\,\upsilon_y dx+\left(k(y)\upsilon_x^2+\upsilon_y^2+\mu^2k(y)\upsilon^2\right)dy\right]+$$

$$+\int\limits_{\Omega_1}\delta(x)\left[(\alpha-4\mu)\,k(y)\upsilon_x^2\;+\alpha\,\upsilon_y^2+\alpha\mu^2k(y)\upsilon^2\right]d\Omega_1+$$

$$+\int\limits_{\Omega_2}\delta(x)\left[2\upsilon_x^2+\alpha\,\upsilon_y^2-\mu\alpha\upsilon^2\right]d\Omega_2=$$

$$=I_1+I_2+I_3+I_4+I_5+I_6+I_7, \tag{15}$$

where $\Gamma_1=AA_0\cup A_0B_0\cup BB_0\cup AB;\;\Gamma_2=AC\cup CB\cup AB$ — are boundaries of the domains Ω_1 and Ω_2, respectively.

Since $\mu<0$ then

$$I_1=\int\limits_{A}^{A_0}\delta(x)\;\left[\upsilon_y^2\,(0,\,y)-\mu\upsilon^2\,(0,\,y)\right]dy\geq 0$$

and due to boundary condition (8)

$$I_2 = \int\limits_{B}^{B_0} \delta(x)\ \left[\mu\, \upsilon^2\,(r,\, y) - \upsilon_y^2\,(r,\, y)\right] dy = 0.$$

Next, in view of boundary condition (8) $\upsilon(x,y)\,|_{A_0B_0} = \upsilon(x,y)\,\big|_{y=\varphi(x)} = 0$, and therefore in the line A_0B_0 the equality: $\upsilon_x + \upsilon_y\varphi'(x) = 0$ holds. Thus,

$$I_3 = \int\limits_{A_0}^{B_0} \delta(x)\left[2\upsilon_x\upsilon_y dx + \left(\upsilon_y^2 - \mu\,\upsilon^2\right) dy\right] = \int\limits_{0}^{r} \delta(x)\left[2\upsilon_x\upsilon_y + \upsilon_y^2\,\varphi'(x)\right] dx =$$

$$= \int\limits_{0}^{r} \delta(x)\left[-2\varphi'(x)\upsilon_y^2 + \varphi'(x)\upsilon_y^2\right] dx = -\int\limits_{0}^{r} \delta(x)\,\varphi'(x)\,\upsilon_y^2 dx$$

and with condition (10) for the function $y = \varphi(x)$ the integral $I_3 \geq 0$.

On the curve $\sigma_1 :\ y = \gamma_1(x)$ taking into account the boundary condition (9) we have

$$I_4 = -\int\limits_{A}^{C} \delta(x)\left[2\upsilon_x\,\upsilon_y dx + \left(k(y)\upsilon_x^2 + \upsilon_y^2 + \mu^2 k(y)\upsilon^2\right) dy\right] =$$

$$= -\int\limits_{0}^{l} \delta(x)\left[2\upsilon_x\,\upsilon_y\ + \left(k(y)\upsilon_x^2 + \upsilon_y^2 + \mu^2 k(y)\upsilon^2\right)\ \gamma_1'(x)\right] dx =$$

$$= -k(y)\int\limits_{0}^{l} \gamma_1'(x)\,\delta(x)\left[\upsilon_x^2 + \mu^2\upsilon^2\right] dx \geq 0.$$

Similarly on the curve $\sigma_2 :\ y = \gamma_2(x)$ get

$$I_5 = -\int\limits_{C}^{B} \delta(x)\left[2\upsilon_x\,\upsilon_y dx + \left(k(y)\upsilon_x^2 + \upsilon_y^2 + \mu^2 k(y)\upsilon^2\right) dy\right] =$$

$$= -\int\limits_{l}^{r} \delta(x)\left[2\upsilon_x\,\upsilon_y + \left(k(y)\upsilon_x^2 + \upsilon_y^2 + \mu^2 k(y)\upsilon^2\right) \gamma_2'(x)\right] dx =$$

$$= \int\limits_{l}^{r} \delta(x)\left[-k(y)\,\gamma_2'(x)\,\upsilon_x^2 - 2\upsilon_x\,\upsilon_y\ - \gamma_2'(x)\,\upsilon_y^2 - \mu^2\,k(y)\,\gamma_2'(x)\,\upsilon^2\right] dx.$$

Due to condition (8) of problem 1: $\upsilon(x,\ y)\,|_{\sigma_2} = \upsilon(x,\ y)\,\big|_{y\ =\ \gamma_2(x)} = 0$. Hence $[\upsilon_x(x,y) + \upsilon_y(x,y)\,\gamma_2'(x)]\,|_{\sigma_2 = CB} = 0$. Considering this, for the integral I_5 we have

$$I_5 = \int\limits_{l}^{r} \delta(x)\left[\gamma_2'(x)\,\upsilon_y^2 - k(y)\,\gamma_2'^3(x)\,\upsilon_y^2\right] dx = \int\limits_{l}^{r} \delta(x)\,\gamma_2'(x)\ \left[1 - k(y)\,\gamma_2'^2(x)\right] \upsilon_y^2 dx.$$

By the last formula it is clear under conditions (11), (13) on the curve $\sigma_2 = CB :\ y = \gamma_2(x)$ for integral I_5 we get the inequality

$$I_5 = \int\limits_{l}^{r} \delta(x)\gamma_2'(x)\left[1 - k(y)\,\gamma_2'^2(x)\right] \upsilon_y^2 dx \geq 0.$$

Thus, under conditions (10)–(13) of theorem 1 the integrals $I_n \geq 0,\ n = \overline{1,5}$. Discarding the nonnegative integrals $I_n,\ \ n = \overline{1,5}$ by (15) we arrive at the inequality

$$2\left(\delta(x)\upsilon_x,\ L_\mu\upsilon\right)_0 \geq$$
$$\geq \int\limits_{\Omega_1} \delta(x)\left[(\alpha-4\mu)\,k(y)\upsilon_x^2+\alpha\upsilon_y^2+\alpha\mu^2k(y)\upsilon^2\right]d\Omega_1+$$
$$+\int\limits_{\Omega_2} \delta(x)\left[2\upsilon_x^2+\alpha\,\upsilon_y^2-\mu\alpha\upsilon^2\right]d\Omega_2=$$
$$=\int\limits_{\Omega} \delta(x)\left[\left(2H(y)+(\alpha-4\mu)\,k(y)H\left(-y\right)\right)\upsilon_x^2+\alpha\upsilon_y^2\right]d\Omega+$$
$$+\alpha\int\limits_{\Omega} \delta(x)\left[\mu^2H(-y)-\mu H(y)\right]\upsilon^2d\Omega, \tag{16}$$

where $H(y)$ is the Heaviside function.

On the other hand, with the Cauchy-Bunyakovskii inequality, for all $\varepsilon > 0$ find

$$2\left(\delta(x)\upsilon_x, L_\mu\upsilon\right)_0 = 2\left(\sqrt{\varepsilon\delta(x)}\,\upsilon_x,\ \frac{\sqrt{\delta(x)}\,L_\mu\upsilon}{\sqrt{\varepsilon}}\right)_0 \leq$$
$$\leq \varepsilon\left\|\sqrt{\delta(x)}\upsilon_x\right\|_0^2+C\left(\varepsilon\right)\ \left\|\sqrt{\delta(x)}\,L_\mu\upsilon\right\|_0^2. \tag{17}$$

By (16) and (17) the inequality

$$\int\limits_{\Omega} \delta(x)\left[\left(2H(y)+(\alpha-4\mu)\ k(y)\,H\left(-y\right)\right)\ \upsilon_x^2+\right.$$
$$\left.+\ \alpha\upsilon_y^2+\alpha\left(\mu^2H\left(-y\right)-\mu H(y)\right)\ \upsilon^2\right]d\Omega\leq$$
$$\leq\varepsilon\ \left\|\sqrt{\delta(x)}\,\upsilon_x\right\|_0^2+C\left(\varepsilon\right)\ \left\|\sqrt{\delta(x)}\,L_\mu\upsilon\right\|_0^2,$$

is implied, whence

$$\int\limits_{\Omega} \delta(x)\ \left[\left(2H(y)+(\alpha-4\mu)\ k(y)\,H\left(-y\right)-\varepsilon\right)\ \upsilon_x^2+\alpha\upsilon_y^2+\right.$$
$$\left.+\ \alpha\left(\mu^2H\left(-y\right)-\mu H(y)\right)\ \upsilon^2\right]d\Omega\leq C_1\left(\varepsilon\right)\ \|\ L_\mu\upsilon\|_0^2, \tag{18}$$

where $C_1\left(\varepsilon\right) = \exp\left(\alpha r\right)C\left(\varepsilon\right)$. Select in the latter inequality the numbers $\varepsilon > 0,\ \alpha > 0,\ \mu < 0$ so that $\varepsilon < \min\left\{(\alpha-4\mu)\,k_1,\ 2\right\}$. Then for the left-hand side the following estimate holds:

$$\int\limits_{\Omega} \delta(x)\ \left[\left(2H(y)+(\alpha-4\mu)\ k(y)\,H\left(-y\right)-\varepsilon\right)\ \upsilon_x^2+\alpha\upsilon_y^2+\right.$$
$$\left.+\ \alpha\left(\mu^2H\left(-y\right)-\mu H(y)\right)\ \upsilon^2\right]d\Omega\geq M\int\limits_{\Omega}\left(\upsilon^2+\upsilon_x^2+\upsilon_y^2\right)d\Omega = M\ \|\upsilon\|_1^2, \tag{19}$$

where $M = \min\left\{2-\varepsilon;\ (\alpha-4\mu)k_1-\varepsilon;\ \alpha;\ \mu^2\alpha,\ |\mu|\alpha\right\}$.

By inequalities (18)–(19) we arrive at the a priori estimate (14). Theorem 1 is proved.

By the Theorem 1 we conclude that if $u = u(x, y)$ in the domain Ω is the solution of original equation (1) for the class $C^1\left(\bar{\Omega}\right)$ with the right-hand side $f(x, y) \in L_2\left(\Omega\right)$ satisfying the boundary conditions

$$u = 0\quad \forall\ (x, y) \in BB_0 \cup \sigma_0 \cup \sigma_2,$$

then we have the estimate

$$\int\limits_{\Omega}\left[\left(u_x-\mu u\right)^2+u_y^2+u^2\right]d\Omega\leq M_1^2\ \|f\|_0^2. \tag{20}$$

By the a priori estimate (20) implies the uniqueness of the regular solution of Problem 1 and the existence of a weak solution for the dual of Problem 1 for any right-hand side $f(x,y) \in L_2(\Omega)$.

We note that boundary condition (9) in the statement of Problem 1 can be replaced either by the condition $\upsilon_x = 0 \quad \forall\,(x,y) \in \sigma_1$ or the condition $\upsilon_n = 0 \quad \forall\,(x,y) \in \sigma_1$, where υ_n is the derivative of the function $\upsilon = \upsilon(x,y)$ in the direction of the outward-pointing normal to the curve $\sigma_1 = AC$.

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Ж.А. Балкизов

Гиперболану облысында сипаттаушылардан алыстаған екінші ретті парабола-гиперболалық типті моделді теңдеу үшін бірінші шеттік есеп

Екінші ретті моделді парабола-гиперболалық типті теңдеу мысалында коэффициенттеріне қатысты А.М. Нахушевтің шарттары орындалмаған жағдайы үшін бірінші шеттік есеп қойылып зерттелген. Коэффициенттеріне қатысты А.М. Нахушевтің шарттары орындалмаған жағдайға қарамастан, жұмыста зерттеліп отырған бірінші шеттік есеп үшін А.М. Нахушевтің жұмыстарында алынған априорлық бағалауға ұқсас априорлық бағалаудың орын алатыны көрсетілген.

Кілт сөздер: параболикалық-гиперболалық теңдеу, бірінші шекаралық есеп, проблеманы шешуге априорлы бағалау.

Ж.А. Балкизов

Первая краевая задача для модельного уравнения параболо-гиперболического типа второго порядка с отходом от характеристик в области гиперболичности

На примере модельного уравнения параболо-гиперболического типа второго порядка, относительно коэффициентов которого нарушены условия А.М. Нахушева, сформулирована и исследована первая краевая задача. Показано, что несмотря на то, что относительно коэффициентов рассматриваемого уравнения нарушены условия А.М. Нахушева, для решения исследуемой в работе первой краевой задачи будет иметь место априорная оценка, аналогичная априорной оценке, полученной А.М. Нахушевым.

Ключевые слова: уравнение смешанного параболо-гиперболического типа, первая краевая задача, априорная оценка решения.

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Hardy-Littlewood theorem for series with general monotone coefficients

In this work we study trigonometric series with general monotone coefficients. Also, we consider $L_q\varphi(L_q)$ space. In particular, when $\varphi(t)\equiv 1$ the space $L_q\varphi(L_q)$ coincides with L_q. Well known the theorem of Hardy and Littlewood about trigonometric series with monotone coefficients. Also known various generalizations of this theorem. In 1982 this theorem was generalized by M.F. Timan for the spaces $L_q\varphi(L_q)$. And in 2007 S.Tikhonov proved Hardy-Littlewood theorem for trigonometric series with general monotone coefficients. In this work we have generalized Hardy-Littlewood theorem for Fourier series of functions $f \in L_q\varphi(L_q)$ with general monotone coefficients. Also, obtained upper-bound estimate of best approximation of functions $f \in L_q$ through its Fourier's coefficients which are general monotone.

Keywords: trigonometric series, Hardy-Littlewood theorem, general monotone sequences, convergence, Fourier's coefficients.

Let $L_q(0,2\pi)$, $1\le q<+\infty$ denotes the space of all 2π- periodic, measurable by Lebesgue functions $f(x)$, for which

$$\|f\|_q=\left(\int_0^{2\pi}|f(x)|^q\,dx\right)^{\frac{1}{q}}<+\infty.$$

Through $E_n(f)_q$ we will designate the best approximation of a function $f\in L_q$ by trigonometrical polynomials of total degree n in the metric of spaces L_q:

$$E_n(f)_q=\inf_{T_n}\|f-T_n\|_q.$$

Let the function $\varphi(t)$ satisfies the following conditions [1]:
a) $\varphi(t)$ is an even, non-negative, non-decreasing on $[0,+\infty)$;
b) $\varphi(t^2)\le C\varphi(t)$, $t\in[0,\infty)$, $C\ge 1$;
c) $\frac{\varphi(t)}{t^\varepsilon}\downarrow$ on $(0,+\infty)$ for some $\varepsilon>0$.
Measurable, 2π-periodic function $f\in L_q\varphi(L_q)$, if

$$\int_0^{\pi}|f(x)|^q\,\varphi\left(|f(x)|^q\right)dx<+\infty.$$

In particular, when $\varphi(t)\equiv 1$ the space $L_q\varphi(L_q)$ coincides with L_q.

We consider the series

$$\sum_{n=1}^{\infty}a_n\cos nx \tag{1}$$

and denote by $f(x)$ the sum of this series.

Definition [2]. The sequence of numbers $\{a_n\}$ is said to be general monotone, or $\{a_n\}\in GMS$, if the relation

$$\sum_{k=n}^{2n-1}|a_k-a_{k+1}|\le C|a_n|$$

holds for all $n\ge 1$, where the constant C is independent of n.

The set of all numerical sequences $\{a_n\}$ such that $a_n\downarrow 0$, $n\to\infty$, is denoted by MS. It is known that $MS\subset GMS$.

We give the following well-known theorem of Hardy-Littlewood

Theorem A [2]. Let $\{a_n\} \in MS$. A necessary and sufficient condition that the function $f(x) \in L_q$, $1 < q < +\infty$, is that

$$\sum_{n=1}^{+\infty} n^{q-2} \cdot a_n^q < +\infty.$$

In 1982 this theorem was generalized by M.F.Timan [1] for the spaces $L_q\varphi(L_q)$.

Theorem B. Let $\varphi(t)$ satisfies conditions a)-c) and $f(x) \in L_1$ is an even function with Fourier series $\sum\limits_{n=1}^{+\infty} a_n \cos nx$, where $\{a_n\} \in MS$. A necessary and sufficient condition that the function $f \in L_p\varphi(L_p)$ for some $p > 1$, is that

$$\sum_{n=1}^{+\infty} n^{p-2} \cdot a_n^p \varphi(n) < +\infty.$$

In 2007 S.Tikhonov [2] proved the following theorem.

Theorem C. Let $\{a_n\}$ be a positive sequence and $\{a_n\} \in GMS$. A necessary and sufficient condition that the function f should belong to L_q, $1 < q < +\infty$, is that inequality

$$\sum_{n=1}^{+\infty} n^{q-2} \cdot a_n^q < +\infty$$

holds.

Our main goal is to prove the theorem of Hardy and Littlewood for the Fourier series of a function $f \in L_q\varphi(L_q)$, the coefficients are generally monotonous.

To obtain the main result we need the following Lemma.

Lemma. Let $f(x) = \sum\limits_{n=1}^{+\infty} a_n \cos nx$, where positive sequence $\{a_n\} \in GMS$ and for some q, $1 < q < +\infty$ converges the series $\sum\limits_{n=1}^{+\infty} n^{q-2} \cdot a_n^q$.

Then, the following inequality holds

$$E_n(f)_q \le C \left[a_n (n+1)^{1-\frac{1}{q}} + \left(\sum_{k=n+1}^{+\infty} k^{q-2} \cdot a_k^q \right)^{\frac{1}{q}} \right], n = 1, 2, ...$$

Proof. From the properties of the best approximation and norms, we have

$$E_n(f)_q \le \|f - S_n(f)\|_q = \|f - S_n(f) + a_n \cdot D_n(\cdot) - a_n \cdot D_n(\cdot)\|_q \le$$
$$\le \|f - S_n(f) + a_n \cdot D_n(\cdot)\|_q + a_n \|D_n(\cdot)\|_q , \qquad (2)$$

where $S_n(f)$ is the partial sum of series (1), $D_n(x) = \sum\limits_{k=1}^{n} \cos kx$ is the Dirichlet kernel.

For the Dirichlet kernel is known the following inequality [2]

$$\|D_n(\cdot)\|_q \le C \cdot n^{1-\frac{1}{q}}. \qquad (3)$$

To estimate the first term in (2) we use the theorem C. Because the sequence of coefficients of the series

$$f(x) - S_n(f)(x) + a_n \cdot D_n(x) = \sum_{k=1}^{+\infty} b_k \cdot \cos kx$$

belongs to GMS. Indeed, for

$$b_k = \begin{cases} a_n, & k = 1, ..., n; \\ a_k, & k = n+1, ... \end{cases}$$

we have

$$\sum_{k=m}^{2m-1} |b_k - b_{k+1}| = 0 < C \cdot b_m, \text{ at } 2m - 1 \le n;$$

$$\sum_{k=m}^{2m-1} |b_k - b_{k+1}| = \sum_{k=m}^{2m-1} |a_k - a_{k+1}| \le C \cdot a_m = C \cdot b_m, \text{ at } m \ge n.$$

If $m < n < 2m-1$, then we assume that $b_k = a_n$, $k = m, ..., m+s$ and $b_k = a_k$, $k = m+s+1, ..., 2m-1$, where s is natural number. Then

$$\sum_{k=m}^{2m-1} |b_k - b_{k+1}| = \sum_{k=m}^{m+s-1} |b_k - b_{k+1}| + \sum_{k=m+s}^{2m-1} |b_k - b_{k+1}| =$$

$$= \sum_{k=m+s}^{2m-1} |a_k - a_{k+1}| < \sum_{k=m+s}^{2(m+s)-1} |a_k - a_{k+1}| \le C \cdot a_{m+s} \le C \cdot b_m.$$

Therefore, by theorem C

$$\|f - S_n(f) + a_n \cdot D_n(\cdot)\|_q \le C \left(a_n^q \sum_{k=1}^{n} k^{q-2} + \sum_{k=n+1}^{\infty} a_k^q \cdot k^{q-2} \right)^{\frac{1}{q}} \le$$

$$\le C \cdot a_n (n+1)^{1-\frac{1}{q}} + \left(\sum_{k=n+1}^{\infty} a_k^q \cdot k^{q-2} \right)^{\frac{1}{q}}. \qquad (4)$$

Now, using inequalities (3) and (4), from (2) we have

$$E_n(f)_q \le C \left[a_n (n+1)^{1-\frac{1}{q}} + \left(\sum_{k=n+1}^{+\infty} k^{q-2} \cdot a_k^q \right)^{\frac{1}{q}} \right], n = 1, 2, ...$$

This completes the proof of Lemma.

Now we prove the main result:

Theorem. Let the function $\varphi(t)$ satisfies the conditions a)-c), and $f(x) \in L_1$ is an even function with Fourier series $\sum\limits_{n=1}^{+\infty} a_n \cos nx$, where $\{a_n\}$ is positive sequence, and $\{a_n\} \in GMS$.

A necessary and sufficient condition that the function f should belong to $L_q\varphi(L_q)$, $q > 1$, is that

$$\sum_{n=2}^{+\infty} n^{q-2} \cdot a_n^q \cdot \varphi(n) < +\infty. \qquad (5)$$

Proof. Suppose inequality (5) holds. Then, from the properties of the function $\varphi(t)$ converges also following series

$$\sum_{n=2}^{+\infty} n^{q-2} \cdot a_n^q.$$

So, by theorem 4.2 of [2] $f \in L_q$. Then applying the Lemma, the inequality of Hardy [1] and the properties of the function $\varphi(t)$, we have for $1 < p_0 < q < +\infty$:

$$\sum_{n=1}^{+\infty} n^{\frac{q}{p_0}-2} \cdot \varphi(n) \cdot E_n^q(f)_{p_0} \le C \cdot \sum_{n=1}^{+\infty} n^{\frac{q}{p_0}-2} \cdot \varphi(n) \cdot \left[a_n (n+1)^{1-\frac{1}{p_0}} + \left(\sum_{k=n+1}^{+\infty} k^{p_0-2} \cdot a_k^{p_0} \right)^{\frac{1}{p_0}} \right]^q \le$$

$$\le C \cdot \sum_{n=1}^{+\infty} n^{\frac{q}{p_0}-2} \cdot \varphi(n) \cdot a_n^q n^{q\left(1-\frac{1}{p_0}\right)} + C \cdot \sum_{n=1}^{+\infty} n^{\frac{q}{p_0}-2} \cdot \varphi(n) \left(\sum_{k=n+1}^{+\infty} k^{p_0-2} \cdot a_k^{p_0} \right)^{\frac{q}{p_0}} \le$$

$$\le C \cdot \sum_{n=1}^{+\infty} n^{q-2} \cdot \varphi(n) \cdot a_n^q + C \cdot \sum_{n=1}^{+\infty} n^{-\left(2-\frac{q}{p_0}\right)} \left(\sum_{k=n+1}^{+\infty} k^{p_0-2} \cdot a_k^{p_0} \varphi^{\frac{p_0}{q}}(k) \right)^{\frac{q}{p_0}} \le$$

$$\leq C \cdot \sum_{n=1}^{+\infty} n^{q-2} \cdot \varphi(n) \cdot a_n^q + C \cdot \sum_{n=1}^{+\infty} n^{\frac{q}{p_0}-2} \left(n \cdot a_n^{p_0} \varphi^{\frac{p_0}{q}}(n) \cdot n^{p_0-2} \right)^{\frac{q}{p_0}} \leq$$

$$\leq C \cdot \sum_{n=1}^{+\infty} n^{q-2} \cdot a_n^q \varphi(n) < +\infty.$$

Hence by theorem 3.3 of [1] we have

$$\int_0^\pi |f(x)|^q \varphi\left(|f(x)|^q\right) dx < +\infty.$$

Now let us prove the opposite. Let $f(x) \in L_q\varphi(L_q)$. Then by theorem 13.1 of [3; 54] we have

$$\int_0^\pi |f(x)|^q \varphi\left(|f(x)|^q\right) dx = \int_0^\pi \left(f^*(t)\right)^q \varphi\left(\left(f^*(t)\right)^q\right) dt < +\infty,$$

where f^* — is non-increasing rearrangement of f.

Let

$$f_1(x) = \int_0^x f(u)du \quad \text{and} \quad f_2(x) = \int_0^x f_1(u)du, \quad \text{for} \quad x \in (0,\pi).$$

Also for $x \in (0,\pi)$ we denote

$$f_3(x) = \int_0^x f^*(u)du \quad \text{and} \quad f_4(x) = \int_0^x f_3(u)du.$$

In [4] proved that for $\frac{\pi}{4(n+1)} \leq x \leq \frac{\pi}{4n}$ the following inequality is satisfied

$$f_4(x) \geq |f_2(x)| \geq f_2(x) \geq \frac{C}{n^2} \sum_{k=\left[\frac{n}{2}\right]}^{n} a_k.$$

Next, arguing as in [4], we have

$$\sum_{n=1}^{+\infty} n^{q-2} \cdot a_n^q \varphi(n) \leq C \cdot \sum_{n=1}^{+\infty} n^{q-2} \cdot \varphi(n) \left(\frac{2}{n} \sum_{k=\left[\frac{n}{2}\right]}^{n} a_k \right)^q =$$

$$= C \cdot \sum_{n=1}^{+\infty} n^{-2} \cdot \varphi(n) \left(\sum_{k=\left[\frac{n}{2}\right]}^{n} a_k \right)^q \leq C \cdot \sum_{n=1}^{+\infty} n^{2q-2} \cdot \varphi(n) \min_{\frac{\pi}{4(n+1)} \leq x \leq \frac{\pi}{4n}} \left(f_4(x)\right)^q \leq$$

$$\leq C \cdot \sum_{n=1}^{+\infty} \varphi(n) \frac{4n(n+1)}{\pi} \int_{\frac{\pi}{4(n+1)}}^{\frac{\pi}{4n}} \left(\frac{\pi}{4x}\right)^{2q-2} \left(f_4(x)\right)^q dx \leq$$

$$\leq C \cdot \sum_{n=1}^{+\infty} \int_{\frac{\pi}{4(n+1)}}^{\frac{\pi}{4n}} \left(\frac{\pi}{4x}\right)^{2q} \varphi\left(\frac{\pi}{x}\right) \left(f_4(x)\right)^q dx \leq$$

$$\leq C \cdot \int_0^\pi x^{-2q} \varphi\left(\frac{\pi}{x}\right) \left(\int_0^x f_3(u)du \right)^q dx =$$

$$= C \int_0^\pi x^{-2q} \varphi\left(\frac{\pi}{x}\right) \left(\int_0^x \left(\int_0^u f^*(t)dt \right) du \right)^q dx \leq$$

$$\leq C \int_0^\pi x^{-q} \left(\int_0^x \left(\int_0^u \varphi^{\frac{1}{q}}\left(\frac{\pi}{t}\right) f^*(t)dt \right) \frac{du}{u} \right)^q dx \leq$$

$$\leq C \int_0^\pi x^{-q+1} \left(\int_0^x \left(\int_0^u \varphi^{\frac{1}{q}}\left(\frac{\pi}{t}\right) f^*(t)dt \right) \frac{du}{u} \right)^q \frac{dx}{x} =$$

$$= C\int_0^{\pi}\left(x^{-1+\frac{1}{q}}\int_0^x\left(\int_0^u \varphi^{\frac{1}{q}}\left(\frac{\pi}{t}\right)f^*(t)dt\right)\frac{du}{u}\right)^q\frac{dx}{x}\le$$

$$\le C\int_0^{\pi}\left(x^{-1+\frac{1}{q}}\int_0^x \varphi^{\frac{1}{q}}\left(\frac{\pi}{t}\right)f^*(t)dt\right)^q\frac{dx}{x}=$$

$$= C\int_0^{\pi}\left(\frac{1}{x}\int_0^x \varphi^{\frac{1}{q}}\left(\frac{\pi}{t}\right)f^*(t)dt\right)^q dx\le C\cdot\int_0^{\pi}\varphi\left(\frac{\pi}{t}\right)\cdot(f^*(t))^q\,dt\le$$

$$\le C\cdot\int_0^{\pi}(f^*(t))^q\,\varphi\left((f^*(t))^q\right)dt<+\infty.$$

This completes the proof of Theorem.

Remark. The proved Theorem is extension of the Theorem B. Also, at $\varphi(t)\equiv 1$ the Theorem C follows from the proved Theorem.

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С.Бітімхан

Жалпы монотонды коэффициентті қатарлар үшін Харди-Литтлвуд теоремасы

Мақалада жалпы монотонды коэффициентті тригонометриялық қатарлар зерттелді. Сонымен бірге $L_q\varphi(L_q)$ кеңістігі қарастырылды. Дербес жағдайда $\varphi(t)\equiv 1$ болғанда $L_q\varphi(L_q)$ кеңістігі L_q кеңістігімен беттеседі. Монотонды коэффициентті тригонометриялық қатарлар үшін Харди мен Литтлвуд теоремасы жақсы белгілі. Сондай-ақ теореманың әртүрлі жалпыламалары да белгілі. 1982 ж. осы теореманы М.Ф.Тиман $L_q\varphi(L_q)$ кеңістігі үшін жалпылады, ал 2007 ж. С.Тихонов Харди-Литтлвуд теоремасын жалпы монотонды коэффициентті тригонометриялық қатарлар үшін дәлелдеді. Бұл жұмыста Харди-Литтлвуд теоремасын $f\in L_q\varphi(L_q)$ функциясының коэффициенттері жалпы монотонды болатын Фурье қатарлары үшін жалпыланды. Сонымен бірге $f\in L_q$ функциясының ең жақсы жуықтауының жоғарыдан бағалауын оның жалпы монотонды болатын Фурье коэффициенттері арқылы алынды.

Кілт сөздер: тригонометриялық қатарлар, Харди-Литтлвуд теоремасы, жалпы монотонды тізбектер, жинақтылық, Фурье коэффициенттері.

С.Битимхан

Теорема Харди-Литтлвуда для рядов с обобщенно-монотонными коэффициентами

В статье исследованы тригонометрические ряды с обобщенно-монотонными коэффициентами. Также рассмотрено пространство $L_q\varphi(L_q)$. В частности, когда $\varphi(t) \equiv 1$, пространство $L_q\varphi(L_q)$ совпадает с L_q. Хорошо известна теорема Харди и Литтлвуда о тригонометрических рядах с монотонными коэффициентами. Также известны различные обобщения этой теоремы. В 1982 г. М.Ф.Тиман обобщил эту теорему для пространства $L_q\varphi(L_q)$, а в 2007 г. С.Тихонов доказал теорему Харди-Литтлвуда для тригонометрических рядов с обобщенно-монотонными коэффициентами. В данной работе обобщили теорему Харди-Литтлвуда для рядов Фурье функции $f \in L_q\varphi(L_q)$ с обобщенно-монотонными коэффициентами. Также получена верхняя оценка наилучшего приближения функции $f \in L_q$ через её коэффициенты Фурье, которые являются обобщенно-монотонными.

Ключевые слова: тригонометрические ряды, теорема Харди-Литтлвуда, обобщенно-монотонные последовательности, сходимость, коэффициенты Фурье.

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On spectral question of the Cauchy-Riemann operator with homogeneous boundary value conditions

In this paper we consider the eigenvalue problem for the Cauchy-Riemann operator with homogeneous Dirichlet type boundary conditions. The statement of the problem is justified to the theorem of M. Otelbaev and A.N. Shynybekov, which implies the correctness of the considered problem. As an example, non-local boundary conditions and Bitsadze-Samarskii type boundary conditions are given. It is taken into account that the above spectral problem for a differential Cauchy-Riemann operator with homogeneous boundary conditions of the Dirichlet type type is reduced to a singular integral, also reduces to a linear integral equation of the second kind with a continuous kernel. And it is also taken into account that the index of the singular integral equation is zero and the Noetherian condition is obtain. It is proved that the considered spectral problem does not have eigenvalues, that is, for any complex ?, has only the zero solution and thus the Cauchy-Riemann spectral problem is a Volterra problem.

Keywords: Cauchy-Riemann operator, Dirichlet type problem, spectral parameter, resolvent set, residues, kernel, homogeneous boundary conditions, Volterra property, Noetherian, Fredholm equation.

Introduction

In the functional space $C(|z| \leq 1)$ we consider an operator K, generated by the differential Cauchy-Riemann operation

$$K\omega(z) = \frac{\partial \omega(z)}{\partial \overline{z}},$$

where $z = x + iy, \overline{z} = x - iy$, $\frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$ on the set

$$D(K) \subset \left\{\omega(x) \in C(|z| \leq 1), \frac{\partial \omega}{\partial \overline{z}} \in C(|z| < 1)\right\}.$$

We assume that the operator K has a non-empty resolvent set $\rho(K)$. Not begging for generality, we assume that

$$0 \in \rho(K), \tag{1}$$

i.e. there is a bounded operator K^{-1}. In [1] the set of the operators $\{K\}$ with the property (1) has been described:

Theorem 1. [1]. For every linear operator K with the property (1) there exists a bounded operator G, which carries continuous functions in the circle $|z| \leq 1$ into holomorphic functions for which imaginary parts are equal to zero when $z = 0$, and also the bounded functional $S(f)$ on the set of continuous functions in the circle $|z| \leq 1$ that uniquely determine domain of the operator K by the formula:

$$D(K) = \left\{\omega(z) \in C(|z| \leq 1), \frac{\partial \omega}{\partial \overline{z}} \in C(|z| < 1)\ , \mathrm{Re}\,\omega(z) = \mathrm{Re}\, G(\frac{\partial \omega}{\partial \overline{z}}),\ |z| = 1;\right.$$

$$\left.\mathrm{Im}\,\omega(0) = \mathrm{Im}\, S(\frac{\partial \omega}{\partial \overline{z}}),\ |z| = 0\right\}.$$

Inversely, the pair of G and S determines , for which (1) is true.

It is known in [2; 151], that the boundary value problem:

$$\frac{\partial \omega}{\partial \overline{z}} = f(z),\ |z| < 1;$$

$$\mathrm{Re}\omega(z)|_{|z|=1} = g(z);$$
$$\mathrm{Im}\omega(0) = C,$$

has a unique solution $\omega(z)$ at any selection

$$f(z) \in C(|z| < 1),\ g(z) \in C(|z| = 1),\ C \in R.$$

Moreover, this solution is obtained by the Schwartz's formula [2] (in the multidimensional case by the Poisson formula):

$$\omega(z) = \frac{1}{2\pi i} \oint\limits_{|t|=1} g(z) \frac{t+z}{t-z} \cdot \frac{dt}{t} + iC + L_{\Phi}^{-1} f(z),$$

where

$$\omega_{\Phi}(z) = L_{\Phi}^{-1} f(z)$$

is a solution of the homogenous boundary value problem for the non-homogenous equation:

$$\frac{\partial \omega_{\Phi}(z)}{\partial \overline{z}} = f(z),\ |z| < 1,$$

with the homogenous boundary value conditions:

$$\mathrm{Re}\omega_{\Phi}(z)|_{|z|=1} = 0, \quad \mathrm{Im}\omega_{\Phi}(0) = 0.$$

Let the operator be given by the Cauchy-Riemann relation and the condition (1) holds, that means existence of the inverse operator K^{-1}, it implies, that the operator equation $K\omega(z) = f(z)$ has the unique solution $\omega(z) = K^{-1} f(z)$.

Denote the real part of the solution $\omega(z)$ on the circle $|z| = 1$ by $g(z)$, and the imaginary part of the solution $\omega(z)$ when $z = 0$ by C .

In [1] as specific boundary value conditions it is chosen the boundary pair:

$$(Gf)(z) = \frac{1}{2\pi i} \oint\limits_{|t|=1} \frac{f(t)}{t-z} dt;$$

$$S(f) = \frac{1}{2\pi i} \oint\limits_{|t|=1} \frac{f(t)}{t} dt.$$

The corresponding eigenvalue problem to this boundary pair has the following form:

$$\frac{\partial \omega}{\partial \overline{z}} = \lambda \omega(z),\ |z| < 1;$$

$$\mathrm{Re}\omega(z) = \mathrm{Re} \frac{1}{2\pi i} \oint\limits_{|t|=1} \frac{\lambda \omega(t)}{t-z} dt, \quad |z| = 1;$$

$$\mathrm{Im}\omega(0) = \mathrm{Im} \frac{1}{2\pi i} \oint\limits_{|t|=1} \frac{\lambda \omega(t)}{t} dt.$$

For this problem in [3] for the spectral parameter λ conditions have been obtained for which the problem is Noetherian in the corresponding function space and is reduced to a linear integral Fredholm equation of the second kind with a continuous kernel. Moreover, formulas, characterizing the approximate structure of solution of the boundary-value problem with shift, have been obtained. The paper [4] is devoted to study of spectrum of elliptic operators. In the general case, the spectrum of an elliptic operator is essentially determined by spectral properties of boundary operator. However, identification dependence of the spectrum of the operator K in initial terms of boundary conditions represents an actual (unresolved) problem. From the general results, such facts are not traced; therefore it is necessary to involve deeper methods, related to specifics of the specific boundary conditions.

Formulation of the problem

In [1] along with nonlocal boundary-value conditions, as specific boundary value conditions the Bitsadze-Samarskii type boundary conditions, that is, the problem «with shift in interior of the domain» as well as the Dirichlet problem type homogeneous boundary value conditions are chosen, that is,

$$(Gf)(z)|_{|z|=1} = 0;$$

$$S(f)|_{z=0} = 0.$$

Then the spectral problem has the form:

$$\frac{\partial \omega(z)}{\partial \overline{z}} = \lambda \cdot \omega(z), \ |z| < 1; \tag{1a}$$

$$\mathrm{Re}\omega(z) = 0, \quad |z| = 1; \tag{2}$$

$$\mathrm{Im}\omega(0) = 0, \quad z = 0, \tag{3}$$

where complex λ is a spectral parameter, which is reduced to singular integral equation with continuous kernel, and the index is calculated, the condition for the Noetherian is established in [5, 6].

This case will be the subject of our research in this paper.

Description of general regular boundary value problems for the differential Cauchy-Riemann expression was developed by J.F. Neiman, M.I. Vishik, A.A. Dezin, in 1982 by M. Otelbayev and A.N. Shynybekov [1].

From another point of view, the problems of solvability and behavior of solution of the boundary value problem for the generalized Cauchy-Riemann equation have been extensively studied in [7–10]. Boundary value problems for the generalized Cauchy-Riemann system with non-smooth coefficients were studied in [11].

Spectral problems with regular but not intensely regular boundary value conditions for the multiple differentiation operator were studied in [12].

Main result of the paper

Denote general solution of the equation (1) by $\Phi(z) = \omega(z)e^{\lambda\overline{z}}$. Since

$$\frac{\partial}{\partial \overline{z}}\left(e^{\lambda\overline{z}} \cdot \omega(z)\right) = 0$$

in the circle $|z| < 1$ the function $\Phi(z)$ is holomorphic. Consequently, the spectral problem (1a)–(3) has the following form:

$$\frac{\partial \Phi(z)}{\partial \overline{z}} = 0, \ |z| < 1; \tag{4}$$

$$\mathrm{Re}\left(e^{\lambda\overline{z}} \cdot \Phi(z)\right)|_{|z|=1} = 0; \tag{5}$$

$$\mathrm{Im}\Phi(0) = 0. \tag{6}$$

Rewrite the real part of the complex number (5) in the form of a half-sum of the complex number and its conjugate, then we get to the relation when $|z| = 1$, with considering $z \cdot \overline{z} = 1$:

$$e^{\lambda\overline{z}} \cdot \Phi(z) + e^{\lambda\overline{z}} \cdot \Phi\left(\frac{1}{z}\right) = 0.$$

Introduce the function:

$$\widetilde{\Phi(z)} = \Phi(z), |z| < 1;$$

$$\widetilde{\Phi(z)} = \Phi\left(\frac{1}{z}\right), |z| > 1.$$

It follows that $\Phi(z) = \overline{\Phi\left(\frac{1}{z}\right)}$ when $|z| = 1$. Then along the unit circle $|z| = 1$ we have $e^{\lambda\overline{z}}\widetilde{\Phi_{+}(z)} + e^{\lambda\overline{z}}\widetilde{\Phi_{-}(z)} = 0$.

Solution has the following form [2; 147]: $\widetilde{\Phi(z)} = e^{\Gamma(z)}$,
where

$$\Gamma(z) = -\frac{1}{2\pi i} \oint\limits_{|t|=1} \frac{\overline{\lambda} t - \lambda \overline{t}}{t - z} dt.$$

When $|z| = 1$ check the condition $\Phi(z) = \overline{\Phi\left(\frac{1}{z}\right)}$.

As a result, we get:

$$\Gamma(z) = \overline{\Gamma\left(\frac{1}{z}\right)} = \frac{1}{2\pi i} \oint\limits_{|t|=1} \frac{\overline{\lambda} t - \lambda \overline{t}}{t - z} dt = -\frac{1}{2\pi i} \oint\limits_{|t|=1} \frac{\overline{\lambda} t - \lambda \overline{t}}{t - \frac{1}{z}} \overline{dt}.$$

Due to $t \cdot \overline{t} = 1, z \cdot \overline{z} = 1$, we consider the expression when $|t| = 1$ and $|z| = 1$:

$$\frac{\lambda \overline{t} - \overline{\lambda} t}{\overline{t} - \frac{1}{z}} = \frac{\lambda \overline{t} - \overline{\lambda} t}{\frac{1}{t} - z} = \frac{\overline{\lambda} t - \lambda \overline{t}}{1 - zt} \cdot t, \overline{dt} = d\overline{t} = d\left(\frac{1}{t}\right) = -\frac{dt}{t^2}.$$

Hence we have

$$\frac{1}{2\pi i} \oint\limits_{|t|=1} (\overline{\lambda} t - \lambda \overline{t}) \frac{dt}{t - z} = \frac{1}{2\pi i} \oint\limits_{|t|=1} \frac{\overline{\lambda} t - \lambda \overline{t}}{1 - zt} \frac{dt}{t}.$$

Due to the obvious equality, were write the last integral in the form:

$$\overline{\lambda} \frac{1}{2\pi i} \oint\limits_{|t|=1} \left(\frac{t}{t - z} - \frac{1}{1 - zt} \right) dt = \lambda \frac{1}{2\pi i} \oint\limits_{|t|=1} \left(\frac{1}{t(t - z)} - \frac{1}{(1 - zt)t^2} \right) dt.$$

By the Deduction Theorem [13] we have that

$$\overline{\lambda} \cdot \left(z - \lim_{t \to z} \frac{t - z}{1 - zt} \right) = \lambda \cdot \left(\frac{1}{-z} + \frac{1}{z} - \lim_{t \to z} \frac{t - z}{(1 - zt)t^2} - \lim_{t \to 0} \frac{d}{dt} \left(\frac{t^2}{(1 - zt)t^2} \right) \right)$$

and also

$$\overline{\lambda} \cdot \left(z - \lim_{t \to z} \frac{1}{-z} \right) = \lambda \cdot \left(-\frac{1}{z^2} \left(-\frac{1}{z} \right) - z \right).$$

It yields that

$$\overline{\lambda} \cdot \left(z + \frac{1}{z} \right) = \lambda \cdot \left(z - \frac{1}{z^3} \right) = 0;$$

that is

$$\overline{\lambda} \cdot \left(\frac{z^2 + 1}{z} \right) + \lambda \cdot \left(\frac{z^4 - 1}{z^3} \right) = 0.$$

The last equality is reduced to

$$\overline{\lambda} + \lambda \cdot \left(\frac{z^2 - 1}{z^2} \right) = 0.$$

Hence, inside the unit circle $|z| = 1$ we get $z^2 \overline{\lambda} + \lambda z^2 - \lambda = 0$.

Taking derivative by φ in the polar coordinates, we have:

$$\frac{\partial}{\partial \varphi} | e^{2i\varphi} \overline{\lambda} + \lambda e^{2i\varphi} - \lambda = 0.$$

Then the following equation is true:

$$2z\overline{\lambda} = -\lambda 2z.$$

As a result we receive:

$$\overline{\lambda} + \lambda = 0.$$

From this case it is not hard to establish that when $\lambda = 0$ condition $z^2(-\lambda) + \lambda z^2 = \lambda$ holds. Thus, it is proved

Theorem 2. The spectral problem (1a)–(3) does not have eigenvalues, that is, for any complex λ, the spectral problem (1a)–(3) has only the zero solution.

Remark. The spectral problem (1a)–(3) turned out to be a Volterra problem.

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Біртекті шеттік шарттармен берілген Коши-Риман операторының спектралдық мәселесі

Мақалада Дирихле текті біртекті шеттік шарттармен берілген Коши-Риман операторының меншікті мәндерін зерттеуге арналған есеп қарастырылған. Есептің қисынды қойылуы М. Өтелбаев пен А.Н. Шыныбековтың теоремасына негізделген. Бұл теорема негізінде Коши-Риман операторы үшін бейлокалды шеттік шарттармен берілген және Бицадзе-Самарский текті шеттік шарттармен берілген есептердің мысалдары көрсетілген. Қарастырылып отырған Дирихле текті біртекті шеттік шарттармен берілген Коши-Риман операторының меншікті мәндерін зерттеуге арналған есебінің сингулярлы интегралдық теңдеуге, сонан соң сызықтық интегралдық екінші текті Фредгольм теңдеуіне редукцияланғаны, сондай-ақ сингулярлы интегралдың теңдеудің индексінің нөлге тең болатындығы және нетерлік шарты туралы мәліметтер ескерілген. Авторлар Коши-Риман дифференциалдық операторы үшін қойылған спектралдық есептің меншікті мәндерінің болмайтындығын дәлелдеген, яғни кез келген кешенді λ үшін тек қана нөлдік шешімі ғана бар болады.

Кілт сөздер: Коши-Риман операторы, Дирихле тектес есеп, спектралдық параметр, резольвенттік жиын, қалындылар, ядро, біртекті шеттік шарттар, вольтерлік, нетерлі, Фредгольм теңдеуі.

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К спектральному вопросу оператора Коши-Римана с однородными краевыми условиями

В статье рассмотрена задача на собственные значения оператора Коши-Римана с однородными краевыми условиями типа задачи Дирихле. Постановка задачи обоснована к теореме М. Отелбаева и А.Н. Шыныбекова, установлена корректность рассматриваемой задачи, в качестве примера указаны нелокальные краевые условия и краевые условия типа Бицадзе-Самарского. Учитывается, что указанная выше спектральная задача для дифференциального оператора Коши-Римана с однородными краевыми условиями типа задачи Дирихле редуцирована к сингулярному интегральному, затем к линейному интегральному уравнению Фредгольма второго рода с непрерывным ядром. А также учтено, что индекс сингулярного интегрального уравнения равен нулю, и установлено условие нетеровости. Доказано, что рассматриваемая спектральная задача не имеет собственных значений, т.е. при любом комплексном λ имеет только нулевое решение.

Ключевые слова: оператор Коши-Римана, задача типа Дирихле, спектральный параметр, резольвентное множество, вычеты, ядро, однородные краевые условия, вольтерровость, нетерово, уравнение Фредгольма.

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On a stability of a solution of the loaded heat equation

Steadily growing interest in study of loaded differential equations is explained by the range of their applications and a circumstance that loaded equations make a special class of functional-differential equations with specific problems. These equations have applications in study of inverse problems of differential equations with important applied interests. In this paper solvability questions of stabilization problems with a boundary for the loaded heat equation are studied in the given bounded domain $\Omega \equiv (-\pi/2, \pi/2)$. The task is to choose boundary conditions (controls), that the solution of the obtained mixed boundary value problem tends to a given stationary solution with the prescribed speed $\exp(-\sigma_0 t)$ as $t \to \infty$. At this the control is required to be a feedback control, i.e. that it reacted to the unintended fluctuations of the system, suppressing the results of their impact on the stabilized solution. Stabilization problems have a direct connection with controllability problems. The paper proposes a mathematical formalization of the concept of feedback, and with its help it solves the problem of stabilizability of a loaded heat equation by dint of feedback control given on the part of the boundary is solved.

Keywords: stability, feedback control, loaded heat equation, boundary value problem, inverse problem, Green function, eigenvalue, eigenfunction.

Introduction. In recent years, an increasing interest in studying loaded differential equations is manifested. In this both the steadily extending field of their applications and the fact, that the loaded equations are a special class of equations with specific problems, played a role.

In this paper, the statement of the inverse problem on the stabilization of solutions for the loaded heat conduction equation using the boundary conditions is given. The theorem on the solvability of the inverse problem is proved and an algorithm for approximate constructing boundary controls in the form of synthesis is developed. The numerical calculations have been carried out, that show the effectiveness of the proposed algorithm (Fig. 1-3).

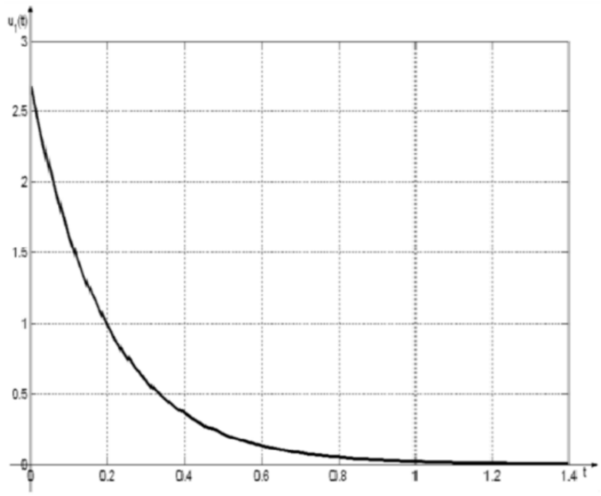

Figure 1. Graphic of $u_1(t)$

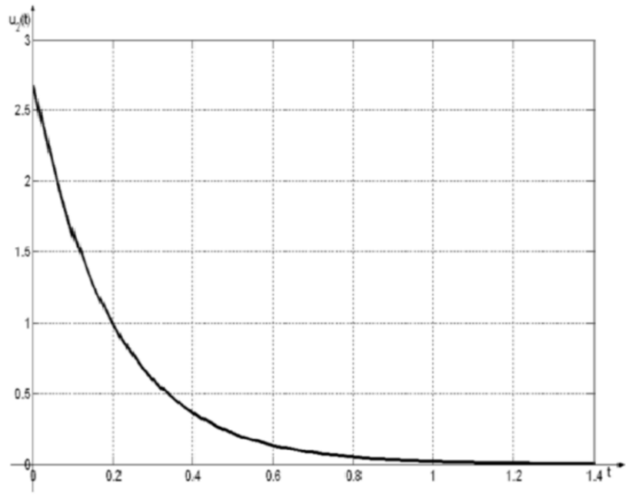

Figure 2. Graphic of $u_2(t)$

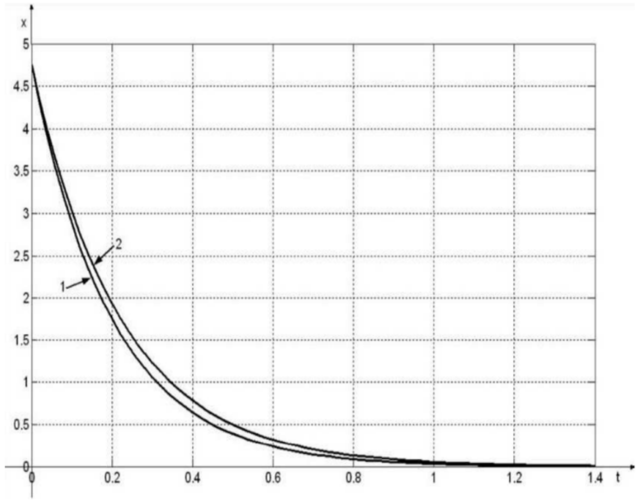

Figure 3. Graphics of $1 - \|y(x,t)\|_{L_2(-\pi/2,\pi/2)}$;
$2 - C_0 \cdot \exp\{-\sigma_0 t\}$, where $C_0 \approx 2,6808 \cdot \sqrt{\pi}$; $\alpha = 5,\ \sigma_0 = 4,5$

Statement of the boundary value problem. Search for such boundary controls $u_1(t), u_2(t) \in L_2(0,\infty)$, that the solution $y(x,t)$ of the boundary value problem

$$y_t(x,t) - y_{xx}(x,t) + \alpha \cdot y(0,t) = 0,\ \ \{x,t\} \in Q; \tag{1}$$

$$y(-\pi/2,t) = u_1(t),\ y(\pi/2,t) = u_2(t),\ y(x,0) = y_0(x), \tag{2}$$

as $t \to \infty$ approach zero as follows:

$$\|y(x,t)\|_{L_2(-\pi/2,\pi/2)} \le C_0 e^{-\sigma_0 t}, \tag{3}$$

where $Q = \{x,t|\ \frac{-\pi}{2} < x < \frac{\pi}{2},\ t > 0\}$, $\alpha \in \mathbf{C}$, σ_0 is the given positive number, $y_0(x) \in L_2(\frac{-\pi}{2},\frac{\pi}{2})$ is the given function.

Equation (1) is called the loaded equation [1–3]. We note that the vast literature is devoted to the inverse problems of the differential equations. Among them, we want to acknowledge the recently published textbook for university students [4], which is apparently the first textbook dedicated to the inverse and ill-posed problems, and in which there is fairly detailed overview of statements current problems and unsolved problems.

On the solvability of the boundary value problem (1)–(2). We write the problem (1)–(2) in the operator form:

$$Ly = \{y_0, u_1, u_2\},$$

where

$$L : L_2(Q) \to E \equiv L_2(-\pi/2,\pi/2) \times L_2(0,\infty) \times L_2(0,\infty),$$

and we give the definition of a strong solution.

Definition 1. The function $y(x,t) \in L_2(Q)$ is called a strong solution of the boundary value problem (1)–(2), if there exists a sequence

$$\{y_s(x,t)\}_{s=1}^{\infty} \subset C^{2,1}_{x,t}(Q) \cap C(\overline{Q}),$$

such that

$$y_s(x,t) \to y(x,t) \text{ in } L_2(Q),\ \ Ly_s \to \{y_0,u_1,u_2\} \text{ in } E \text{ at } s \to \infty.$$

The following theorem holds

Theorem. For any given controls $u_1(t), u_2(t) \in L_2(0,\infty)$ and any initial function $y_0(x) \in L_2(\frac{-\pi}{2},\frac{\pi}{2})$ of boundary value problem (1)–(2) has the unique strong solution $y(x,t) \in L_2(Q)$, and $y(x,t) \in W(0,\infty)$, where

$$W(0,\infty) = \{v|\ v \in L_2(0,\infty; W_2^1(-\pi/2,\pi/2)), v_t \in L_2(0,\infty; W_2^{-1}(-\pi/2,\pi/2))\}.$$

Proof. We transform boundary value problem (1)–(2) to the following loaded integral equation

$$y(x,t)=\int\limits_{-\pi/2}^{\pi/2}y_0(\xi)G(x,\xi,t)d\xi-\alpha\int\limits_0^t y(0,\tau)\int\limits_{-\pi/2}^{\pi/2}G(x,\xi,t-\tau)d\xi\,d\tau+$$

$$+\int\limits_0^t u_1(\tau)H_1(x,t-\tau)d\tau-\int\limits_0^t u_2(\tau)H_2(x,t-\tau)d\tau, \tag{4}$$

where the Green function G has the form

$$G(x,\xi,t)=\frac{2}{\pi}\sum_{n=1}^{\infty}\sin n(x+\pi/2)\sin n(\xi+\pi/2)\exp\{-n^2t\},$$

and the functions H_1 and H_2 are expressed in terms the Green function by the formulas:

$$H_1(x,t)=\frac{\partial}{\partial\xi}G(x,\xi,t)_{|\xi=-\pi/2},\quad H_2(x,t)=\frac{\partial}{\partial\xi}G(x,\xi,t)_{|\xi=\pi/2}.$$

In turn, from (4) for the unknown function $\mu(t)=y(0,t)$ we obtain the following integral equation

$$\mu(t)+\alpha\int\limits_0^t K(t-\tau)\mu(\tau)d\tau=F(t),\ t>0, \tag{5}$$

where the kernel of the integral operator has the form

$$K(t)=\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n-1}\exp\{-(2n-1)^2t\}, \tag{6}$$

the right-hand side of the equation represents the sum $F(t)=F_0(t)+F_1(t)+F_2(t)$,
where

$$F_0(t)=\frac{2}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}\exp\{-(2n-1)^2t\}\int\limits_{-\pi/2}^{\pi/2}y_0(\xi)\sin(2n-1)(\xi+\pi/2)d\xi, \tag{7}$$

$$F_j(t)=\int\limits_0^t A(t-\tau)u_j(\tau)d\tau,\ j=1,2, \tag{8}$$

the kernel $A(t)$ is determined by the formula:

$$A(t)=\frac{2}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}(2n-1)\exp\{-(2n-1)^2t\}. \tag{9}$$

We note that expressions (6) and (9) are called the Dirichlet series with real exponents [5; 111].

We show, that the function $K(t)\in L_1(0,\infty)$, and the functions $F_j(t)$, $j=0,1,2$, belong to the space $L_2(0,\infty)$. Indeed, we have

$$\int\limits_0^{\infty}|K(t)|dt=\frac{4}{\pi}\sum_{n=1}^{\infty}\int\limits_0^{\infty}\left[\frac{\exp\{-(4n-3)^2t\}}{(4n-3)}-\frac{\exp\{-(4n-1)^2t\}}{(4n-1)}\right]dt<+\infty$$

according to the formula 0.234.4 from [6; 9]: $\sum\limits_{n=1}^{\infty}(-1)^{n-1}/(2n-1)^3=\pi^3/32$.

We use the Cauchy inequality [7; 28]:

$$\left(\sum_{n=1}^{\infty} a_n b_n\right)^2 \le \sum_{n=1}^{\infty} a_n^2 \sum_{n=1}^{\infty} b_n^2,$$

where

$$a_n = \frac{2}{\pi} \int\limits_{-\pi/2}^{\pi/2} y_0(\xi) \sin(2n-1)(\xi + \pi/2)\, d\xi, \quad b_n = \exp\{-(2n-1)^2 t\},$$

and also the equality 0.234.2 from [6; 9]: $\sum\limits_{n=1}^{\infty} 1/(2n-1)^2 = \pi^2/8$,

$$\int\limits_0^{\infty} |F_0(t)|^2 dt \le C_0^2 \cdot \frac{4}{\pi^2} \sum_{n=1}^{\infty} \int\limits_0^{\infty} \exp\{-2(2n-1)^2 t\} dt \le \frac{\|y_0\|_0^2}{4} < +\infty,$$

where

$$C_0^2 = \sum_{n=1}^{\infty} \left[\int\limits_{-\pi/2}^{\pi/2} y_0(\xi) \sin(2n-1)(\xi + \pi/2)\, d\xi \right]^2 \le \|y_0\|_0^2.$$

Further, the functions F_1, F_2 are square-summable on the positive semiaxis, if the absolute value of functional series (9) is integrable. To prove the latter we rewrite series (9) as the sum of the differences:

$$A(t) = \frac{2}{\pi} \sum_{n=1}^{\infty} \left[(4n-3) \exp\{-(4n-3)^2 t\} - (4n-1) \exp\{-(4n-1)^2 t\} \right].$$

We note that each of these differences

$$(4n-3) \exp\{-(4n-3)^2 t\} - (4n-1) \exp\{-(4n-1)^2 t\} \tag{10}$$

represents a alternating function of the variable t, which changes sign once from negative to positive at the point $t_n = [8(2n-1)]^{-1} \ln (4n-1)/(4n-3)$, and it is evident that $t_1 > t_2 > ... > t_n > ...,\ t_n \to 0+$ at $n \to \infty$.

So, the integral of the absolute value of each difference (10) on the semiaxis is equal to:

$$I_n = \int\limits_0^{\infty} \left| (4n-3) \exp\{-(4n-3)^2 t\} - (4n-1) \exp\{-(4n-1)^2 t\} \right| dt =$$

$$= 2 \left[\frac{1}{4n-3} \exp\{-(4n-3)^2 t_n\} - \frac{1}{4n-1} \exp\{-(4n-1)^2 t_n\} \right] - \left[\frac{1}{4n-3} - \frac{1}{4n-1} \right]. \tag{11}$$

Note that, by simple analysis to the maximum, we have that the following equality holds:

$$\frac{1}{4n-3} \exp\{-(4n-3)^2 t_n\} - \frac{1}{4n-1} \exp\{-(4n-1)^2 t_n\} =$$

$$= \max_{0 \le t \le \infty} \left[\frac{1}{4n-3} \exp\{-(4n-3)^2 t\} - \frac{1}{4n-1} \exp\{-(4n-1)^2 t \right].$$

We take the sum of the right hand side of (11) from 1 to ∞ and multiply the result by $2/\pi$ (see formula (9). As a result, taking into account the well-known equality

$$\frac{\pi}{4} = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{2n-1},$$

we have

$$\int_0^\infty |A(t)|\,dt \le \frac{4}{\pi}\sum_{n=1}^\infty \frac{(-1)^{n-1}}{2n-1}\exp\{-(2n-1)^2\bar{t}_n\} - \frac{1}{2},$$

where for all odd $n : \bar{t}_n = \bar{t}_{n+1} = t_n$. The series on the right side of the last inequality converges by Leibnitz Theorem for alternating series [8; 302]. A positivity of the right side follows from relation (7).

Now it remains to use the convolution property (8) to obtain the desired properties of functions $F_j(t) \in L_2(0,\infty)$ [9; 9].

The existence and uniqueness of strong solution. Assume that functions $y_0(x)$, $u_j(t)$, $j = 1, 2$, satisfy the conditions of Theorem. And assume that problem (1)–(2) has two distinct solutions $y_1(x,t)$ and $y_2(x,t)$. Then the difference $\tilde{y}(x,t) = y_1(x,t) - y_2(x,t)$ is the solution of the following homogeneous boundary value problem:

$$\begin{cases} \tilde{y}_t(x,t) - \tilde{y}_{xx}(x,t) + \alpha\tilde{y}(0,t) = 0, \; x,t \in Q; \\ \tilde{y}(x,0) = 0, \; \tilde{y}(-\pi/2,t) = \tilde{y}(\pi/2,t) = 0. \end{cases} \tag{12}$$

By taking a inner product of (12) with $\tilde{y}(x,t)$ in $L_2(-\pi/2,\pi/2)$, we have

$$\frac{1}{2}\frac{d}{dt}\|\tilde{y}(x,t)\|_0^2 + \|\tilde{y}_x(x,t)\|_0^2 \le |\alpha|\sqrt{\pi}|\tilde{y}(0,t)| \cdot \|\tilde{y}(x,t)\|_0. \tag{13}$$

Here and further we denote by $\|\cdot\|_0$ and $(\cdot,\cdot)_0$ the norm and the inner product in $L_2(-\pi/2,\pi/2)$, respectively. Then, using the Friedrichs', Hölder's, Cauchy inequalities to the right side of (13), we have

$$\frac{d}{dt}\|\tilde{y}(x,t)\|_0^2 + \|\tilde{y}_x(x,t)\|_0^2 \le |\alpha|^2\pi^2\|\tilde{y}(x,t)\|_0^2.$$

Hence, by Gronwall inequality [10], we have

$$\tilde{y}(x,t) \equiv 0 \in L_\infty((0,\infty); L_2(-\pi/2,\pi/2)) \cap L_2((0,\infty); \overset{\circ}{W}{}_2^1(-\pi/2,\pi/2)),$$

i.e., the boundary value problem (1)–(2) has no more than one solution.

Hence it follows that the integral equation (5) has no more than one solution. Otherwise, if the integral equation (5) has more than one solution, the boundary value problem (1)–(2) according to relation (4) would also have more than one solution, which is impossible, as we have just proved. This means that integral equation (5) in the class $L_2(0,\infty)$ can have only one solution. The uniqueness is proved.

The foregoing proof of the uniqueness without changes holds for the homogeneous boundary value problem adjoint to (12):

$$\begin{cases} -\tilde{p}_t(x,t) - \tilde{p}_{xx}(x,t) + \overline{\alpha}\cdot\delta(x) \otimes \int_{-\pi/2}^{\pi/2} \tilde{p}(\xi,t)d\xi = 0, \; \{x,t\} \in Q; \\ \tilde{p}(x,\infty) = 0, \; \tilde{p}(-\pi/2,t) = \tilde{p}(\pi/2,t) = 0. \end{cases} \tag{14}$$

We transform the boundary value problem (14) to the following loaded integral equation

$$\tilde{p}(x,t) = -\overline{\alpha}\int_t^\infty G(0,x,\tau-t)\int_{-\pi/2}^{\pi/2} \tilde{p}(\xi,t)d\xi\,d\tau, \tag{15}$$

where the Green function G has the form:

$$G(\xi,x,t) = \frac{2}{\pi}\sum_{n=1}^\infty \sin n(\xi+\pi/2)\sin n(x+\pi/2)\exp\{-n^2 t\}.$$

Implies that (15) integral equation the corresponding to the boundary-value problem (14) adjoint to the equation (5)

$$\nu(t) + \overline{\alpha}\int_t^\infty K(\tau-t)\nu(\tau)d\tau = 0, \; t > 0, \text{ where } \nu(t) = \int_{-\pi/2}^{\pi/2} \tilde{p}(\xi,t)d\xi \tag{16}$$

and the kernel of the integral operator (in accordance with formula (6) has the form

$$K(t)=\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n-1}\exp\{-(2n-1)^2t\}.$$

Since the uniqueness holds for the adjoint boundary value problem (14), then integral equation (16) in $L_2(0,\infty)$ can have only the trivial solution.

So, according to the theory of integral equations, in $L_2(0,\infty)$ integral equation (5) has a unique solution for all $F(t)\in L_2(0,\infty)$. Consequently, it follows the existence of a unique strong solution of boundary value problem (1)–(2). It remains to show that under the conditions of Theorem 1 the solution of problem (1)–(2), represented by formula (4) belongs to the class $L_2(Q)$.

We write a detailed expression for solution (4):

$$y(x,t)=\frac{2}{\pi}\sum_{n=1}^{\infty}\int\limits_{-\pi/2}^{\pi/2}y_0(\xi)\sin n(\xi+\pi/2)d\xi\cdot\sin n(x+\pi/2)\exp\{-n^2t\}-$$

$$-\frac{4\alpha}{\pi}\sum_{n=1}^{\infty}\frac{\sin{(2n-1)(x+\pi/2)}}{2n-1}\int\limits_0^t\exp\{-(2n-1)^2(t-\tau)\}y(0,\tau)\,d\tau+$$

$$+\frac{2}{\pi}\sum_{n=1}^{\infty}n\cdot\sin n(x+\pi/2)\int\limits_0^t\exp\{-n^2(t-\tau)\}u_1(\tau)\,d\tau+$$

$$+\frac{2}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}n\cdot\sin n(x+\pi/2)\int\limits_0^t\exp\{-n^2(t-\tau)\}u_2(\tau)\,d\tau\equiv\sum_{j=1}^{4}y_j(x,t). \tag{17}$$

Hence the required property of the solution follows. Indeed, the first summand is estimated as follows (using Cauchy inequality):

$$|y_1(x,t)|=\frac{2}{\pi}\left|\sum_{n=1}^{\infty}a_nb_n\right|\le\frac{2}{\pi}\left(\sum_{n=1}^{\infty}a_n^2\right)^{1/2}\left(\sum_{n=1}^{\infty}b_n^2\right)^{1/2},$$

where

$$a_n=\int\limits_{-\pi/2}^{\pi/2}y_0(\xi)\cdot\sin n(\xi+\pi/2)\,d\xi,\ \ \sum_{n=1}^{\infty}|a_n|^2\le\|y_0(x)\|_0^2<\infty,$$

$$b_n(x,t)=\sin n(x+\pi/2)\exp\{-n^2t\},\ \ \sum_{n=1}^{\infty}\|b_n(x,t)\|_{L_2(Q)}^2=\frac{\pi}{4}\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^3}{24}.$$

Thus, we obtain:

$$\|y_1(x,t)\|_{L_2(Q)}<\infty,\text{i.e.}y_1(x,t)\in L_2(Q).$$

For the second summand $y_2(x,t)$ we take:

$$a_n=\frac{\sin{(2n-1)(x+\pi/2)}}{2n-1},\ \ b_n=\int\limits_0^t\exp\{-(2n-1)^2(t-\tau)\}y(0,\tau)\,d\tau.$$

Taking into account the recent notations and applying the Cauchy inequality as in the case of the first summand, as a result of simple calculations, we obtain:

$$\|y_2(x,t)\|_{L_2(Q)}<\infty,\ \text{ i.e. }\ y_2(x,t)\in L_2(Q).$$

We estimate the third summand. For this we rewrite it in the form:

$$y_3(x,t) = \frac{2}{\pi}\sum_{n=1}^{\infty} \sin n(x+\pi/2) \int_0^t n \exp\{-n^2(t-\tau)\} u_1(\tau)\, d\tau =$$

$$= \frac{2}{\pi}\sum_{m=1}^{\infty} \sin 2mx \cdot S_{1m}(t) + \frac{2}{\pi}\sum_{m=1}^{\infty} \cos{(2m-1)x} \cdot S_{2m}(t), \tag{18}$$

where

$$S_{1m}(t) = \int_0^t (-1)^m 2m \exp\{-4m^2(t-\tau)\} u_1(\tau)\, d\tau,$$

$$S_{2m}(t) = \int_0^t (-1)^{m-1}(2m-1) \exp\{-(2m-1)^2(t-\tau)\} u_1(\tau)\, d\tau.$$

Further, for the first summand of the right part of (18) we have:

$$\sum_{m=1}^{\infty} S_{1m}(t) = \sum_{n=1}^{\infty} \int_0^t \big[-2(2n-1)\exp\{-4(2n-1)^2(t-\tau)\} +$$

$$+ 4n \exp\{-16n^2(t-\tau)\}\big]\, u_1(\tau)\, d\tau = \sum_{n=1}^{\infty} S^0_{1n}(t). \tag{19}$$

We note that in the last representation each summand in the form of the integrand function enclosed in square brackets changes sign from positive to negative only once, and the point of changing the sign is determined by the formula:

$$t_n = \frac{1}{4(4n-1)} \ln \frac{2n}{2n-1}, \;\; t_1 > t_2 > \ldots > t_n \to 0 \;\text{ at }\; n \to \infty.$$

We estimate the norm of the first sum in (18) taking into account (19):

$$\frac{2}{\pi}\left\| \sum_{m=1}^{\infty} \sin 2mx \cdot S_{1m}(t) \right\|_{L_2(Q)} \le \sqrt{\frac{2}{\pi}} \sum_{n=1}^{\infty} \|S^0_{1n}(t)\|_{L_2(0,\infty)}. \tag{20}$$

We now estimate each summand represented as a convolution $S^0_{1n}(t)$:

$$\|S^0_{1n}(t)\|^2_{L_2(0,\infty)} \le \int_0^{\infty} \Bigg| \int_0^t \big[-2(2n-1)\exp\{-4(2n-1)^2(t-\tau)\} \;+$$

$$+\; 4n \exp\{-16n^2(t-\tau)\}\big]\, u_1(\tau)\, d\tau \Bigg|^2 dt \le$$

$$\le \|u_1(t)\|^2_{L_2(0,\infty)} \cdot \left| \int_0^{\infty} \left|-2(2n-1)\exp\{-4(2n-1)^2 t\} \;+\; 4n \exp\{-16n^2 t\}\right| dt \right|^2 .$$

Now we compute the second factor on the right hand side in the last inequality:

$$\int_0^{\infty} \left|-2(2n-1)\exp\{-4(2n-1)^2(t-\tau)\} + 4n\exp\{-16n^2(t-\tau)\}\right| dt =$$

$$= \int_0^{t_n} \left[-2(2n-1)\exp\{-4(2n-1)^2(t-\tau)\} + 4n\exp\{-16n^2(t-\tau)\}\right] dt +$$

$$+ \int_{t_n}^{\infty} \left[2(2n-1)\exp\{-4(2n-1)^2(t-\tau)\} - 4n\exp\{-16n^2(t-\tau)\}\right] dt =$$

$$= \left[\frac{\exp\{-4(2n-1)^2 t_n\}}{2n-1} - \frac{\exp\{-16n^2 t_n\}}{2n}\right] - \left[\frac{1}{2(2n-1)} - \frac{1}{4n}\right] > 0,$$

since a simple research to maximum of the function shows

$$\varphi_n(t) = \frac{\exp\{-4(2n-1)^2 t\}}{2(2n-1)} - \frac{\exp\{-16n^2 t\}}{2\cdot 2n}, \ \varphi_n(t_n) = \max_{0\le t\le\infty}\{\varphi_n(t)\}.$$

Taking into account the calculations for the right hand side of (20), we obtain:

$$\sqrt{2/\pi}\sum_{n=1}^{\infty}\|S_{1n}^0(t)\|_{L_2(0,\infty)} \le \sqrt{2/\pi}\|u_1(t)\|_{L_2(0,\infty)}\times$$

$$\times\left\{\sum_{n=1}^{\infty}\left[\frac{\exp\{-4(2n-1)^2 t_n\}}{2n-1} - \frac{\exp\{-16n^2 t_n\}}{2n}\right] < \infty,\right.$$

since the series on the right hand side converge (by the Leibnitz theorem as alternating series).

Analogous calculations (carried out for the first summand of the right part of (18) are valid for the second summand of the right part of (18). Thus, we obtain the desired estimate for the third summand in (17) $\|y_3(t)\|_{L_2(Q)} < \infty$.

Now we show, that $y_4(x,t)$ in (17) belongs to $L_2(Q)$. We have:

$$y_4(x,t) = \frac{2}{\pi}\sum_{n=1}^{\infty}\sin n(x+\pi/2)\cdot S_n(t),$$

where

$$S_n(t) = \int_0^t (-1)^{n-1} n\exp\{-n^2(t-\tau)\}u_2(\tau)\,d\tau.$$

Hence we obtain the following estimate:

$$\|y_4(x,t)\|_{L_2(Q)} \le \sqrt{2/\pi}\sum_{n=1}^{\infty}\|S_n(t)\|_{L_2(0,\infty)}.$$

To obtain an estimate for the convolution $S_n(t)$ we rewrite it in the form:

$$\sum_{n=1}^{\infty} S_n(t) = \sum_{n=1}^{\infty} S_n^0(t),$$

where

$$S_n^0(t) = \int_0^t \left[(2n-1)\exp\{-(2n-1)^2(t-\tau)\} - 2n\exp\{-4n^2(t-\tau)\}\right] u_2(\tau)\,d\tau.$$

By arguing as in the estimate of $y_3(x,t)$, we obtain the required estimate $\|y_4(t)\|_{L_2(Q)} < \infty$.

Thus, the first assertion of Theorem is completely proved.

The next section is devoted to the proof of the second assertion of Theorem, that is to the establishment of additional differential properties of the solution of problem (1)–(2).

On additional smoothness of the solution. We note that according to the theorem on traces [11; 32–33, 265–269] for the given functions $u_j(t) \in L_2(0,\infty),\ j=1,2$, there exists a function $w(x,t) \in W(0,\infty)$, where

$$W(0,\infty) = \{v|\, v \in L_2(0,\infty; W_2^1(-\pi/2,\pi/2)), v_t \in L_2(0,\infty; W_2^{-1}(-\pi/2,\pi/2))\},$$

such that

$$w(-\pi/2,t) = u_1(t),\ w(\pi/2,t) = u_2(t).$$

The boundary value problem (1)–(2) takes the form:

$$\begin{cases} (y-w)_t(x,t) - (y-w)_{xx}(x,t) + \alpha(y-w)(0,t) = f_1(x,t),\ \{x,t\} \in Q; \\ (y-w)(x,0) = y_1(x),\ (y-w)(-\pi/2,t) = (y-w)(\pi/2,t) = 0, \end{cases} \tag{21}$$

where

$$\begin{cases} f_1 = -w_t(x,t) + w_{xx}(x,t) - \alpha w(0,t) \in L_2(0,\infty; W_2^{-1}(-\pi/2,\pi/2)); \\ y_1 = y_0(x) - w(x,0) \in L_2(-\pi/2,\pi/2). \end{cases} \tag{22}$$

Earlier, in [12] on the basis of a priori estimates established there and the application of Galerkin method it was proven that boundary value problem (21) for any given functions $f_1(x,t)$ and $y_1(x)$, satisfying the conditions (22), has a solution $(y-w)(x,t) \in W(0,\infty)$, namely corresponding to (21) boundary value problem (1)–(2) has a solution $y(x,t) \in W(0,\infty)$.

Hence the second assertion of Theorem follows. Thus, the proof of Theorem is completed.

However, relation (3) requires the choice of boundary controls that would provide the decrease of L_2-average values of the solution not slower than some exponent by time. Fourier method provides this requirement by choice of those exponents $\{\exp\{-\lambda_k t\},\ k \in \mathbf{Z}\}$ in the representation of solution through a series, where numbers λ_k, are defined by positive eigenvalues of the corresponding spectral problem, and which are not less than the exponent of decrease in the exponent of condition (3).

Thus inverse problem (1)–(2) will be solved, if we find a way of constructing the controls $u_j(t),\ j=1,2$, that provides the existence only the exponents of the form $\{\exp\{-\lambda_k t\},\ k \in \mathbf{Z}\}$ (where $\lambda_k \geq \sigma_0$ in (3)), in the presentation for the solution in the form of a series.

The following section of work is devoted to constructing and justifying the algorithm of choice of the desired boundary control functions $u_j(t),\ j=1,2$, in the problem (1)–(2) and its numerical realization.

Solving the problem of stabilization by extension of domain for independent variables. We consider in the domain $Q_1 = \{x,t| -\pi < x < \pi,\ t > 0\}$ the additional problem

$$z_t(x,t) - z_{xx}(x,t) + \alpha \cdot z(0,t) = 0,\ \ \{x,t\} \in Q_1; \tag{23}$$

$$z(-\pi,t) = z(\pi,t),\ z_x(-\pi,t) = z_x(\pi,t),\ z(x,t)|_{t=0} = z_0(x), \tag{24}$$

where $z_0(x)$ is a function that must be defined.

We will seek a solution of problem (23)–(24) in the form

$$z(x,t) = \sum_{k\in\mathbf{Z}} Z_k(t)\varphi_k(x). \tag{25}$$

where $\{\varphi_k(x),\ k \in \mathbf{Z}\}$ is the basis of the space $L_2(-\pi,\pi)$ and $\mathbf{Z} = \{0,\pm1,\pm2,...\}$.

For this, we consider the spectral problem corresponding to problem (23)–(24):

$$-\varphi''(x) + \alpha \cdot \varphi(0) = \lambda\varphi(x); \tag{26}$$

$$\varphi(-\pi) = \varphi(\pi),\ \varphi'(-\pi) = \varphi'(\pi). \tag{27}$$

We introduce the following notation $\mathbf{Z}' = \mathbf{Z}\backslash\{0\}$. For problem (26)–(27) we consider the following two cases.

1^0. *The case when there is no such $k \in \mathbf{Z}$, that $\alpha = k^2$.* The general solution of spectral problem (26)–(27) has the form:

$$\varphi_k(x) = A_k e^{i\sqrt{\lambda_k}x} + D_k \tag{28}$$

and substituting (28) into (26) we find $D_k = \frac{\alpha A_k}{\lambda_k - \alpha}$, here we take $A_k = 1$. Then it is necessary to take $\lambda_k = k^2$, to satisfy conditions (27). Hence we can write the final form of the solution of equation (26)

$$\varphi_k(x) = e^{ikx} + \frac{\alpha}{\lambda_k - \alpha}, \quad k \in \mathbf{Z}'.$$

For $k = 0 :\ \varphi_0(x) = \text{const}$, $\lambda_0 = \alpha$; that is for the eigenvalue $\lambda_0 = \alpha$ it is possible to take the eigenfunction $\varphi_0(x) = 1$.

Thus, we have the following system of eigenfunctions and eigenvalues

$$\{\varphi_k(x),\ \lambda_k;\ k \in \mathbf{Z}\} = \left\{1,\ \lambda_0 = \alpha;\ e^{ikx} + \frac{\alpha}{k^2 - \alpha},\ \lambda_k = k^2,\ k \in \mathbf{Z}'\right\}. \tag{29}$$

We note that the obtained system of eigenfunctions (29) is complete in the space $L_2(-\pi, \pi)$, constitutes a basis, but it is not orthogonal. Completeness of the system of eigenfunctions (29) follows from the known theorem of Paley-Wiener [7; 224–227]. For (29) we will find a biorthogonal sequence in the following form

$$\{\psi_k(x),\ k \in \mathbf{Z}\} = \left\{f_0(x),\ e^{ikx},\ k \in \mathbf{Z}'\right\},$$

where $f_0(x)$ is unknown function.

Using basis (29) we will seek the unknown function $f_0(x)$ in the form:

$$f_0(x) = C_0 + \sum_{n \in \mathbf{Z}'} C_n \left(e^{inx} + \frac{\alpha}{n^2 - \alpha}\right),$$

from orthogonality conditions:

$$(1,\ f_0(x)) = 1; \quad \left(e^{ikx} + \frac{\alpha}{k^2 - \alpha},\ f_0(x)\right) = 0,\ \ k \in \mathbf{Z}'.$$

From these conditions, we have:

$$(1,\ f_0(x)) = \int_{-\pi}^{\pi} \left[C_0 + \sum_{n \in \mathbf{Z}'} C_n \left(e^{inx} + \frac{\alpha}{n^2 - \alpha}\right)\right] dx = 2\pi \cdot \left[C_0 + \sum_{n \in \mathbf{Z}'} C_n \cdot \frac{\alpha}{n^2 - \alpha}\right] = 1.$$

Hence it follows $C_0 = \frac{1}{2\pi} - \sum_{n \in \mathbf{Z}'} C_n \cdot \frac{\alpha}{n^2 - \alpha}$. Further

$$\left(e^{ikx} + \frac{\alpha}{k^2 - \alpha},\ C_0 + \sum_{n \in \mathbf{Z}'} C_n \left(e^{inx} + \frac{\alpha}{n^2 - \alpha}\right)\right) = 0,\ \ k \in \mathbf{Z}';$$

$$C_0 \cdot \frac{\alpha}{k^2 - \alpha} + C_k + \frac{\alpha}{k^2 - \alpha} \cdot \left(\frac{1}{2\pi} - C_0\right) = 0,\ \ k \in \mathbf{Z}'.$$

Here we find $C_k :\ C_k = -\frac{1}{2\pi} \cdot \frac{\alpha}{k^2 - \alpha},\ k \in \mathbf{Z}'$. Using the values C_k we rewrite C_0 :

$$C_0 = \frac{1}{2\pi} \cdot \left[1 + \sum_{n \in \mathbf{Z}'} \left(\frac{\alpha}{n^2 - \alpha}\right)^2\right].$$

Further, using the value C_0 we write the desired function f_0:

$$f_0(x) = \frac{1}{2\pi} \cdot \left[1 - \sum_{n \in \mathbf{Z}'} \frac{\alpha}{n^2 - \alpha} \cdot e^{inx}\right] = -\frac{1}{2\pi} \cdot \sum_{n \in \mathbf{Z}} \frac{\alpha}{n^2 - \alpha} \cdot e^{inx}.$$

Therefore, for basis (29) a biorthogonal sequence is the following sequence:

$$\{\psi_k(x),\ k \in \mathbf{Z}\} = \left\{-\frac{1}{2\pi} \sum_{n \in \mathbf{Z}} \frac{\alpha}{n^2 - \alpha} \cdot e^{inx},\ e^{ikx},\ k \in \mathbf{Z}'\right\}, \tag{30}$$

which defines in the space $L_2(-\pi, \pi)$ a biorthogonal basis.

2^0. *The case when there exists such number $k \in \mathbf{Z}$, that $\alpha = k^2$.* Let us k_0 be such number, namely $\alpha = k_0^2$. The general solution of spectral problem (26)–(27) has the form:

$$\varphi_k(x) = A_k e^{i\sqrt{\lambda_k}x} + D_k, \tag{31}$$

and substituting (31) into (26) we find $D_k = \frac{\alpha A_k}{\lambda_k - \alpha}$, here we take $A_k = 1$. Then it is necessary to take $\lambda_k = k^2$, to satisfy conditions (27). Hence we write the final form of the solution of equation (26)

$$\varphi_k(x) = e^{ikx} + \frac{\alpha}{\lambda_k - \alpha}; \quad k \in \mathbf{Z}' \backslash \{\pm k_0\}.$$

For $k_0 : \varphi_{k_0}(x) = \text{const}$, $\lambda_{k_0} = \alpha = k_0^2$; that is for the eigenvalue $\lambda_{k_0} = \alpha = k_0^2$ it is possible to take the eigenfunction $\varphi_{k_0}(x) = 1$. Further, the system of eigenfunctions and eigenvalues are complete, if we find the associated functions that satisfy the following conditions

$$-\overline{\varphi}''_{k_0}(x) + \alpha \cdot \overline{\varphi}_{k_0}(0) - k_0^2 \overline{\varphi}_{k_0}(x) = k_0^2; \tag{32}$$

$$\overline{\varphi}_{k_0}(-\pi) = \overline{\varphi}_{k_0}(\pi), \ \overline{\varphi}'_{k_0}(-\pi) = \overline{\varphi}'_{k_0}(\pi). \tag{33}$$

The general solution of spectral problem (32)–(33) has the form

$$\overline{\varphi}_{k_0}(x) = \mathrm{C} + A_1 e^{ik_0 x} + A_2 e^{-ik_0 x}. \tag{34}$$

Substituting the general solution (34) into (32) we find $\alpha(A_1 + A_2) = k_0^2$, here we take $A_1 + A_2 = 1$. The associated functions are $\{e^{\pm ik_0 x}\}$.

Thus, we have the eigenvalues and the corresponding eigenfunctions

$$\{\varphi_k(x), \ \lambda_k; \ k \in \mathbf{Z} \backslash \{\pm k_0\}\} =$$

$$= \left\{1, \ \lambda_0 = k_0^2; \ e^{ikx} + \frac{\alpha}{k^2 - \alpha}, \ \lambda_k = k^2, \ k \in \mathbf{Z}' \backslash \{\pm k_0\}\right\} \tag{35}$$

and the associated functions

$$\{\varphi_{\pm k_0}(x), \ \lambda_0\} = \left\{e^{\pm ik_0 x}, \ \lambda_0 = k_0^2 = \alpha\right\}. \tag{36}$$

Here, the constant is an eigenfunction corresponding to the eigenvalue $\lambda_0 = k_0^2 = \alpha$. Furthermore, we note that zero is not an eigenvalue. In this case, the system of eigenfunctions is not complete and not orthogonal in the space $L_2(-\pi, \pi)$.

Combining (35) and (36), we obtain the complete system [7; 224–227]:

$$\{\varphi_k(x), \ \lambda_k; \ k \in \mathbf{Z}\} = \left\{1, \ \lambda_0 = k_0^2; \ e^{\pm ik_0 x}, \ \lambda_0 = k_0^2 = \alpha; \right.$$

$$\left. e^{ikx} + \frac{\alpha}{k^2 - \alpha}, \ \lambda_k = k^2, \ k \in \mathbf{Z}' \backslash \{\pm k_0\}\right\}. \tag{37}$$

For (37) the biorthogonal sequence is

$$\{\psi_k(x); \ k \in \mathbf{Z}\} = \left\{f_0(x), \ e^{ikx}, \ k \in \mathbf{Z}'\right\},$$

where it is necessary to find unknown function $f_0(x)$ by the following way:

$$f_0(x) = C_0 + \sum_{n \in \mathbf{Z}' \backslash \{\pm k_0\}} C_n \left(e^{inx} + \frac{\alpha}{n^2 - \alpha}\right) + C_{k_0} e^{ik_0 x} + C_{-k_0} e^{-ik_0 x},$$

from orthogonality conditions

$$(1, f_0(x)) = 1; \ \left(e^{ikx} + \frac{\alpha}{k^2 - \alpha}, f_0(x)\right) = 0, \ \ k \in \mathbf{Z}' \backslash \{\pm k_0\};$$

$$(e^{\pm ik_0 x}, f_0(x)) = 0.$$

From these conditions we find:

$$(1, f_0(x)) = 2\pi \cdot \left[C_0 + \sum_{n\in\mathbf{Z}'\setminus\{\pm k_0\}} C_n \cdot \frac{\alpha}{n^2-\alpha} \right] = 1,$$

$$C_0 = \frac{1}{2\pi} - \sum_{n\in\mathbf{Z}'\setminus\{\pm k_0\}} C_n \cdot \frac{\alpha}{n^2-\alpha}.$$

Further

$$\left(e^{ikx} + \frac{\alpha}{k^2-\alpha}, f_0(x)\right) = \left(e^{ikx} + \frac{\alpha}{k^2-\alpha},\ C_0 + \sum_{n\in\mathbf{Z}'\setminus\{\pm k_0\}} C_n \left(e^{inx} + \frac{\alpha}{n^2-\alpha}\right)\right) = 0;$$

$$k \in \mathbf{Z}'\setminus\{\pm k_0\}.$$

Hence it follows

$$C_0 \cdot \frac{\alpha}{k^2-\alpha} + C_k + \frac{\alpha}{k^2-\alpha} \cdot \left(\frac{1}{2\pi} - C_0\right) = 0,\ \ k \in \mathbf{Z}'\setminus\{\pm k_0\}.$$

Here we find C_k:

$$C_k = -\frac{1}{2\pi} \cdot \frac{\alpha}{k^2-\alpha},\ \ k \in \mathbf{Z}'\setminus\{\pm k_0\}.$$

Using the values C_k we rewrite C_0:

$$C_0 = \frac{1}{2\pi} \cdot \left[1 + \sum_{n\in\mathbf{Z}'\setminus\{\pm k_0\}} \left(\frac{\alpha}{n^2-\alpha}\right)^2 \right].$$

Next, using the values C_0 we write the desired function f_0:

$$f_0(x) = -\frac{1}{2\pi} \sum_{n\in\mathbf{Z}} \frac{\alpha}{n^2-\alpha} \cdot e^{inx}.$$

So for (37) the biorthogonal system is

$$\{\psi_k(x),\ k \in \mathbf{Z}\} = \left\{ -\frac{1}{2\pi} \sum_{n\in\mathbf{Z}} \frac{\alpha}{n^2-\alpha} \cdot e^{inx},\ \ e^{ikx},\ \ k \in \mathbf{Z}' \right\}. \tag{38}$$

This system also defines a biorthogonal basis in the space $L_2(-\pi,\pi)$.

To determine the Fourier coefficients of expansion (25) we have Cauchy problem:

$$Z_k'(t) + \lambda_k Z_k(t) = 0,\ \ Z_k(0) = z_{0k},\ \ k \in \mathbf{Z}, \tag{39}$$

where z_{0k} are the expansion coefficients of the function $z_0(x)$ on system $\{\varphi_k(x)\}$.

The solution of Cauchy problem (39) has the form: $Z_k(t) = z_{0k}e^{-\lambda_k t},\ k \in \mathbf{Z}$.

We will further assume that in the space $L_2(-\pi,\pi)$ we have:

– basis $\{\varphi_k(x),\ k \in \mathbf{Z}\}$, composed of the system of eigenfunctions (29) or of the system of eigenfunctions and associated functions (37);

– and the corresponding biorthogonal basis $\{\psi_k(x),\ k \in \mathbf{Z}\}$, (30) or (38).

Then the solution of original initial-boundary value problem (23)–(24) can be written in form (25):

$$z(x,t) = z_{00}e^{-\alpha t}\varphi_0(x) + \sum_{k\in\mathbf{Z}'} z_{0k}e^{-k^2 t}\varphi_k(x), \tag{40}$$

where

$$z_{0k} = \int_{-\pi}^{\pi} \overline{\psi_k(x)} z_0(x)dx,\ \ k \in \mathbf{Z},$$

are Fourier coefficients $z_0(x)$, where $\psi_k(x),\ k\in\mathbf{Z}$, are defined by the formulas (30) and (38). From (39) and (40) it follows directly that if

$$z_{0k}=0 \ \text{ at } \ k^2<\sigma_0 \tag{41}$$

and

$$z_{00}\neq 0 \ \text{ at } \ \operatorname{Re}\alpha\geq\sigma_0;\ \ z_{00}=0 \ \text{ at } \ \operatorname{Re}\alpha<\sigma_0, \tag{42}$$

then solution (40) of problem (23)–(24) will satisfy the inequality

$$\|z(x,t)\|_{L_2(-\pi,\pi)}\leq Ce^{-\sigma_0 t}.$$

We denote by $\mathbf{Z}_0$ ($\mathbf{Z}_0\subset\mathbf{Z}$) the set of indices k that satisfy conditions (41) and (42).

Now, with the restriction operator $\zeta_{-\pi/2}$ and $\zeta_{\pi/2}$ we find the desired controls

$$u_1(t)=\zeta_{-\pi/2}\{z(x,t)\},\ \ u_2(t)=\zeta_{\pi/2}\{z(x,t)\}.$$

It remains to construct an extension operator of the function $y_0(x)$ up to the function $z_0(x)$, defined on the interval $(-\pi,\pi)$,

$$E:\ L_2(-\pi/2,\pi/2)\to L_2(-\pi,\pi),\ \text{i.e.}\ (\zeta_{(-\pi/2,\pi/2)}Ey_0)(x)\equiv y_0(x), \tag{43}$$

so that the Fourier coefficients z_{0k} of function $z_0=Ey_0$ (43) would satisfy conditions (41) and (42). Here we use the notation $\zeta_{(-\pi/2,\pi/2)}$ for the restriction operator $\zeta_{(-\pi/2,\pi/2)}:\ L_2(-\pi,\pi)\to L_2(-\pi/2,\pi/2)$.

By arguing as in the lemma of [13] we obtain the following lemma.

Lemma. For each $\sigma_0>0$ there exists a continuous extension operator E in (43), that for all $y_0(x)\in$ $\in L_2(-\pi/2,\pi/2)$ equality holds

$$\int\limits_{-\pi}^{\pi}\overline{\psi_k(x)}(Ey_0)(x)dx=0,\ \forall\, k\in\mathbf{Z}_0:\ \ |k|<\sqrt{\sigma_0}. \tag{44}$$

Proof. We define the operator E (43) by the formula

$$Ey_0(x)=\begin{cases} y_0(x), & x\in(-\pi/2,\pi/2);\\ z_1(x), & x\in(-\pi,-\pi/2)\cup(\pi/2,\pi),\end{cases}$$

where the function $z_1(x)$ to be determined. By virtue (44) $z_1(x)$ must satisfy the system of equations:

$$\int\limits_{(-\pi,-\pi/2)\cup(\pi/2,\pi)}\overline{\psi_k(x)}\cdot z_1(x)dx=-\int\limits_{(-\pi/2,\pi/2)}\overline{\psi_k(x)}\cdot y_0(x)dx\equiv-\widehat{y}_0(k),\ \ k\in\mathbf{Z}_0. \tag{45}$$

We seek the function $z_1(x)$ in the form:

$$z_1(x)=\sum_{j\in\mathbf{Z}_0}\widehat{z}_1(j)\psi_j(x). \tag{46}$$

Substituting (46) into (45), we obtain a system of equations to determine $\widehat{z}_1(j)$:

$$\sum_{j\in\mathbf{Z}_0}a_{kj}\widehat{z}_1(j)=-\widehat{y}_0(k),\ \ k\in\mathbf{Z}_0, \tag{47}$$

where $\widehat{y}_0(k)$ is defined in (45), and the coefficients a_{kj} are determined by relations:

$$a_{kj}=\int\limits_{(-\pi,-\pi/2)\cup(\pi/2,\pi)}\overline{\psi_k(x)}\cdot\psi_j(x)dx,\ \ k,\,j\in\mathbf{Z}_0. \tag{48}$$

The matrix $A=\|a_{kj}\|$ is positive. Indeed, if

$$\Psi=\{\widehat{\psi}_k,\ k\in\mathbf{Z}_0\}\ \text{and}\ \psi=\sum_{k\in\mathbf{Z}_0}\widehat{\psi}_k\cdot\psi_k(x),$$

then by virtue of (48)

$$(A\Psi,\Psi)=\sum_{k,j\in\mathbf{Z}_0}a_{kj}\cdot\widehat{\psi}_j\cdot\overline{\widehat{\psi}}_k=\sum_{k,j\in\mathbf{Z}_0}\int\limits_{(-\pi,-\pi/2)\cup(\pi/2,\pi)}\psi_j(x)\cdot\overline{\psi_k(x)}dx\cdot\widehat{\psi}_j\cdot\overline{\widehat{\psi}}_k=$$

$$=\int\limits_{(-\pi,-\pi/2)\cup(\pi/2,\pi)}\sum_{j\in\mathbf{Z}_0}\psi_j(x)\cdot\widehat{\psi}_j\cdot\sum_{k\in\mathbf{Z}_0}\overline{\widehat{\psi}}_k\cdot\overline{\psi_k(x)}dx=$$

$$=\int\limits_{(-\pi,-\pi/2)\cup(\pi/2,\pi)}\overline{\psi(x)}\cdot\psi(x)dx=\int\limits_{(-\pi,-\pi/2)\cup(\pi/2,\pi)}|\psi(x)|^2dx\geq 0. \qquad (49)$$

If for some Ψ in (49) the equality holds, then

$$\psi(x)=\sum_{k\in\mathbf{Z}_0}\widehat{\psi}_k\psi_k(x)\equiv 0 \text{ and hence } \widehat{\psi}_k=0,\ \forall\, k\in\mathbf{Z}_0.$$

Hence $\det\|a_{kj}\|\neq 0$ and therefore system (47) and formula (46) uniquely determine operator (43), satisfying all the conditions of Lemma.

An algorithm for solving the inverse problem. The results of the preceding sections allow us to implement the following algorithms of approximate constructing the boundary control functions (and even in the form of synthesis, processing their random perturbations), providing monotonic decrease in time, not slower than the given exponent according to formula (4) in $L_2(-\pi/2,\pi/2)$–norm of the solution. The latter is achieved by fulfillment of requirements (41) and (42).

Step 1. According to original boundary value problem (1)–(2) at half-strip of the width π with non-homogeneous Dirichlet boundary conditions and initial condition on the interval $(-\pi/2,\pi/2)$, given by the function $y_0(x)$, auxiliary boundary value problem (23)–(24) is posed on the extended half-strip of the width which is equal to 2π, with periodicity conditions (instead of the Dirichlet conditions) and the initial function $z_0(x)$ on the interval $(-\pi,\pi)$. The function $z_0(x)$ will be defined as the continuation of the given function $y_0(x)$.

Thus, in auxiliary boundary problem (23)–(24) it is necessary to complete the definition of function $z_0(x)$ on the interval $(-\pi,\pi)$, so that for the solutions $z(x,t)$ of problem (23)–(24) requirement (4) would be fulfilled. In this case, condition (4) holds for its restriction $y(x,t)$ and the required boundary controls $u_1(t)$ and $u_2(t)$ will be determined as traces of the function $z(x,t)$ when $x=\pm\pi/2$.

Step 2. Constructing the complete biorthogonal systems of functions on the interval $(-\pi,\pi)$ by solving the corresponding spectral problems.

Step 3. We find the coefficients of the expansion of the required function $z_0(x)$ on the interval $(-\pi,\pi)$ by complete biorthogonal system that constructed in the preceding step, so that conditions (41) and (42) were hold. We note that conditions (41) and (42) provide the fulfillment of requirement (4) to solve auxiliary boundary value problem (23)–(24).

Step 4. According to solution $z(x,t)$ that is obtined of auxiliary boundary value problem (23)–(24) we find the solution $y(x,t)$ of original boundary value problem (1)–(2), satisfying required condition (4). We find the boundary controls $u_1(t)$ and $u_2(t)$ as traces of the solution $z(x,t)$, that is

$$u_1(t)=z(x,t)_{|x=-\pi/2},\quad u_2(t)=z(x,t)_{|x=\pi/2}.$$

The main step of the algorithm is the third. The constructive realizability of step 3 is mathematically justified by Lemma.

Conclusion. In this paper the statement of the inverse problem to stabilize the solution of the loaded heat conduction equation using boundary conditions is given. Theorem on solvability of the stated inverse problem is proved. An algorithm of approximate construction of boundary controls in the form of synthesis is developed. Numerical calculations were carried out, that showed the effectiveness of the proposed algorithm. We note that within this work the load is determined at the point $x=0$. This unessential condition, the results can be easily extended to the case of an arbitrary point in the interval $(-\pi/2,\pi/2)$.

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Жүктелген жылуөткізгіштік теңдеуі шешімінің стабилизациясы туралы

Жүктелген дифференциалдық теңдеулерді зерттеуге үнемі артып келе жатқан қызығушылық жүктелген теңдеулер нақты есептерге қатысты функционалды-дифференциалдық теңдеулердің арнайы класын қалыптастыруы сынды қосымшалары мен жағдаяттарына байланысты түсіндіріледі. Бұл теңдеулер маңызды қолданбалы тартымдылығы бар дифференциалдық теңдеулердің кері есептерін зерттеуге арналған қосымшаларға ие. Мақалада $\Omega \equiv (-\pi/2, \pi/2)$ шектелген облысында жүктелген жылуөткізгіштік теңдеуі үшін шекара арқылы стабилизациялау есептерінің шешілетіндігі мәселелері зерттелген. Мәселе шекаралық шарттарды (басқаруды) таңдау кезінде алынған аралас шеттік есептің шешімі $t \to \infty$ болғанда берілген $\exp(-\sigma_0 t)$ жылдамдықпен белгілі стационар шешімге ұмтылуында. Сонымен қатар басқару кері байланысты болуы талап етіледі, яғни ол жүйенің күтілмеген флуктуацияларына жауап бере отырып, олардың шешімнің стабилизациясына әсер етуі нәтижелерін басуы керек. Стабилизация есептері басқарымдылық мәселелерімен тікелей байланысты. Авторлар кері байланыс ұғымын математикалық формализациялауды ұсынады және де оның көмегімен жүктелген жылуөткізгіштік теңдеуі шекара аймағында берілген кері байланысты басқару арқылы шешіледі.

Кілт сөздер: стабилизация, жүктелген жылуөткізгіштік теңдеуі, меншікті мән, меншікті функция.

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О стабилизации решения нагруженного уравнения теплопроводности

Постоянно растущий интерес к изучению нагруженных дифференциальных уравнений объясняется их приложением и тем обстоятельством, что нагруженные уравнения образуют особый класс функционально-дифференциальных уравнений с конкретными задачами. Эти уравнения имеют приложения для изучения обратных задач дифференциальных уравнений с важными прикладными интересами. В статье исследованы вопросы разрешимости задач стабилизации с границей для нагруженного уравнения теплопроводности в заданной ограниченной области $\Omega \equiv (-\pi/2, \pi/2)$. Задача заключается в выборе граничных условий (управлений); решение полученной смешанной краевой задачи стремится к заданному стационарному решению с заданной скоростью $\exp(-\sigma_0 t)$ при $t \to \infty$. При этом требуется, чтобы управление было с обратной связью, т.е. чтобы оно реагировало на непредусмотренные флуктуации системы, подавляя результаты их воздействия на стабилизируемое решение. Задачи стабилизации имеют непосредственную связь с проблемами управляемости. В работе предложена математическая формализация понятия обратной связи, и с его помощью решается задача о стабилизируемости нагруженного уравнения теплопроводности посредством управления с обратной связью, заданного на части границы.

Ключевые слова: стабилизация, управление с обратной связью, нагруженное уравнение теплопроводности, краевая задача, обратная задача, функция Грина, собственное значение, собственная функция.

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On the ill-posed problem for the Poisson equation

A boundary value problem in a two-dimensional rectangular region for the Poisson equation is studied in the paper. The original ill-posed boundary value problem is transformed to the optimal control problem. The paper gives a brief overview of the problem under study, defines the formulation of the original boundary value problem and optimization problems, proves the existence of a solution to the regularized optimization problem, determines the formulation of the adjoint boundary value problem, studies the optimality conditions, and presents the application of the variable separation method. The necessary and sufficient conditions of optimality in terms of the conjugate boundary value problem are established in the paper, and a strong criterion for the solvability of the ill-posed boundary value problem is obtained. Boundary value problems for the Poisson equation arise in many sections of physics, mechanics, and other applied sciences. So, the stress function in the torsion problem of elastic rods is the solution of the Dirichlet problem, and the height of the liquid rise in the cylindrical capillary is the solution of the Neumann problem. But in many cases practitioners are interested in ill-posed problems for the Poisson equation and their solvability, which determines the relevance of the problem studied in the article.

Keywords: Poisson equation, ill-posed problem, optimal control, variational inequality, two-dimensional rectangular area.

Introduction. Recently among the experts on equations of mathematical physics interest in problems that are ill-posed by J. Hadamard has significantly increased [1]. Due to the ill-posed problems classic work by J. Hadamard [1], A.N. Tikhonov [2], M.M. Lavrent'ev [3] and many others can be noted, which have drawn the attention of researchers for ill-posed problems and have made a significant contribution to the development of this important area of mathematics. In this paper we study the ill-posed problem [1]–[8] for the Poisson equation in two-dimensional rectangular domain. The correctness criterion of homogeneous mixed Cauchy problem for the Poisson equation in a rectangular domain was established in the paper of T.Sh. Kalmenov, U.A. Iskakova [6]. In paper [8] the ill-posed problem for the heat equation is considered. The general regularization method for constructing an approximate solution of ill-posed problems of mathematical physics was proposed by A.N. Tikhonov [2]. In the book R. Lattes, J.-L. Lions [4] for regularization of ill-posed boundary value problems the quasiinversion method is proposed. Features and questions of the regularization of Cauchy problems for abstract differential equations with the operator coefficients are studied by I.V. Mel'nikova and U.A. Anufrieva [8].

Statement of the problem. We consider the boundary value problem

$$y_{tt}(x,t) + y_{xx}(x,t) = f(x,t); \tag{1}$$

$$y(0,t) = 0, \;\; y(\pi,t) = 0; \tag{2}$$

$$y(x,-1) = \varphi_1(x), \;\; y_t(x,-1) = \varphi_2(x), \tag{3}$$

in the domain $\Omega = \{x,\, t \,|\, 0 < x < \pi, -1 < t < 1\}$ with the additional condition

$$y_t(x,1) \in \mathcal{U}_g, \;\text{ where } \mathcal{U}_g \text{ is a closed convex set of } L_2(0,\pi). \tag{4}$$

It is assumed that the data in the problem (1)–(3) satisfies the following conditions:

$$f \in L_2(\Omega), \;\; \varphi_1 \in H_0^1(0,\pi), \;\; \varphi_2 \in L_2(0,\pi). \tag{5}$$

In the book R.Lattes, J.-L.Lions [4], it is indicated that problem (1)–(3) is ill-posed in the space $L_2(\Omega)$.

In this paper for solving the ill-posed problem we apply methods of optimal control.

The optimization problem. For the investigation of the problem (1)–(4), we formulate according to it the following optimization problem:

$$y_{tt}(x,t)+y_{xx}(x,t)=f(x,t); \tag{6}$$

$$y(0,t)=y(\pi,t)=0; \tag{7}$$

$$y_t(x,-1)=\varphi_2(x),\ y_t(x,1)=\psi(x), \tag{8}$$

with functional of optimality:

$$\mathcal{J}(\psi)=\int\limits_0^\pi |y_x(x,-1)-\varphi_1'(x)|^2dx\to\min_{\psi\in\mathcal{U}_g}. \tag{9}$$

We note, in optimization problem (6)–(9) the function $\psi(x)$ plays the role of control function. In addition, further in the work it will be shown that boundary problem (6)–(8) is well-posed, namely it is uniquely solvable for any given functions $\psi\in\mathcal{U}_g\subset L_2(0,\pi),\ f\in L_2(\Omega)$.

As it is known from the theory of optimal control optimization problem (6)–(9) is also ill-posed. The ill-posedness of this problem is shown in the following: functional to be minimized (9) is not strictly convex. Therefore, to small change of the value of the minimized functional (9) the significant change of the control function $\psi(x)$ can correspond in the admissible set $\mathcal{U}_g$, or to single value of functional (9) the set of admissible controls can correspond. For such optimization problems, there is an effective regularization method of Tikhonov [2]. To study our problem, we will use stabilizer of Tikhonov [2].

Regularized optimization problem. Effective tool is the method of regularization. In our case

$$\alpha\int\limits_0^\pi |\psi(x)|^2\,dx\ \ (\alpha>0)$$

will serve as a stabilizer.

We consider the problem of minimizing the following functional

$$\mathcal{J}_\alpha(y,\psi)=\int\limits_0^\pi |y_x(x,-1)-\varphi_1'(x)|^2dx+\alpha\int\limits_0^\pi |\psi(x)|^2dx\to\min_{\psi\in\mathcal{U}_g}. \tag{10}$$

Thus, we have the regularized optimization problem (6)–(8), (10). Due to the presence of the stabilizer the problem has become strictly convex, namely we get well-posed optimization problem. Therefore, for each value $\alpha>0$ this problem has the unique optimal solution that delivers the minimum value to minimized functional (10). However, it does not exclude the fact that the minimum value problem of functional (10) can be strictly greater than zero.

For optimal control problem (6)–(8), (10) we will establish optimality conditions. We introduce the concept of optimal control.

Definition 1. An element $\overline{\psi}\in L_2(0,\pi)$ which satisfies the condition

$$\mathcal{J}_\alpha(\overline{\psi})=\inf_{\psi\in\mathcal{U}_g}\mathcal{J}_\alpha(\psi)$$

is called the optimal control.

We denote the solution of problem (6)–(8) by $y(x,t;\psi)$ corresponding to the given control $\psi(x)\in\mathcal{U}_g$.

So $y(x,t;0)$ corresponds to the solution of problem (6)–(8) when $\psi(x)\equiv 0$. Then, we get

$$\pi(\psi_1,\psi_2)=\int\limits_0^\pi [y_x(x,-1;\psi_1)-y_x(x,-1;0)]\times$$

$$\times[y_x(x,-1;\psi_2)-y_x(x,-1;0)]dx+\alpha\cdot\int\limits_0^\pi \psi_1(x)\cdot\psi_2(x)dx;$$

$$L(\psi_1)=\int\limits_0^\pi [\varphi_1'(x)-y_x(x,-1;0)][y_x(x,-1;\psi_1)-y_x(x,-1;0)]dx.$$

Here, $\pi(\psi_1,\psi_2)$ is the bilinear functional on $\mathcal{U}_g$, $L(\psi_1)$ is the continuous linear functional on admissible set of controls $\mathcal{U}_g$, as it will be shown below, that the solution $y(x,t;\psi)$ of problem (6)–(8) is not only continuous but it is continuously differentiable on control ψ. Using the notation, functional (10) can be rewritten as

$$\mathcal{J}_\alpha(\psi)=\pi(\psi,\psi)-2L(\psi)+\int\limits_0^\pi |y_x(x,-1;0)-\varphi_1'(x)|^2dx.$$

The existence of solution of the regularized problem and the variational inequality. The following theorem holds [9].

Theorem 1. As $\pi(\psi,\psi)$ is the continuous symmetric quadratic functional on a $\mathcal{U}_g$ and satisfies the condition

$$\pi(\psi,\psi)\geq c\|\psi\|^2,\ \ (c=\mathrm{const}>0), \tag{11}$$

then for problem (6)–(8), (10) exists only $\overline{\psi}\in\mathcal{U}_g$:

$$\mathcal{J}_\alpha(\overline{\psi})=\inf_{\psi\in\mathcal{U}_g}\mathcal{J}_\alpha(\psi).$$

The inequality (11) holds, as

$$\pi(\psi,\psi)=\int\limits_0^\pi |y_x(x,-1;\psi)-y_x(x,-1;0)|^2dx+\alpha\cdot\int\limits_0^\pi \psi^2(x)dx.$$

The solution of optimization problem (6)-(8), (10) we denote by

$$\overline{\psi}(x)=\arg\min_{\psi\in\mathcal{U}_g}\mathcal{J}_\alpha(\psi).$$

Further, according to the theory of strictly convex optimization problems the following optimality criterion formulated in terms of the directional derivative is valid.

Proposition 1 (Variational inequality). The function $\overline{\psi}\in\mathcal{U}_g$ *is a function of the optimal control if and only if the following inequality holds:*

$$\langle\mathcal{J}_{\alpha\psi}(\overline{\psi}),\psi-\overline{\psi}\rangle\geq 0,\ \ \forall\psi\in\mathcal{U}_g,$$

namely we have

$$\int\limits_0^\pi \left[y_x\left(x,-1;\overline{\psi}\right)-\varphi_1'(x)\right]\cdot\frac{\partial}{\partial x}\left\{y_\psi\left(x,-1;\overline{\psi}\right)\cdot\left[\psi(x)-\overline{\psi}(x)\right]\right\}dx+$$

$$+\alpha\cdot\int\limits_0^\pi \overline{\psi}(x)\cdot\left[\psi(x)-\overline{\psi}(x)\right]dx\geq 0,\ \ \forall\ \psi\in\mathcal{U}_g. \tag{12}$$

We now carry out the necessary further transformations of variational inequality (12). For this purpose, we rewrite the boundary value problem (6)–(8) in the operator form $\mathcal{A}y=\mathcal{F}=\{f,\varphi_2,\psi\}$. As for any admissible controls boundary value problem (6)–(8) is uniquely solvable, then its solution $y(x,t;\psi)$ can be written in the following form $y(x,t;\psi)=\mathcal{A}^{-1}\mathcal{F}=\mathcal{A}_0^{-1}f+\mathcal{A}_1^{-1}\varphi_2+\mathcal{A}_2^{-1}\psi$.

Next, we take the derivative of this solution in the direction of $\psi-\overline{\psi}$. We have

$$y_\psi(x,t;\overline{\psi})\cdot[\psi-\overline{\psi}]=\mathcal{A}^{-1}(\psi-\overline{\psi})=$$

$$=\mathcal{A}_0^{-1}f+\mathcal{A}_1^{-1}\varphi_2+\mathcal{A}_2^{-1}\psi-[\mathcal{A}_0^{-1}f+\mathcal{A}_1^{-1}\varphi_2+\mathcal{A}_2^{-1}\overline{\psi}]=y(x,t;\psi)-y(x,t;\overline{\psi})$$

or

$$y_\psi\left(x,t;\overline{\psi}\right)\cdot\left[\psi-\overline{\psi}\right]=y(x,t;\psi)-y\left(x,t;\overline{\psi}\right).$$

Thus inequality (12) has the form:

$$\int\limits_0^\pi \left[y_x\left(x,-1;\overline{\psi}\right)-\varphi_1'(x)\right]\cdot\left[y_x(x,-1;\psi)-y_x\left(x,-1;\overline{\psi}\right)\right]dx+$$

$$+\alpha \cdot \int_0^\pi \overline{\psi}(x) \cdot \left[\psi(x) - \overline{\psi}(x)\right] dx \geq 0, \;\; \forall \; \psi \in \mathcal{U}_g. \tag{13}$$

The adjoint boundary value problem. For further study of regularized optimization problem (6)–(8), (10), we introduce the adjoint boundary value problem

$$\begin{cases} v_{tt}(x,t) + v_{xx}(x,t) = 0, \;\; x \in (0,\pi), \;\; t \in (-1,1); \\ v(0,t) = v(\pi,t) = 0, \;\; t \in (-1,1); \\ \int_\eta^x v_t(\xi,-1)d\xi = -y_x\left(x,-1;\overline{\psi}\right) + \\ +\varphi_1'(x) + y_\eta\left(\eta,-1;\overline{\psi}\right) - \varphi_1'(\eta), \;\; \forall \; 0 < \eta < x < \pi; \\ v_t(x,1) = 0. \end{cases} \tag{14}$$

For its formal conclusion we consider the following expression

$$\int_{-1}^1 \int_0^\pi \left[\tilde{y}_{tt}\left(x,t\right) + \tilde{y}_{xx}\left(x,t\right)\right] \cdot v(x,t;\overline{\psi}) dx dt = 0,$$

where $\tilde{y}(x,t) = y(x,t;\psi) - y(x,t;\overline{\psi})$.

We transform this expression, considering adjoint boundary value problem (14)

$$\int_{-1}^1 \int_0^\pi [\tilde{y}_{tt}(x,t) + \tilde{y}_{xx}(x,t)] \cdot v(x,t;\overline{\psi}) dx dt = \int_0^\pi [\psi(x) - \overline{\psi}(x)] v(x,1;\overline{\psi}) dx +$$

$$+ \int_0^\pi \left[y\left(x,-1;\psi\right) - y\left(x,-1;\overline{\psi}\right)\right] \cdot v_t(x,-1;\overline{\psi}) dx = 0. \tag{15}$$

Thus, we obtain desired adjoint boundary problem (14).

Optimality conditions. By applying the equality

$$\int_\eta^x v_t(\xi,-1)d\xi = -y_x\left(x,-1;\overline{\psi}\right) + \varphi_1'(x) + y_\eta\left(\eta,-1;\overline{\psi}\right) - \varphi_1'(\eta), \;\; \forall \; 0 < \eta < x < \pi,$$

we rewrite expression (15) as follow

$$\int_0^\pi \left[y_x\left(x,-1;\overline{\psi}\right) - \varphi_1'(x)\right] \cdot \left[y_x\left(x,-1;\psi\right) - y_x\left(x,-1;\overline{\psi}\right)\right] dx = -\int_0^\pi v(x,1) \cdot [\psi(x) - \overline{\psi}(x)] dx,$$

then from relation (13), we finally obtain the desired variational inequality

$$\int_0^\pi \left[-v(x,1;\overline{\psi}) + \alpha \cdot \overline{\psi}(x)\right] \cdot \left[\psi(x) - \overline{\psi}(x)\right] dx \geq 0, \;\; \forall \; \psi \in \mathcal{U}_g. \tag{16}$$

Thus, on the basis of Proposition 1 we have established the optimality conditions, which can be formulated as the following proposition:

Proposition 2. The element $\overline{\psi}(x)$ is the optimal solution to the problem (6)–(8), (10), if and only if it satisfies boundary value problems (6)–(8), (14), and variational inequality (16).

Application of the method of separation of variables. For resolving the conditions of an optimality (6)–(8), (14) and (16) we use a method of separation of variables. We will search solutions of boundary value problems (6)–(8) and (14) in the form

$$y(x,t) = \sum_{k=1}^\infty y_k(t) X_k(x), \quad v(x,t) = \sum_{k=1}^\infty v_k(t) X_k(x),$$

where

$$X_k(x)=\frac{\sin kx}{\sqrt{\pi/2}},\quad \lambda_k=k^2,\ \ k=1,2,\ldots, \tag{17}$$

are systems orthonormalized eigenfunctions and eigenvalues for a spectral problem:

$$X''(x)=\lambda\cdot X(x),\quad X(0)=X(\pi)=0.$$

From (6)–(8), (14) and (16) we accordingly obtain

$$\begin{cases} y_k''(t)-k^2y_k(t)=f_k(t),\ t\in(-1,1); \\ y_k'(-1)=\varphi_{2k};\ \ y_k'(1)=\overline{\psi}_k;\ \ k=1,2,\ldots; \end{cases} \tag{18}$$

$$\begin{cases} v_k''(t)-k^2v_k(t)=0,\ t\in(-1,1); \\ v_k'(-1)=k^2[y_k(-1)-\varphi_{1k}];\ \ v_k'(1)=0;\ \ k=1,2,\ldots; \end{cases} \tag{19}$$

$$\left[-v_k(1)+\alpha\cdot\overline{\psi}_k\right]\cdot\left[\psi_k-\overline{\psi}_k\right]\geq 0,\ \text{ for }\forall\ \psi_k,\ k=1,2,\ldots, \tag{20}$$

where $f_k(t)$, φ_{1k}, φ_{2k}, $\overline{\psi}_k$, ψ_k, $k=1,2,\ldots$ are Fourier-coefficients of functions $f(x,t)$, $\varphi_1(x)$, $\varphi_2(x)$ and $\overline{\psi}(x)$, $\psi(x)$ on system (17).

Assume us write solutions of boundary value problems (18) and (19):

$$y_k(t)=\overline{\psi}_k\cdot\frac{\cosh k(t+1)}{\sinh 2k}-\varphi_{2k}\cdot\frac{\cosh k(1-t)}{k\sinh 2k}+\int_{-1}^{1}G_k(t,\tau)\cdot f_k(\tau)d\tau; \tag{21}$$

$$v_k(t)=-[y_k(-1)-\varphi_{1k}]\cdot\frac{k\cosh k(1-t)}{\sinh 2k}; \tag{22}$$

where

$$G_k(t,\tau)=\begin{cases} -\dfrac{\cosh k(1-t)\cdot\cosh k(1+\tau)}{\sinh 2k},\ -1<\tau<t<1; \\ -\dfrac{\cosh k(1-\tau)\cdot\cosh k(1+t)}{\sinh 2k},\ -1<t<\tau<1. \end{cases}$$

From (21)–(22) and (20) we find

$$-v_k(1)=[y_k(-1)-\varphi_{1k}]\cdot\frac{k}{\sinh 2k},$$

$$y_k(-1;\overline{\psi}_k)=-\varphi_{2k}\frac{\coth 2k}{k}+\overline{\psi}_k\frac{1}{\sinh 2k}+\int_{-1}^{1}G_k(-1,\tau)f_k(\tau)d\tau,$$

$$\left[A_{k\alpha}\overline{\psi}_k-\varphi_{1k}-\varphi_{2k}\frac{\coth 2k}{k}+\int_{-1}^{1}G_k(-1,\tau)f_k(\tau)d\tau\right]\cdot\left[\psi_k-\overline{\psi}_k\right]\geq 0\ \text{ for all }\ \psi_k, \tag{23}$$

where $A_{k\alpha}=\dfrac{k+\alpha\sinh^2 2k}{k\sinh 2k}$, $k=1,2,\ldots$

Now we put, that $\mathcal{U}_g\equiv L_2(0,\pi)$. Since the functions $\psi(x)$ do not have any restrictions except for belonging to the space $L_2(0,\pi)$, from (23) we can find the optimal values of Fourier coefficients $\overline{\psi}_k$, $k=1,2,\ldots$:

$$\overline{\psi}_k=A_{k\alpha}^{-1}\left[\varphi_{1k}+\varphi_{2k}\frac{\coth 2k}{k}-\int_{-1}^{1}G_k(-1,\tau)f_k(\tau)d\tau\right]. \tag{24}$$

Further, as $\alpha\to 0$ (21) and (24) imply that

$$y_{k0}(t)=\lim_{\alpha\to 0}y_k(t)=\varphi_{1k}\cosh k(1+t)+\varphi_{2k}\frac{\sinh k(1+t)}{k}-$$

$$- \cosh k(1+t) \int_{-1}^{1} G_k(-1,\tau) f_k(\tau) d\tau + \int_{-1}^{1} G_k(t,\tau) \cdot f_k(\tau) d\tau; \quad (25)$$

$$\overline{\psi}_{k0} = \lim_{\alpha\to 0} \overline{\psi}_k = \varphi_{1k} \sinh 2k + \varphi_{2k} \frac{\cosh 2k}{k} - \sinh 2k \int_{-1}^{1} G_k(-1,\tau) f_k(\tau) d\tau. \quad (26)$$

Additionally, the solutions $y_k(t)$ found under formula (21) according to optimal Fourier coefficients $\overline{\psi}_k$, $k = 1, 2, \ldots$ (24) must satisfy to limiting relations: $\lim_{\alpha\to 0} y_k(-1) = \varphi_{1k}$, which really hold. And it is coordinated with a condition $y(x,-1) = \varphi_1(x)$ from (3).

Thus, for a finding of the exact solution of problem (6)–(8) according to (26) we construct the following series:

$$\overline{\psi}(x) = \sum_{k=1}^{\infty} \sqrt{2/\pi} \sinh 2k \left[\varphi_{1k} + \varphi_{2k} \frac{\coth 2k}{k} - \int_{-1}^{1} G_k(-1,\tau) f_k(\tau) d\tau \right] \sin kx$$

and for initial Cauchy-Dirichlet problem (1)–(3) we obtain the solution on the basis of formulas (25).

Conclusion. From equalities (25) and (26) the following directly holds:

Firstly, with growth of index k and at $\alpha \to 0$ the Fourier-coefficients of the function $\overline{\psi}(x)$ and, respectively, the solution $y_k(t)$ can increase without limit if this growth is not be «suppressed» with corresponding more rapid decrease of the absolute values of the coefficients φ_{1k}, φ_{2k} and values of norms $\|f_k(t)\|_{L_2(-1,1)}$.

Secondly, boundary problem (1)–(3) under conditions (5) has unique L_2-strong solution [10] if and only if

$$\{\exp\{2k\} \cdot \varphi_{1k}\}_{k=1}^{\infty}, \quad \{k^{-1} \exp\{2k\} \cdot \varphi_{2k}\}_{k=1}^{\infty}; \quad \{\exp\{2k\} \cdot \|f_k(\tau)\|_{L_2(-1,1)}\}_{k=1}^{\infty} \subset l_2. \quad (27)$$

Thus, it is clear not only the meaning of regularization in problem (6)–(8) and (10), but also the nature of incorrectness in Cauchy-Dirichlet problem (1)–(3) [6, 7]. And regularization allows us to find an approximate solution.

Thirdly, we consider the example of Hadamard [11; 37]. To receive analogue of the Hadamard example in problem (1)–(3) it is necessary to accept:

$$f(x,t) = 0, \ \varphi_1(x) = 0, \ \varphi_2(x) = \sqrt{2/\pi} \cdot k \cdot \exp\{-\sqrt{k}\} \sin kx, \ k \in \mathbf{N}.$$

Really, the solution of Cauchy-Dirichlet problem for Laplace equation has the form:

$$y(x,t) = \sqrt{2/\pi} \cdot \exp\{-\sqrt{k}\} \sin kx \cdot \sinh k(t+1), \ k \in \mathbf{N}. \quad (28)$$

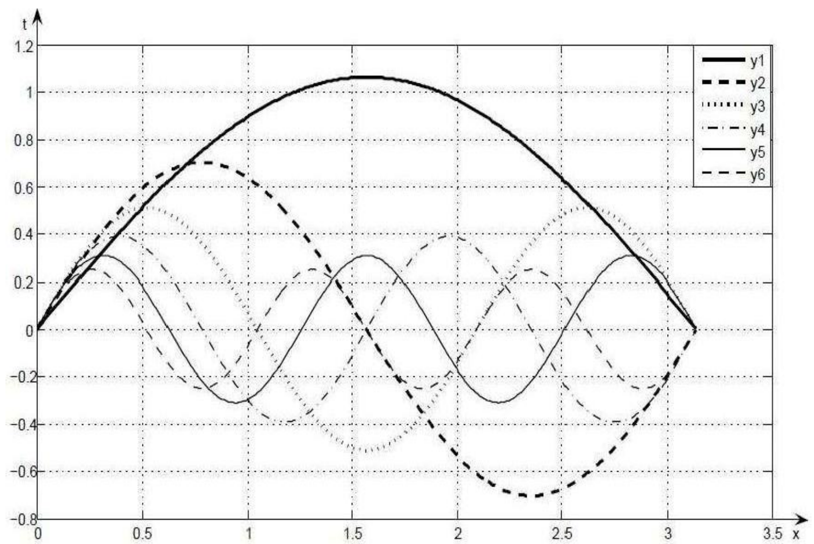

Figure. Graph of solution $y_k(x,t)$ at $k = \overline{1,6}$ of (28)

In Figure are shows the graphs of the solution $yk(x,t)$ at $k = 1,6$ of (28). This solution of a problem in example of Hadamard considered by us is unique. Moreover, as $k \to \infty$ the function $\varphi_2(x)$ approaches uniformly zero and that not only, but also all its derivatives approache zero and it belongs to space $L_2(0,\pi)$. However the solution (28) at any $t > -1$ has the form of a sinusoid with an arbitrarily large amplitude and does not belong to space $L_2((0,\pi)\times(-1,1))$.

In order to the function $\varphi_2(x)$ satisfied to condition (27), it is necessary and sufficient, that the Fourier-coefficients φ_{2k} had the asymptotic behavior for large k of order $\exp\{-(2+\varepsilon)k\}$ where $\varepsilon > 0$. In example of Hadamard considered by us we have asymptotic which is only equal to $\exp\{-\sqrt{k}\}$, and it is obviously not enough for a correctness of Cauchy-Dirichlet problem for Poisson equation.

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Пуассон теңдеуі үшін корректі емес есеп туралы

Мақалада Пуассон теңдеуі үшін екіөлшемді тікбұрышты облыста шекаралық есеп қарастырылды. Корректі емес шеттік есеп тиімді басқару есебіне келтірілді. Авторлар зерттелетін мәселеге қысқаша шолу жасап, бастапқы шеттік есеп пен тиімділік есептерінің қойылымын анықтап, регуляризацияланған тиімділік есебінің шешімінің бар болуын дәлелдей отырып, түйіндес шеттік есептің қойылымын анықтап, тиімділік шарттарын зерттеген, сондай-ақ айнымалыларды бөліктеу әдісінің қолданылуы келтірілген. Жұмыста тиімділіктің қажетті және жеткілікті шарттары түйіндес шеттік есеп терминдерінде табылып, корректі емес шеттік есептің шешімділігінің әлді критерийі алынған. Пуассон теңдеуі үшін шеттік есептер физиканың, механика және де басқа қолданбалы ғылымдардың көптеген салаларында кездеседі. Серпімді өзектің бұралуы туралы есебіндегі кернеу функциясы Дирихле есебінің шешімі, ал цилиндрлік капиллярдағы сұйықтың көтерілуінің биіктігі Нейман есебінің шешімі болып табылады. Дегенмен көптеген жағдайда практиктерді Пуассон теңдеуі үшін корректі емес есептер және олардың шешімділігі мәселелері қызықтырады, сондықтан мақаладағы зерттелген мәселенің өзектілігі артады.

Кілт сөздер: Пуассон теңдеуі, корректі емес есеп, түйіндес шекаралық есеп, тиімді басқару, тиімділік шарттары, регуляризация, вариациялық теңсіздік, екіөлшемді тікбұрышты облыс.

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О некорректной задаче для уравнения Пуассона

В работе рассмотрена краевая задача в двумерной прямоугольной области для уравнения Пуассона. Изученная некорректная краевая задача сводится к задаче оптимального управления. Авторами дан краткий обзор исследуемой проблемы, определены постановка исходной краевой задачи и задачи оптимизации, доказано существование решения регуляризированной задачи оптимизации, определена постановка сопряженной краевой задачи, исследованы условия оптимальности, представлено применение метода разделения переменных. В работе установлены необходимые и достаточные условия оптимальности в терминах сопряженной краевой задачи, а также получен сильный критерий разрешимости некорректной краевой задачи. Краевые задачи для уравнения Пуассона возникают во многих разделах физики, механики и других прикладных наук. Так, функция напряжений в задаче о кручении упругих стержней является решением задачи Дирихле, а высота подъёма жидкости в цилиндрическом капилляре — решением задачи Неймана. Во многих случаях практиков интересуют некорректные задачи для уравнения Пуассона и вопросы их разрешимости, что определяет актуальность исследуемой в статье проблемы.

Ключевые слова: уравнение Пуассона, некорректная задача, сопряженная граничная задача, оптимальное управление, условия оптимальности, регуляризация, вариационное неравенство, двумерная прямоугольная область.

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Some integral estimates on the cones of functions with the monotonicity conditions

In this paper we obtain estimates for the integrals of monotone functions arising in the study of the covering of various cones of functions with monotonicity conditions. We apply the method of covering of the cones with the help of generalized Hardy operator. Sharp conditions are found on the kernels of representations for the validity of given estimates on the cones. The proofs are based on the reduction of integral estimates on the cones of monotone functions to ones on the family of characteristic functions of intervals. The obtained results can be used in finding the condition for the mutual covering of cones associated with decreasing rearrangement of the generalized Bessel and Riesz potentials.

Keywords: functional norm, cones of functions with monotonicity conditions, covering of cones.

When studying the embedding of the potential spaces in rearrangement-invariant spaces, various cones of functions with monotonicity conditions arise. In terms of such cones, we can formulate the embedding criteria for the space of generalized Bessel and Riesz potentials in rearrangement invariant spaces [1–3]. In this case, an important role is played by problems of ordinal covering of cones.

Let (S, Σ, μ) be a measure space. Here Σ is σ-algebra of subsets of the set S, on which is determined a non-negative σ- finite, $\sigma-$ aditive measure μ. $L_0 = L_0(S, \Sigma, \mu)$ denotes the set μ-measurable real-valued functions $f : S \to R, L_0^+ = \{f \in L_0 : f \geq 0\}$.

Definition 1. [4] Mapping $\rho : L_0^+ \to [0, \infty]$ is called a functional norm (short: FN), if for all $f, g, f_n \in L_0^+$, $n \in N$ the conditions are fulfilled:

$(P1)$ $\rho(f) = 0 \Rightarrow f = 0$, $\mu-$ almost everywhere (short: $\mu-$ a.e.);

$\rho(\alpha f) = \alpha\rho(f), \alpha \geq 0; \rho(f + g) \leq \rho(f) + \rho(g)$ (property of the norm);

$(P2)$ $f \leq g$, ($\mu-$ a.e.) $\Rightarrow \rho(f) \leq \rho(g)$ (monotonicity of the norm);

$(P3)$ $f_n \uparrow f \Rightarrow \rho(f_n) \to \rho(f) (n \to \infty)$(Fatou property);

$(P4)$ $0 < \mu(\sigma) < \infty \Rightarrow \int\limits_{\sigma} f d\mu \leq c_\sigma \rho(f), f \in L_0^+$. (Local integrability);

$(P5)$ $0 < \mu(\sigma) < \infty \Rightarrow \rho(\chi_\sigma) < \infty$ (finiteness of the FN for characteristic functions (χ_σ) for the sets of finite measure).

Here $f_n \uparrow f$ means that $f_n \leq f_{n+1}, \lim\limits_{n\to\infty} f_n = f$ ($\mu-$ a.e.)

Definition 2. Let ρ be a functional norm. Set $X = X(\rho)$ of functions in L_0, for which $\rho(|f|) < \infty$ is called a Banach function space, generated by a FN ρ. For $f \in X$ we set

$$\|f\|_X = \rho(|f|).$$

Let relations of partial order and equivalence be introduced on L_0^+ :
$f \prec g$ with properties of transitivity, i.e. $f \prec f$;

$$f \prec g, \quad g \prec h \Rightarrow f \prec h; \;\; f \approx g \Leftrightarrow f \prec g \prec f.$$

We assume, that the order relation is subordinated to pointwise estimate μ-a.e., i.e.

$$1) \;\; f \leq g, \quad (\mu - \texttt{a.e.}) \Rightarrow f \prec g; \qquad 2) \;\; f_n \uparrow f \quad \Rightarrow \quad f_n \underline{\uparrow} \; f. \tag{1}$$

Here $f_n \underline{\uparrow} f$ means that $f_n \prec f_{n+1}$; $f = [\sup] f_n$ i.e. $f_n \prec f$ $n \in N$ and, if $f_n \prec \hat{f}$, $n \in N$ then $f \prec \hat{f}$.
A basic example of the order relation: $f \prec g \Leftrightarrow f \leq g$, μ-a.e. $\Rightarrow \rho(f) \leq \rho(g)$.

We are interested in the relation of the order associated with the decreasing rearrangement of functions. Denote for $f \in L_0$

$$\lambda_f(y) = \mu\{x \in S: \ |f(x)| > y\}, \quad y \in [0, \infty) \tag{2}$$

— Lebesgue distribution function. Through $\overset{\bullet}{L}_0$ denote the set of functions $f \in L_0$ for which $\lambda_f(y)$ is not identical to infinity, i.e. $\exists\, y_0 \in [0,\infty): \quad \lambda_f(y_0) < \infty$. For $f \in \overset{\bullet}{L}_0$ we introduce a decreasing rearrangement f^* as a right inverse function of a decreasing function λ_f, i.e.

$$f^*(t) = \inf\{y \in [0,\infty): \quad \lambda_f(y) \le t\}, \quad t \in R_+ = (0,\infty). \tag{3}$$

It is known that $0 \le f^* \downarrow; \quad f^*(t+0) = f^*(t), \quad t \in R_+$; f^* is equimeasurable with $|f|$, i.e. $\mu_1\{t \in R_+: \ f^*(t) > y\} = \lambda_f(y), \quad y \in [0,\infty)$. In addition, for $f \in \overset{\bullet}{L}_0$ we have: $\lambda_f(y) \to 0, \ (y \to +\infty) \Leftrightarrow$ $\Leftrightarrow |f(x)| < \infty, \ (\mu - \mathtt{a.e.})$ on S.

We define the order relations for the functions from $\overset{\bullet}{L}{}_0^+$:

$$1) \quad f \prec g \Leftrightarrow \quad f^*(t) \le g^*(t); \quad t \in (0, \mu(S)); \tag{4}$$

$$2) \quad f \prec g \Leftrightarrow \int\limits_0^t f^* d\tau \le \int\limits_0^t g^* d\tau; \quad t \in (0, \mu(S)). \tag{5}$$

The order relation (5) is subordinate to (4); both are subordinate to pointwise estimation μ-a.e.. The equivalence of functions by order relation (4) means equimeasurability.

Definition 3. Let ρ be a FN. We say that ρ is consistent with the order relation $\prec$, if for $f, \ g \in L_0^+, \ f \prec g$ we have $\rho(f) \le \rho(g)$.

Let us note that by property $(P2)$ any FN is consistent with a pointwise estimate:

$$f \le g \quad (\mu - \mathtt{a.e.}) \Rightarrow \rho(f) \le \rho(g). \tag{6}$$

Definition 4. A FN ρ is rearrangement-invariant if it is compatible with the order relation (4) i.e.

$$f^* \le g^* \Rightarrow \rho(f) \le \rho(g). \tag{7}$$

BFS $X = X(\rho)$, generated by rearrangement-invariant FN ρ, we call as rearrangement-invariant space (short: RIS).

Let $K, \ M \subset L_0^+$ be cones of functions [5], equipped with non-degenerate positive homogeneous functionals ρ_K, and ρ_M i.e.

$$\rho_K: \quad K \to [0, \quad \infty); \quad h \in K \Rightarrow \alpha h \in K, \ \alpha \ge 0; \ \rho_K(\alpha h) = \alpha \rho_K(h);$$

$\rho_K(h) = 0 \Rightarrow h = 0, \ \mu$ - a.e. and analogous for $\rho_M(h) \ h \in M$.

Let on the L_0^+ given order relation$\prec$. Following [6], we introduce the notions of ordinal covering and order equivalence of cones.

Definition 5. Cone M covers the cone K with the covering constants $c_0 \in R_+$ and $c_1 \in [0,\infty)$ if for any $h_1 \in K$ there exits $h_2 \in M$ such that

$$\rho_M(h_2) \le c_0 \rho_K(h_1), \qquad h_1 \prec h_2 + c_1 \rho_K(h_1). \tag{8}$$

Designation of an ordinal covering: $K \prec M$.

Definition 6. We call the cones $K, \ M$ order-equivalent, if they mutually over each other.

The designation of ordinal equivalence:

$$K \approx M \Leftrightarrow K \prec M \prec K.$$

If the order relation $\prec$ coincides with a pointwise estimate of the functions μ- a.e., we will talk about pointwise covering of the cones and write $K \le M$. So when $K \le M$ (8) takes the form

$$\rho_M(h_2) \le c_0 \rho_K(h_1), \qquad h_1 \le h_2 + c_1 \rho_K(h_1), \quad (\mu - \mathtt{a.e.}). \tag{9}$$

Pointwise equivalence of the cones signifies their mutual pointwise covering and is denoted: $K \cong M$. So,

$$K \cong M \Leftrightarrow K \leq M \leq K. \tag{10}$$

Let $T \in (0, \infty]$. Through $\Omega(T)$ denote the class of functions on $(0,\ T)$:

$$\Omega(T) = \left\{ \varphi : \quad 0 < \varphi(t) \downarrow; \quad \int_0^t \varphi d\xi < \infty; \quad \varphi(t+0) = \varphi(t), \quad t \in (0,T) \right\}. \tag{11}$$

We introduce the functions of two variables $t, \tau \in (0,T)$

$$f_\varphi(t,\tau) = \min\{\varphi(t), (\varphi(\tau)\} = \begin{cases} \varphi(t), & \tau \in (0,t]; \\ \varphi(\tau), & \tau \in (t,T); \end{cases} \tag{12}$$

$$\tilde{f}_\varphi(t,\tau) = \begin{cases} \frac{1}{t}\int\limits_0^t \varphi(\xi)d\xi, & \tau \in (0,t]; \\ \varphi(\tau), & \tau \in (t,T). \end{cases} \tag{13}$$

It is clear that $f_\varphi(t,\tau)$ decreases and is continuous from the right by t and on τ. Further,

$$\varphi \in \Omega(T) \Rightarrow \frac{1}{t}\int_0^t \varphi(\xi)d\xi \geq \varphi(t), \qquad t \in (0,T), \tag{14}$$

so that

$$0 \leq f_\varphi(t,\tau) \leq \tilde{f}_\varphi(t,\tau), \qquad t, \tau \in (0,T) \tag{15}$$

and $\tilde{f}_\varphi(t,\tau)$ decreases by τ on the $(0,T)$. Now space with a measure (S, Σ, μ) is given as follows: $S = (0,T)$; Σ there is $\sigma-$ algebra of Lebesgue-measurable subsets of $(0,T)$, μ is Lebesgue measure. Let $E = E(0,T)$ be an RIS of measurable functions on $(0,T)$ with decreasing rearrangements with respect to Lebesgue measure μ;

$$E^{\downarrow} = E^{\downarrow}(0,T) = \{g \in E(0,T) : \qquad 0 \leq g \downarrow \quad \texttt{on} \quad (0,T)\}. \tag{16}$$

We introduce the cones of functions from $L_0^+(0,T)$:

$$K(T) = K_{\varphi,E}(T) = \left\{ h(t) \equiv h(g;t) := \int_0^T f_\varphi(t,\tau)g(\tau)d\tau : \quad g \in E^{\downarrow} \right\}; \tag{17}$$

$$\tilde{K}(T) = \tilde{K}_{\varphi,E}(T) = \left\{ \tilde{h}(t) \equiv \tilde{h}(g;t) := \int_0^T \tilde{f}_\varphi(t,\tau)g(\tau)d\tau : \quad g \in E^{\downarrow} \right\}. \tag{18}$$

Cones K and $\tilde{K}$ are equipped with functionals: for $h \in K, \quad \tilde{h} \in \tilde{K}$

$$\rho_K(h) = \inf\left\{\|g\|_E : \quad g \in E^{\downarrow}; \quad h(g;t) = h(t), \quad t \in (0,T)\right\}; \tag{19}$$

$$\rho_{\tilde{K}}(\tilde{h}) = \inf\left\{\|g\|_E : \quad g \in E^{\downarrow}; \quad \tilde{h}(g;t) = \tilde{h}(t), \quad t \in (0,T)\right\}. \tag{20}$$

We denote for $\varphi > 0, \varphi \downarrow$,

$$B_\varphi := \sup_{t \in (0,T)} \frac{\int\limits_0^t \varphi(\xi)d\xi}{\frac{1}{t}\int\limits_0^t \varphi(\xi)\xi d\xi}. \tag{21}$$

When investigating the problems of the mutual covering of cones $K_{\varphi,E}(T)$ and $\tilde{K}_{\varphi,E}(T)$ the following statements of independent interest can be used.

Theorem 1. Let $\varphi \in \Omega(T)$ see.(11) and $t \in (0,T)$. The following estimates are valid:

$$\int\limits_0^t \varphi(\xi)d\xi \le \sup_{\rho\in(0,t]} \left\{ \frac{1}{\rho} \int\limits_0^\rho \left[\varphi(\tau)\tau + \int\limits_\tau^t \varphi(\xi)d\xi \right] d\tau \right\} \le 2\int\limits_0^t \varphi(\xi)d\xi; \tag{22}$$

$$\frac{1}{t}\int\limits_0^t \varphi(\tau)\tau d\tau \le \inf_{\rho\in(0,t]} \left\{ \frac{1}{\rho} \int\limits_0^\rho \left[\varphi(\tau)\tau + \int\limits_\tau^t \varphi(\xi)d\xi \right] d\tau \right\} \le \frac{2}{t}\int\limits_0^t \varphi(\tau)\tau d\tau. \tag{23}$$

Proof.
At $\rho \in (0,t]$ we have

$$\int\limits_0^\rho \left[\varphi(\tau)\tau + \int\limits_\tau^t \varphi(\xi)d\xi \right] d\tau = \int\limits_0^\rho \varphi(\tau)\tau d\tau + \int\limits_0^\rho \left(\int\limits_\tau^t \varphi(\xi)d\xi \right) d\tau = \int\limits_0^\rho \varphi(\tau)\tau d\tau +$$

$$+ \int\limits_0^\rho \varphi(\xi) \left(\int\limits_0^\xi d\tau \right) d\xi + \int\limits_\rho^t \varphi(\xi) \left(\int\limits_0^\rho d\tau \right) d\xi = 2\int\limits_0^\rho \varphi(\tau)\tau d\tau + \rho \int\limits_\rho^t \varphi(\xi)d\xi.$$

We denote by

$$\Psi(\rho) = \frac{1}{\rho}\int\limits_0^\rho \varphi(\tau)\tau d\tau + \int\limits_\rho^t \varphi(\xi)d\xi, \qquad \rho \in (0,t];$$

$$Q(\rho) = \frac{1}{\rho}\int\limits_0^\rho \left[\varphi(\tau)\tau + \int\limits_\tau^t \varphi(\xi)d\xi \right] d\tau, \quad \rho \in (0,t]. \tag{24}$$

Then

$$\Psi(\rho) \le Q(\rho) \le 2\Psi(\rho).$$

We note that the function $\Psi(\rho)$ decreases monotonically on $(0,t]$. Indeed, since
$\Psi'(\rho) = -\frac{1}{\rho^2}\int\limits_0^\rho \varphi(\tau)\tau d\tau < 0$ therefore $\Rightarrow \Psi(\rho) \downarrow$.
Consequently,

$$\sup_{\rho\in(0,t]} \Psi(\rho) = \Psi(+0) = \int\limits_0^t \varphi(\xi)d\xi.$$

In the formula for $\Psi(+0)$ we took into account that

$$\frac{1}{\rho}\int\limits_0^\rho \varphi(\tau)\tau d\tau \le \int\limits_0^\rho \varphi(\tau)d\tau \to 0 \quad (\rho \to +0).$$

Moreover

$$\inf_{\rho\in(0,t]} \Psi(\rho) = \Psi(t) = \frac{1}{t}\int\limits_0^t \varphi(\tau)\tau d\tau. \tag{25}$$

From this and (24) follow the estimates (22), (23).
Corollary 1. Under the conditions of Theorem 1, for $\rho \in (0,t]$, $t \in (0,T)$ the estimate holds

$$\frac{1}{t}\int\limits_0^t \varphi(\tau)\tau d\tau \le \frac{1}{\rho}\int\limits_0^\rho \left[\varphi(\tau)\tau + \int\limits_\tau^t \varphi(\xi)d\xi \right] d\tau \le 2\int\limits_0^t \varphi(\xi)d\xi. \tag{26}$$

If $B_\varphi < \infty$ (see.(21)), then with $\rho \in (0, t]$, $t \in (0, T)$ the estimates hold

$$\frac{1}{B_\varphi} \int_0^t \varphi(\xi) d\xi \le \frac{1}{\rho} \int_0^\rho \left[\varphi(\tau)\tau + \int_\tau^t \varphi(\xi) d\xi \right] d\tau \le 2 \int_0^t \varphi(\xi) d\xi; \tag{27}$$

$$\frac{1}{t} \int_0^t \varphi(\tau)\tau d\tau \le \frac{1}{\rho} \int_0^\rho \left[\varphi(\tau)\tau + \int_\tau^t \varphi(\xi) d\xi \right] d\tau \le \frac{2B_\varphi}{t} \int_0^t \varphi(\tau)\tau d\tau. \tag{28}$$

Theorem 2. Under the conditions of Theorem 1, we denote (see (17), (18)):

$$\hat{C_{\varphi,t}} = \sup_{g \in E^{\downarrow}} \left[\frac{\tilde{h}(g;t)}{\frac{1}{t} \int_0^t h(g;\xi) d\xi} \right]; \qquad \hat{C}_\varphi = \sup_{g \in (0,t)} \hat{C_{\varphi,t}}. \tag{29}$$

Then the following estimates hold

$$\frac{1}{2} B_\varphi \le \hat{C}_\varphi \le B_\varphi + 1. \tag{30}$$

Proof. Let $h \in K_{\varphi,E}(T)$. $g \in E^{\downarrow}(0, T)$, be such that

$$h(t) = h(g;t) = \int_0^T f_\varphi(t,\tau) g(\tau) d\tau; \qquad \|g\|_E \le 2\rho_K(h). \tag{31}$$

It follows that

$$\int_0^t h(g;\xi) d\xi = \int_0^T g(\tau) \left(\int_0^t f_\varphi(\xi,\tau) d\xi \right) d\tau, \qquad t \in (0, T).$$

According to (12)

$$f_\varphi(\xi,\tau) = \varphi(\tau), \quad \xi \in (0,\tau]; \qquad f_\varphi(\xi,\tau) = \varphi(\xi), \quad \xi \in (\tau, T),$$

so that

$$\int_0^t h(g;\xi) d\xi = \int_0^T g(\tau) \left[\left(\varphi(\tau)\tau + \int_\tau^t \varphi(\xi) d\xi \right) \chi_{(0,t)}(\tau) + t\varphi(\tau)\chi_{(t,T)}(\tau) \right] d\tau.$$

It follows from (13) that

$$t\tilde{h}(g;t) = \int_0^T g(\tau) \left[\left(\int_0^t \varphi d\xi \right) \chi_{(0,t]}(\tau) + t\varphi(\tau)\chi_{(t,T)}(\tau) \right] d\tau.$$

So,

$$\hat{C}_{\varphi,t} = \sup_{g \in E^{\downarrow}} \left[\frac{\int_0^T g(\tau) \left[\left(\int_0^t \varphi d\xi \right) \chi_{(0,t)}(\tau) + t\varphi(\tau)\chi_{(t,T)}(\tau) \right] d\tau}{\int_0^T g(\tau) \left[\left(\varphi(\tau)\tau + \int_\tau^t \varphi d\xi \right) \chi_{(0,t)}(\tau) + t\varphi(\tau)\chi_{(t,T)}(\tau) \right] d\tau} \right]. \tag{32}$$

All terms in the numerator and denominator in (32) are nonnegative, and the second summands coincide. Therefore, denoting

$$\hat{D}_{\varphi,t} = \sup_{g \in E^{\downarrow}} \left[\frac{\int_0^T g(\tau) \left[\left(\int_0^t \varphi d\xi \right) \chi_{(0,t)}(\tau) \right] d\tau}{\int_0^T g(\tau) \left[\left(\varphi(\tau)\tau + \int_\tau^t \varphi d\xi \right) \chi_{(0,t]}(\tau) + t\varphi(\tau)\chi_{(t,T)}(\tau) \right] d\tau} \right], \tag{33}$$

we get
$$\hat{D}_{\varphi,t} \le \hat{C}_{\varphi,t} \le \hat{C}_{\varphi,t} + 1. \tag{34}$$

Now denote
$$\hat{E}_{\varphi,t} = \sup_{\rho\in(0,T)} \left[\frac{\int\limits_0^\rho \left[\left(\int\limits_0^t \varphi d\xi \right) \chi_{(0,t)}(\tau) \right] d\tau}{\int\limits_0^\rho \left[\left(\varphi(\tau)\tau + \int\limits_\tau^t \varphi d\xi \right) \chi_{(0,t)}(\tau) + t\varphi(\tau)\chi_{(t,T)}(\tau) \right] d\tau} \right]. \tag{35}$$

As for any RIS $E(0,T)$ we have $g(\tau) = \chi_{(0,\rho)}(\tau) \in E^{\downarrow}(0,T)$ at $\rho \in (0,T)$ it is obvious that $\hat{D}_{\varphi,t} \ge \hat{E}_{\varphi,t}$. In fact, these quantities coincide (see, for example, [5]);

$\hat{D}_{\varphi,t} = \hat{E}_{\varphi,t}$, so that
$$\hat{E}_{\varphi,t} \le \hat{C}_{\varphi,t} \le \hat{E}_{\varphi,t} + 1; \tag{36}$$

moreover, by virtue of (22)
$$\hat{E}_{\varphi,t} = \max\left\{ \hat{E}^0_{\varphi,t}, \hat{E}^1_{\varphi,t} \right\}, \tag{37}$$

where

$$\hat{E}^{(0)}_{\varphi,t} := \sup_{\rho\in(0,t)} \left[\frac{\int\limits_0^\rho \left[\left(\int\limits_0^t \varphi(\xi) d\xi \right) \chi_{(0,t]}(\tau) \right] d\tau}{\int\limits_0^\rho \left(\varphi(\tau)\tau + \int\limits_\tau^t \varphi(\xi) d\xi \right) \chi_{(0,t]}(\tau) + t\varphi(\tau)\chi_{(t,T)}(\tau) d\tau} \right] = \sup_{\rho\in(0,t]} \left[\frac{\rho \left(\int\limits_0^t \varphi(\xi) d\xi \right)}{\int\limits_0^\rho \left(\varphi(\tau)\tau + \int\limits_\tau^t \varphi(\xi) d\xi \right) d\tau} \right];$$

$$\hat{E}^{(1)}_{\varphi,t} := \sup_{\rho\in(t,T)} \left[\frac{\int\limits_0^\rho \left[\left(\int\limits_0^t \varphi(\xi) d\xi \right) \chi_{(0,t]}(\tau) \right] d\tau}{\int\limits_0^\rho \left(\varphi(\tau)\tau + \int\limits_\tau^t \varphi(\xi) d\xi \right) \chi_{(0,t]}(\tau) + t\varphi(\tau)\chi_{(t,T)}(\tau) d\tau} \right] =$$
$$= \sup_{\rho\in(t,T)} \left[\frac{t \int\limits_0^t \varphi(\xi) d\xi}{\int\limits_0^t \left(\varphi(\tau)\tau + \int\limits_\tau^t \varphi d\xi \right) d\tau + t \int\limits_t^\rho \varphi(\tau) d\tau} \right].$$

In $\hat{E}^{(1)}_{\varphi,t}$ the upper bound is achieved when $\rho \in (t,T)$ has the minimum value $\rho = t$, so that
$$\hat{E}^{(1)}_{\varphi,t} \le \hat{E}^{(0)}_{\varphi,t}.$$

Thus,
$$\hat{E}_{\varphi,t} = \hat{E}^{(0)}_{\varphi,t} = \sup_{\rho\in(0,t]} \left[\frac{\int\limits_0^t \varphi d\xi}{\frac{1}{\rho} \int\limits_0^\rho \left(\varphi(\tau)\tau + \int\limits_\tau^t \varphi d\xi \right) d\tau} \right]. \tag{38}$$

Further, taking into account that
$$\sup_{\rho\in(0,t]} \frac{\int\limits_0^t \varphi(\xi) d\xi}{\frac{1}{\rho} \int\limits_0^\rho \left(\varphi(\tau)\tau + \int\limits_\tau^t \varphi(\xi) d\xi \right) d\tau} = \frac{\int\limits_0^t \varphi(\xi) d\xi}{\inf\limits_{\rho\in(0,t]} \frac{1}{\rho} \int\limits_0^\rho \left(\varphi(\tau)\tau + \int\limits_\tau^t \varphi(\xi) d\xi \right) d\tau} \tag{39}$$

and applying Theorem 1 in the denominator, we obtain, by virtue of (23),
$$\frac{t \int\limits_0^t \varphi d\xi}{2 \int\limits_0^t \varphi(\tau)\tau d\tau} \le \hat{E}_{\varphi,t} \le \frac{t \int\limits_0^t \varphi d\xi}{\int\limits_0^t \varphi(\tau)\tau d\tau}.$$

Therefore, on the base of (36), we have

$$\frac{t\int\limits_0^t \varphi d\xi}{2\int\limits_0^t \varphi(\tau)\tau d\tau} \le \hat{C}_{\varphi,t} \le \frac{t\int\limits_0^t \varphi d\xi}{\int\limits_0^t \varphi(\tau)\tau d\tau} + 1, \qquad \rho \in (0,T),$$

from whence

$$\frac{1}{2}B_\varphi \le \hat{C}_\varphi \le B_\varphi + 1.$$

Theorem 2 is proved.

Corollary 2. Under the conditions of Theorem 2, the estimates hold

$$\tilde{h}(g;t) \le \hat{C}_{\varphi,t}\frac{1}{t}\int\limits_0^t h(g;\xi)d\xi, \tag{40}$$

$$\tilde{h}^*(g;t) \le \hat{C}_{\varphi,t}\frac{1}{t}\int\limits_0^t h(g;\xi)d\xi, \tag{41}$$

$$\left\|\tilde{h}(g;\cdot)\right\|_{L_\infty(t,T)} \le \hat{C}_{\varphi,t}\frac{1}{t}\int\limits_0^t h(g;\xi)d\xi, \tag{42}$$

for all $t \in (0,T)$, $\quad g \in E^{\downarrow}(0,T)$. Here $\hat{C}_\varphi \le B_\varphi + 1$.

Indeed, (40) follows from (29), and for $\hat{C}_\varphi$ the estimate (30) is valid. Moreover, on the right-hand side of (40) there is a positive, continuous decreasing function, as the mean integral with respect to $(0,t)$ of a decreasing function $h(g;t)$, so

$$\left[\frac{1}{\tau}\int\limits_0^t h(g;\xi)d\xi\right]^* = \frac{1}{\tau}\int\limits_0^t h(g;\xi)d\xi;$$

$$\left\|\frac{1}{t}\int\limits_0^\tau h(g;\xi)d\xi\right\|_{L_\infty(t,T)} = \frac{1}{t}\int\limits_0^\tau h(g;\xi)d\xi.$$

Therefore, (40) $\Rightarrow$ (41), (42).

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Н.А. Бокаев, М.Л. Гольдман, Г.Ж. Қаршығина

Кейбір монотонды конустармен байланысты функциялар үшін бағалаулар

Мақалада монотонды шарттарға ие, әртүрлі функция конустарының көмкерілулерін (жабылуларын) зерттеу кезінде кездесетін монотондық функциялар үшін интегралдық бағалаулар алынған. Жалпыланған Харди операторы көмегімен конустардың операторлық көмкерілуі (жабылу) әдісі қолданылды. Келтірілген бағалаулардың дұрыстығы конустарды ұсынатын ядроларға нақты шарттар келтіру арқылы көрсетілген. Дәлелдеу аралықтардың сипаттамалық функцияларының арасындағы монотонды функциялар конустарының интегралдық бағалауларын редукциялаға негізделген. Алынған нәтижелер жалпыланған Бессел және Рисс түріндегі кемімелі ауыстырымдармен байланысқан конустардың өзара көмкерілуі (жабылу) шартарын іздестіру кезінде қолданылды.

Кілт сөздер: функционалды норма, монотонды функциялар конусы, кемімелі алмастырылымды конустар, конустардың реттік бүркенуі.

Н.А. Бокаев, М.Л. Гольдман, Г.Ж. Каршыгина

Некоторые интегральные оценки на конусах функций с условиями монотонности

В статье получены интегральные оценки для монотонных функций, возникающие при изучении накрывания различных конусов функций с условиями монотонности. Использован метод операторного накрывания конусов с помощью обобщенного оператора Харди. Найдены точные условия на ядра представлений конусов, обеспечивающие справедливость приведенных оценок. Доказательства основаны на редукции интегральных оценок на конусах монотонных функций к оценкам на семействе характеристических функций интервалов. Полученные результаты могут быть применены при нахождении условий взаимного накрывания конусов, связанных с убывающими перестановками обобщенных потенциалов Бесселя и Рисса.

Ключевые слова: функциональная норма, конусы функций с условиями монотонности, порядковое накрывание конусов.

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Pregeometry on the subsets of Jonsson theory's semantic model

One of the interesting achievements among investigations from modern model theory is the implement of local properties of the geometry of strongly minimal sets. In E. Hrushovski have proved remarkable results under this ideas and this one had impacted an essential infuence for development of methods and ideas of research for global properties of structures.These new model-theoretical features and approvals play an important role in E. Hrushovski's proof of the Mordell–Lang Conjecture for function fields.

In this article, we are trying to redefine the basic concepts of the above mentioned ideas on the formul subsets of some extentional-closed model for some fixed Jonsson theory. With the help of new concepts in the frame of Jonssoness features, pregeometry is given on all subsets of Jonsson theory's semantic model. Minimal structures and, correspondingly, pregeometry and geometry of minimal structures are determined. We consider the concepts of dimension, independence, and basis in the Jonsson strongly minimal structures for Jonsson theories.

Keywords: Jonsson theory, Jonsson set, semantic model, Jonsson minimal structure, Jonsson pregeometry, Jonsson dimension, Jonsson basis, Jonsson independence, modularity.

In the paper [1, 2] strongly minimal Jonsson sets were studied. A natural generalization will be a consideration of Jonsson analogs of strongly minimal arbitrary subsets of semantic model of some fixed Jonsson theory.

In order that we could transfer from the apparatus of Model theory of developed for complete theories the basic concepts connected with the concept of strongly minimal for fixed formula subsets of semantic model of the above Jonsson theory, we need the semantic model to be saturated in its power, that is, the theory must be perfect. We note that similar approaches to the work with the transfer of basic model-theoretic concepts for some fixed Jonsson theory and its semantic model were considered in the following works [3–5].

We recall the necessary definitions. We will assume everywhere that the language is always countable.

Definition 1 [6]. A theory T is called Jonsson if:

1) The theory T has an infinite models;

2) The theory T is inductive;

3) The theory T has the joint embedding property (JEP);

4) The theory T has the amalgamation property (AP).

Further, throughout this article, all considered theories will be an existential complete perfect Jonsson theory in a countable language $\mathcal{L}$.

In this paper, we redefine the basic concepts connected with strongly minimal for complete theories in the framework of the study of some Jonsson theory. All main definitions belong to A. Yeshkeyev and they are taken from the above sources. All results that are given without proof also belong to A. Yeshkeyev and they can be extracted from [2].

Let $\mathcal{M}$ be some an existentially closed submodel of semantic model for fixed Jonsson theory in the language $\mathcal{L}$, $\phi(\overline{v})$ is an $\mathcal{L}_M$ –formula, we will let $\phi(\mathcal{M})$ denote the elements of M that satisfy ϕ.

Definition 2. Let $D \subseteq M^n$ be an infinite Δ-definable set, where $\Delta \subseteq \mathcal{L}$. As a rule, throughout this article under Δ we shall consider the set of all existential formulas of the given language. We say that a set D is Jonsson minimal in $\mathcal{M}$ if for any Δ-definable $Y \subseteq D$ either Y is finite or $D \backslash Y$ is finite.

If $\phi(\overline{v}, \overline{a})$ is the formula that defines D, then we also say that $\phi(\overline{v}, \overline{a})$ is Jonsson minimal.

We say that D and ϕ are Jonsson strongly minimal if ϕ is Jonsson minimal in any existentially closed extension $\mathcal{N}$ of $\mathcal{M}$.

We say that a theory T is Jonsson strongly minimal if the formula $v = v$ is Jonsson strongly minimal (i.e., if $\mathcal{M} \in ModE_T$ then M is Jonsson strongly minimal).

Let $\mathcal{M}$ be some an existentially closed submodel of semantic model for fixed Jonsson theory in the language $\mathcal{L}$ and $D \subseteq M$ be Jonsson strongly minimal.

We will consider acl_D, the algebraic closure relation restricted to D. Recall that b is Jonsson algebraic over A if there is a formula $\phi(x,\overline{a}) \in \Delta$ with $\overline{a} \in A$ such that $\phi(\mathcal{M},\overline{a})$ is finite.
For $A \subseteq D$, we let $\mathrm{acl}_D(A) = \{b \in D : b$ is Jonsson algebraic over $A\}$.

The following properties of Jonsson algebraic closure are true for any subset D of semantic model for theory T.

Lemma 1.

1) $\mathrm{acl}(\mathrm{acl}(A)) = \mathrm{acl}(A) \supseteq A$.

2) If $A \subseteq B$, then acl(A)⊆acl(B).

3) If $a \in$ acl(A), then $a \in$ acl(A_0) for some finite $A_0 \subseteq A$.

Lemma 2 (Exchange Principle) Suppose that D ⊂ M is Jonsson strongly minimal, A ⊆ D, and a, b ∈ D. If a ∈ $\mathrm{acl}(A\ U\{b\}) \setminus \mathrm{acl}(A)$, then b ∈ $\mathrm{acl}(A \cup \{a\})$.

In any Jonsson strongly minimal set, we can define a notion of independence. We fix $\mathcal{M} \in ModE_T$ and D is Jonsson strongly minimal set in $\mathcal{M}$.

Definition 3. We say that A ⊆D is Jonsson independent if a ∈ $\mathrm{acl}(A \setminus\{a\})$ for all a ∈A. If $C \subset D$, we say that A is Jonsson independent over C if a ∉ $\mathrm{acl}(C\ \mathrm{U}\ (A \setminus \{a\}))$ for all a ∈ A.

It is turn out that cardinality is the only one opportunity to distinguish independent subsets of D.

Definition 4. We say that A is a Jonsson basis for $Y \subseteq D$ if $A \subseteq Y$ is Jonsson independent and $acl(A) = acl(Y)$.

Clearly, any maximal Jonsson independent subset of Y is a Jonsson basis for Y.

Lemma 3. Let $A,\ B \subseteq D$ be Jonsson independent with $A \subseteq acl\ (B)$.

1) Suppose that $A_0 \subseteq A$, $B_0 \subseteq B$, $A_0 \cup B_0$ is Jonsson basis for *acl(B)* and $a \in A\backslash A_0$. Then, there is $b \in B_0$ such that $A_0 \cup \{a\} \cup (B_0 \backslash \{b\})$ is Jonsson basis for *acl(B)*.

2) $|A| \leq |B|$.

3) If A and B are Jonsson bases for $Y \subseteq D$, then $|A| = |B|$.

Definition 5. If $Y \subseteq D$, then the Jonsson dimension of Y is the cardinality of a Jonsson basis for Y.

We denote the Jonsson dimension of Y through Jdim(Y).

Note that if D is uncountable, then Jdim(D) $= |D|$ because our language is countable and *acl(A)* is countable for any countable $A \subseteq D$.

For Jonsson strongly minimal theories, every model is determined up to isomorphism by its Jonsson dimension.

Theorem 1. Suppose T is a Jonsson strongly minimal theory.

$\mathcal{M}, \mathcal{N} \in ModE_T$ then $\mathcal{M} \cong \mathcal{N}$ if and only if $Jdim\,(M) = Jdim(N)$.

Corollary 1. If T is a Jonsson strongly minimal theory, then T^* is κ -categorical for $\kappa \geq \aleph_0$ and $I(T,\aleph_0) \leq \aleph_0$, where T^* is the center of T and $I(T,\aleph_0)$ denote the number of existentially closed countable models of T.

Analogously to the Baldwin-Lachlan result, we obtain the following theorem.

Theorem 2. If T^* is perfect existentially complete Jonsson theory, then $I(T,\aleph_0) = \aleph_0$.

In this article, we will use Jonsson strongly minimal sets and consider some properties of combinatorial geometry of algebraic closure.

It is well known that in the proof of Morley's theorem on uncountable categoricity, the properties of an algebraic closure on strongly minimal sets are used in an essential way. In this article, we turn to Jonsson strongly minimal sets and with their help we study the combinatorial geometry of algebraic closure.

We give the following definitions.

Definition 6. Let X be subset of semantic model of fixed Jonsson theory and let cl :$\mathcal{P}(X) \longrightarrow \mathcal{P}(X)$ be an operator on the power set of X. We say that (X, cl) is a Jonsson pregeometry if the following conditions are satisfied.

1. If $A \subseteq X$, then $A \subseteq cl(A)$ and $cl(cl(A)) = cl(A)$.
2. If $A \subseteq B \subseteq X$, then $cl(A) \subseteq cl(B)$.
3. (Exchange) $A \subseteq X$, $a, b \in X$, and $a \in cl(A \cup \{b\})$, then $a \in cl(A) b \in cl(A \cup \{a\}$.
4. (Finite character) If $A \subseteq X$ and $a \in cl(A)$, then there is a finite $A_0 \subseteq A$ such that $a \in cl(A_0)$.

We say that $A \subseteq X$ is closed if $cl(A) = A$.

We may notice (By Lemmas 1 and 2) that if D is Jonsson strongly minimal, we can specify a Jonsson pregeometry by defining $cl(A) = acl(A) \cap D$ for $A \subseteq D$. We can generalize basic ideas about independence and dimension from Jonsson strongly minimal sets to arbitrary Jonsson pregeometries.

Definition 7. If (X, cl) is a Jonsson pregeometry, we say that A is Jonsson independent if $a \notin cl(A \backslash \{a\})$ for all $a \in A$ and that B is a J -basis for Y if $B \subseteq Y$ is J-independent and $Y \subseteq acl(B)$. The natural generalization of Lemma 3 is true for all J-pregeometries.

Lemma 4. If (X, cl) is a J -pregeometry, $Y \subseteq X, B_1, B_2 \subseteq Y$, and each B_i is a J -basis for Y, then $|B_1| = |B_2|$.

We call $|B_i|$ the J -dimension of Y and write $J\ \dim(Y) = |B_i|$.

If $A \subseteq X$, we also consider the localization $cl_A(B) = cl(A \cup B)$.

Lemma 5. If (X, cl) is a J -pregeometry, then (X, cl_A) is a J -pregeometry.

If (X, cl) is a J -pregeometry, we say that $Y \subseteq X$ is J -independent over A if Y is J -independent in (X, cl_A). We let Jdim(Y/A) be the J -dimension of Y in the localization (X, cl_A). We call $J\ \dim(Y/A)$ the J -dimension of Y over A.

Definition 8. We say that a J-pregeometry (X, cl) is a J-geometry if cl$(\oslash) = \oslash$ and $cl(\{x\}) = \{x\}$ for any $x \in X$.

If (X, cl) is a J-pregeometry, then we can naturally define a J-geometry. Let $X_0 = X/cl(\oslash)$. Consider the relation $\sim$ on X_0 given by $a \sim b$ if and only if $cl(\{a\}) = cl(\{b\})$. By exchange, $\sim$ is an equivalence relation. Let $\hat{X}$ be $X_0/\sim$. Define $\overset{\wedge}{cl}$ on $\hat{X}$ by $\overset{\wedge}{cl}(A/\sim) = \{b/\sim : b \in cl(A)\}$.

Lemma 6. If (X, cl) is a J-pregeometry, then $(\hat{X}, \overset{\wedge}{cl})$ is a J-geometry.

We distinguish some properties of J-pregeometries that will play an important role.

Definition 9. Let (X, cl) be J-pregeometry. We say that (X, cl) is trivial if $cl(A) = \cup_{a \in A} cl(\{a\})$ for any $A \subseteq X$. We say that (X, cl) is modular if for any finite-dimensional closed $A, B \subseteq X$

$$J\dim(A \cup B) = J\dim(A) + J\dim(B) - J\dim(A \cap B).$$

We say that (X, cl) is locally modular if (X, cl_a) is modular for some $a \in X$.

Theorem 3. Let (X, cl) be J- pregeometry. The following are equivalent.

1. (X, cl) is modular.

2. If $A \subseteq X$ is closed and nonempty, $b \in X$, and $x \in cl(A, b)$, then there is $a \in A$ such that $x \in cl(a, b)$.

3. If $A, B \subseteq X$ are closed and nonempty, and $x \in cl(A, B)$, then there are $a \in A$ and $b \in B$ such that $x \in cl(a, b)$.

Proof. Let us prove 1) $\Rightarrow$ 2). By the definition of the closure, we conclude that $J\dim A$ is finite. There are two cases $x \in cl(b)$ and $x \notin cl(b)$. The case $x \in cl(b)$ is proved trivially. It is sufficient for us to consider the case when $x \notin cl(b)$. As consequence, from definition of modularity $J\dim(A, b, x) = J\dim A +$ $+ J\dim(b, x) - J\dim(A \cap cl(b, x))$ and $J\dim(A, b, x) = J\dim(A, b) = J\dim A + J\dim b - J\dim(A \cap cl(b))$. Since $J\dim(b, x) = J\dim(b) + 1$, then there exists $a \in A$ such that $a \in cl(b, x) \backslash cl(b)$. Consequently, by the Exchange Principle, we obtain $x \in cl(b, a)$.

Let us prove 2) $\Rightarrow$ 3). Without loss of generality, we consider case, $J\dim A < \omega$ and $J\dim B < \omega$. Further we will prove by induction on $J\dim A$. If $J\dim A$ is zero, then the condition 3) is satisfied. Let $A = cl(A_0, a)$ where $J\dim A = J\dim A - 1$. Hence $x \in cl(A_0, B, a)$. From 2) it follows that, there is $c \in cl(A_0, B)$ such that $x \in cl(c, a)$. By induction, there is $a_0 \in A_0$ and $b \in B$ such that $c \in cl(c_0, b)$. Continuing further, by condition 2) there is $a^* \in cl(a_0, a) \subseteq A$ such that $x \in cl(b, a)$.

Let us prove 3) $\Rightarrow$ 1). We may assume that $A, B \subseteq X$ are finite-dimensional and closed. We will show condition 1) by induction on $J\dim A$. If $J\dim A = 0$, then we are done. Assume that $A = cl(A_0, a))$, where $J\dim A_0 = J\dim A - 1$ and we suppose, by induction, that

$$J\dim(A_0, B) = J\dim A_0 + J\dim B - J\dim(A_0 \cap B).$$

First of all, suppose that $a \in cl(A_0, B)$. Hence $J\dim(A_0, B) = J\dim(A, B)$ and, since $a \notin A_0$, $J\dim A = J\dim A_0 + 1$. Since $a \in cl(A_0, B)$, then by 3) there is $a_0 \in A_0$ and $b \in B$ such that $a \in cl(a_0, b)$. Because $a \notin cl(a_0)$, by the Exchange Principle, $b \in cl(a, a_0)$. So $b \in A$. But $b \notin A_0$, because otherwise $a \in A_0$. Therefore, $J\dim(A \cap B) = J\dim(A_0, B) + 1$, as we would like.

Further, assume that $a \notin cl(A_0, B)$. We must show that $A \cap B = A_0 \cap B$. Suppose that $b \in B$ and $b \in cl(A_0, a) \backslash cl(A_0)$. Then, by the Exchange Principle, $a \in cl(A_0, b)$, we have obtained a contradiction.

All undefined concepts related to Jonsson theory can be found in [7].

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М.Т. Касыметова

Йонсондық теорияның семантикалық моделінің ішкі жиындарындағы прегеометрия

Қазіргі модельдер теориясының нәтижелері арасындағы ең қызықты жетістіктерінің біріне қатты минималды жиындардың геометриясының локалды қасиеттерін жүзеге асыру жатады. Осы идеяларға қатысты жұмысында Э. Хрущовскийдың тамаша нәтижелерге ие болды. Олар құрылымдардың ғаламды қасиеттерін зерттеу үшін әдістердің және идеялардың дамуына елеулі ықпал етеді. Бұл теория-модельдік ерекшеліктері мен мәлімдемелері функциялар өрістер үшін Хрущовскийдың Морделл-Ланг гипотезасын дәлелдеуінде маңызды рөл атқарады. Мақалада автор жоғарыда көрсетілген идеялардың негізгі ұғымдарын кейбір бекітілген йонсондық теория үшін экзистенционалдытұйық модельдерінің формулалық ішкі жиындары арқылы анықтауға тырысты. Сонымен қатар жаңа ұғымдар көмегімен ерекшеліктердің йонсондылылығының аясында йонсондық теорияның семантикалық моделінің барлық ішкі жиындарында прегеометрия ұғымы берілді. Минималды құрылымдар және сәйкесінше прегеометрия және минималды структуралардың геометриясы анықталды. Йонсондық теориялар үшін йонсондық қатты минималды құрылымдарда өлшемділік, тәуелсіздік және базис ұғымдары қарастырылды.

Кілт сөздер: йонсондық теория, йонсондық жиын, семантикалық модель, йонсондық минималды құрылым, йонсондық прегеометрия, йонсондық өлшем, йонсондық базис, йонсондық тәуелсіздік, модулярлық.

М.Т. Касыметова

Предгеометрия на подмножествах семантической модели йонсоновской теории

Одним из интересных достижений среди исследований современной теории моделей является реализация локальных свойств геометрии сильно минимальных множеств. В работе Э. Хрущовского показаны замечательные результаты по этим идеям, и это оказало существенное влияние на разработку методов и идей исследования глобальных свойств структур. Эти новые теоретико-модельные особенности и утверждения играют важную роль в доказательстве Э. Хрущовского гипотезы Морделла-Ланга для функций полей. В этой статье мы пытаемся переопределить основные понятия упомянутых выше идей на формульных подмножествах некоторой экзистенциально-замкнутой модели для некоторой фиксированной йонсоновской теории. С помощью новых понятий в рамках особенностей йонсоновости предгеометрия дается на всех подмножествах семантической модели йонсоновской теории. Определены минимальные структуры и, соответственно, предгеометрия, и геометрия минимальных структур. Рассмотрены понятия размерности, независимости и базиса в йонсоновских сильно минимальных структурах для йонсоновских теорий.

Ключевые слова: йонсоновская теория, йонсоновское множество, семантическая модель, йонсоновская минимальная структура, йонсоновская предгеометрия, йонсоновская размерность, йонсоновский базис, йонсоновская независимость, модулярность.

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ε-Approximation of the temperatures model of inhomogeneous melts with allowance for energy dissipation

Accumulated facts and information about the Navier-Stokes equations, together with a large number of experiments and approximate calculations, made it possible to reveal some discrepancies between the mathematical model of a viscous melt and real phenomena in the nature of real molten systems. There are many reasons for this. One of them is the nonlinearity of the Navier-Stokes equations. And for non-linear equations it is known that in non-stationary problems, a solution satisfying it can exist not on the entire interval $t \geq 0$. Over a finite period of time, it can either go to infinity, or crumble.A solution lose regularity and no satisfying the equations and begin branching. It is mathematically proved that if this solution exists for $t \geq 0$, then it may not seek to solve the stationary problem when stabilizing the boundary conditions and external influences. The solutions of the nonstationary problem obtained even with a smooth initial regime and smooth external influences can become less regular with time, and then generally go into irregular or turbulent regimes. The actual implementation of this or that branch of the solution depends on extraneous reasons not taken into account in the Navier-Stokes equations. In the proposed paper, we constructed a numerical scheme with good convergence. The regularization of the initial systems of differential equations by ε-approximation is constructed. The Galerkin method is implemented ensuring the correctness of boundary value problems for an incompressible viscous flow both numerically and analytically. A splitting scheme for the Navier-Stokes equations with a weak approximation is constructed. An approximation is constructed for stationary and nonstationary models of an incompressible melt, which leads to nonlinear equations of hydrodynamics to a system of equations of Cauchy-Kovalevskaya type.

Keywords: energy dissipation, system approximation, Cauchy inequality, Galerkin method, a priori estimates.

It is known that the Navier-Stokes equations are analytically insoluble, and if they are solvable, then in relatively simple cases.

To solve the Navier-Stokes equations, a nonconformal finite element method was proposed in [1]. It is based on the modification of the Navier-Stokes equations with the introduction of weight functions. The solution of the Navier-Stokes equations with respect to natural variables, from which pressure is excluded, is considered in [2]. This approach is already known. Thus, the Navier-Stokes equations system is integrated numerically using a finite-difference method using splines. Other methods are also being sought. An attempt is made to derive the Navier-Stokes equation itself using the variational method, as presented in [3].

We investigate the initial-boundary value problem for nonstationary Navier-Stokes equations in this section. Let us consider the temperature model of an inhomogeneous melt [4] in the region $\Omega \subset R^2$.

$$\rho\left(\frac{\partial \upsilon}{\partial t} + (\upsilon \cdot \nabla)\,\upsilon\right) = \mu\Delta\upsilon - \nabla p + e\theta\rho + \rho f; \tag{1}$$

$$\frac{\partial \rho}{\partial t} + (\upsilon \cdot \nabla)\,\rho = 0; \tag{2}$$

$$div\upsilon = 0; \tag{3}$$

$$\rho\left(\frac{\partial \theta}{\partial t} + (\upsilon \cdot \nabla)\,\theta\right) = div\left(\lambda\left(\theta\right)\nabla\theta\right) + \mu\sigma; \tag{4}$$

$$\sigma = \sum_{i,j=1}^{2}\left(\frac{\partial \upsilon_i}{\partial x_j} + \frac{\partial \upsilon_j}{\partial x_i}\right)^2; \tag{5}$$

with initial-boundary conditions

$$\upsilon\,|_{t=0}\, = \upsilon_0\,(x)\,, \rho\,|_{t=0}\, = \rho_0\,(x)\,, \theta\,|_{t=0}\, = \theta_0\,(x)\,;$$

$$\upsilon\,|_S\, = 0, \frac{\partial \theta}{\partial n}\,|_S\, = 0, t \in [0, T]\,, \tag{6}$$

where $\sigma-$ energy dissipation; $\upsilon\,(x,t)-$ velocity vector; $\theta\,(x,t)-$ temperature field; $\rho\,(x,t)-$ density field; $p\,(x,t)-$ pressure field; $f\,(x,t)-$ mass force vecto; $\mu-$ melt viscosity; $\lambda\,(\theta)-$ thermal conductivity coefficient; n is the outer normal to the boundary S, $e = \{0, 1\}$.

Solvability of the problem (1)–(6) was studied in Sh.S. Smagulova and A.B. Kazhikhova [5, 6].

The system of equations (1)–(5) is non-evolutionary, therefore direct application of the method of fractional steps is difficult. In connection with this we study the approximation of the system (1)–(5) by a system of evolutionary type and study the existence theorem for the solution of the auxiliary problem. Let the motion of the melt occur in a bounded region $\Omega \subset R^2$with a sufficiently smooth boundary S. For simplicity we assume that the boundary S is impermeable and there is no mass transfer between the melt and the external medium.

We consider a system of equations with a small parameter approximating the system of equations (1)–(5):

$$\rho^\varepsilon\left(\frac{\partial \upsilon^\varepsilon}{\partial t} + (\upsilon^\varepsilon\cdot\nabla)\,\upsilon^\varepsilon\right) = \mu\Delta\upsilon^\varepsilon - \nabla p^\varepsilon + e\theta^\varepsilon\rho^\varepsilon + \rho^\varepsilon f - \rho^\varepsilon\frac{\upsilon^\varepsilon}{2}div\upsilon^\varepsilon; \tag{7}$$

$$\frac{\partial \rho^\varepsilon}{\partial t} + (\upsilon^\varepsilon\cdot\nabla)\,\rho^\varepsilon = 0; \tag{8}$$

$$\varepsilon p^\varepsilon + div\upsilon^\varepsilon = 0; \tag{9}$$

$$\rho^\varepsilon\left(\frac{\partial \theta^\varepsilon}{\partial t} + (\upsilon^\varepsilon\cdot\nabla)\,\theta^\varepsilon\right) = div\left(\lambda\left(\theta^\varepsilon\right)\nabla^\varepsilon\theta^\varepsilon\right) + \mu\sigma^\varepsilon; \tag{10}$$

$$\sigma^\varepsilon = \sum_{i,j=1}^{2}\left(\frac{\partial \upsilon_i^\varepsilon}{\partial x_j} + \frac{\partial \upsilon_j^\varepsilon}{\partial x_i}\right)^2; \tag{11}$$

with initial-boundary conditions:

$$\upsilon^\varepsilon\,|_{t=0}\, = \upsilon_0\,(x)\,, \rho^\varepsilon\,|_{t=0} = \rho_0\,(x)\,,\ \theta^\varepsilon\,|_{t=0}\, = \theta_0\,(x)\,;$$

$$\upsilon^\varepsilon\,|_S\, = 0, \frac{\partial \theta^\varepsilon}{\partial n}\,|_S\, = 0, t \in [0, T]\,, \tag{12}$$

Before proceeding to the proof of the theorem, we formulate an important definition.

*Definition.*A function $(\upsilon, p, \rho, \theta)$,that is summable together with the derivatives entering into the system of equations (1)–(6) satisfying (1)–(5) almost everywhere in the corresponding measure is called a strong solution of the problem (1)–(6).

The definition of the strong solution of the problem (7)–(12) is defined similarly.

Theorem 1. Let $f \in L_p\,(Q)\,, \Omega \subset E^2$,$\upsilon_0\,(x) \in W_p^1\,(\Omega)\,, 0 < m \leq \rho_0\,(x) \leq M < \infty$,$\lambda\,(\theta)-$ is continuously differentiable with respect to

θ,$\rho_0\,(x) \in W_p^1\,(\Omega)\,, p > 2, \lambda\,(\theta) \sim \theta^2$,at$\theta \to \infty$,$\theta_0\,(x) \in L_\infty\,(\Omega)\,, \theta_0 \in L_p\,(\Omega)\,, \varepsilon > 0, S \in C^2, \mu > 0$.

Then a unique strong solution of problem (7)–(12) exists and we have the estimate for the solution, where C — the constant does not depend on

$$\left\|\frac{\partial \upsilon^\varepsilon}{\partial t}\right\|_{L_p(0,T,L_p(\Omega))} + \|\upsilon^\varepsilon\|_{L_p\left(0,T,W_p^2(\Omega)\right)} + \frac{1}{\varepsilon}\,\|div\upsilon^\varepsilon\|_{L_p(0,T,L_p(\Omega))} + \|\rho^\varepsilon\|_{W_p^{1,1}(Q)} + \|\theta^\varepsilon\|_{W_p^{2,1}(Q)} \leq C < \infty.$$

The proof of the theorem is constructed from three stages: obtaining a priori estimates, applying Galerkin's method for constructing approximate solutions and limiting the transition.

A priori estimates.

We have by the maximum principle:

$$0 < m \leq \rho_0^\varepsilon\,(x) \leq M < \infty.$$

We multiply equation (7) scalarly in $\upsilon^{\varepsilon}(x,t)$ space and integrate by parts. Applying the Cauchy inequality:

$$\left|\int_{\Omega}\rho^{\varepsilon}\left(f,\upsilon^{\varepsilon}\right)dx\right|\leq\left(\int_{\Omega}\rho^{\varepsilon}\left|\upsilon^{\varepsilon}\right|^{2}dx\right)^{\frac{1}{2}}\left(\int_{\Omega}\rho^{\varepsilon}\left|f\right|^{2}dx\right)^{\frac{1}{2}}.$$

We have the estimate on the basis of imbedding theorems:

$$\left\|\upsilon_{x}^{\varepsilon}\right\|_{L_{p}(0,T,L_{p}(\Omega))}+\frac{1}{\varepsilon}\left\|div\upsilon^{\varepsilon}\right\|_{L_{p}(0,T,L_{p}(\Omega))}\leq C<\infty. \tag{13}$$

We obtain further multiplying (10) by θ_{t}^{ε}, integrating by parts Ω:

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega}\lambda(\theta^{\varepsilon}){\theta_{x}^{\varepsilon}}^{2}dx+\int_{\Omega}\rho^{\varepsilon}{\theta_{t}^{\varepsilon}}^{2}dx=$$

$$=\int_{\Omega}\frac{1}{2}\left\{\lambda'\left(\theta^{\varepsilon}\right){\theta_{x}^{\varepsilon}}^{2}\cdot\theta_{t}^{\varepsilon}\right\}dx+\int_{\Omega}\mu\sigma^{\varepsilon}\theta_{t}^{\varepsilon}dx-\int_{\Omega}\rho^{\varepsilon}\left(\upsilon^{\varepsilon}\cdot\nabla\right)\theta^{\varepsilon}\cdot\theta_{t}^{\varepsilon}dx.$$

We estimate the integrals on the right-hand side and integrate with respect to the variable t:

$$\max_{0\leq t\leq T}\left\|\theta_{x}^{\varepsilon}\right\|_{2,\Omega}^{2}+\left\|\theta_{t}^{\varepsilon}\right\|_{2,Q}^{2}\leq C.$$

We will write down the energy equation:

$$\rho^{\varepsilon}\theta_{t}^{\varepsilon}-\left(\lambda\left(\theta^{\varepsilon}\right)\Delta\theta^{\varepsilon}\right)=\mu\sigma^{\varepsilon}-\rho^{\varepsilon}\left(\upsilon^{\varepsilon}\cdot\nabla\right)\theta^{\varepsilon}+\lambda'\left(\theta^{\varepsilon}\right)\cdot{\theta_{x}^{\varepsilon}}^{2}$$

and multiply it by $\frac{1}{\rho}\Delta\theta^{\varepsilon}$. We have After integration Ω:

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega}{\theta_{x}^{\varepsilon}}^{2}dx+\int_{\Omega}\lambda\left(\theta^{\varepsilon}\right)\frac{1}{\rho}\left(\Delta\theta^{\varepsilon}\right)^{2}dx=\int_{\Omega}\left\{\rho\left(\upsilon^{\varepsilon}\cdot\nabla\right)\theta^{\varepsilon}-\mu\sigma-\lambda'\left(\theta^{\varepsilon}\right){\theta_{x}^{\varepsilon}}^{2}\right\}\times\frac{1}{\rho}\Delta\theta^{\varepsilon}dx.$$

We find after integrating over the variable t, estimating the integrals on the right-hand side:

$$\max_{0\leq t\leq T}\left\|\theta_{x}^{\varepsilon}\right\|_{2,\Omega}^{2}+\left\|\Delta\theta_{t}^{\varepsilon}\right\|_{2,Q}^{2}\leq C.$$

We conclude as a result:

$$\left\|\theta^{\varepsilon}\right\|_{W_{p}^{2,1}(Q)}\leq C<\infty. \tag{14}$$

We obtain multiplying equation (8) by u, integrating by parts Ω:

$$\frac{1}{2}\frac{d}{dt}\left\|\nabla\rho^{\varepsilon}\right\|_{2,\Omega}^{2}+\int_{\Omega}\left(\upsilon^{\varepsilon}\cdot\nabla\right)\rho^{\varepsilon}\Delta\rho^{\varepsilon}dx=0.$$

With allowance for the estimate (13) by virtue of the maximum principle it follows that:

$$\left\|\rho^{\varepsilon}\right\|_{W_{p}^{1,1}(Q)}\leq C<\infty. \tag{15}$$

We have estimating p^{ε} from the negative norm, as in [7]:

$$\left\|p^{\varepsilon}\right\|\leq C\left\|\nabla p^{\varepsilon}\right\|<\infty. \tag{16}$$

It is known that if $\upsilon^{\varepsilon},p^{\varepsilon}-$solution of the following linear Stokes problem:

$$\mu\Delta\upsilon^{\varepsilon}-\nabla p^{\varepsilon}=f; \tag{17}$$

$$\varepsilon p^{\varepsilon}+div\upsilon^{\varepsilon}=0;$$

$$\upsilon^{\varepsilon}\left|_{S=0}\right.=0,\frac{\partial\rho^{\varepsilon}}{\partial n}\left|_{S}\right.=0,$$

then provided that $f\in L_{p}\left(\Omega\right)$the following inequality holds:

$$\left\|\upsilon^{\varepsilon}\right\|_{W_{p}^{2}\bigcap\overset{0}{\underset{P^{1}}{W}}}+\left\|p^{\varepsilon}\right\|_{W_{p^{1}}}\leq C\left\|f\right\|_{L_{p}}. \tag{18}$$

We take the function as a function f in problem (17):

$$f=-\rho^{\varepsilon}\left(\frac{\partial \upsilon^{\varepsilon}}{\partial t}+\left(\upsilon^{\varepsilon}\cdot\nabla\right)\upsilon^{\varepsilon}\right)-\ell\theta^{\varepsilon}\rho^{\varepsilon}-\rho^{\varepsilon}f-\rho^{\varepsilon}\frac{\upsilon^{\varepsilon}}{2}div\upsilon^{\varepsilon}.$$

We estimate the right side by the Cauchy inequality using the maximum principle:

$$\|f\|^2_{L_p(\Omega)}\leq CM\left(\|\upsilon^{\varepsilon}_t\|^2+\int_{\Omega}\left(|\upsilon^{\varepsilon}|^2\left|\nabla\upsilon^{\varepsilon}\right|^2+|f|^2+\ell\left|\theta^{\varepsilon}\right|^2\right)dx\right). \tag{19}$$

We have the following estimate taking into account the inequalities of the embedding and the obtained estimates (13)–(14):

$$\int_{\Omega}|\upsilon^{\varepsilon}|^2\left|\nabla\upsilon^{\varepsilon}\right|^2dx\leq\max_{\Omega}|\upsilon^{\varepsilon}|^2\int_{\Omega}\left|\nabla\upsilon^{\varepsilon}\right|^2dx\leq\|\upsilon^{\varepsilon}\|_{L_p(\Omega)}\|\upsilon^{\varepsilon}\|_{W_p^2(\Omega)\bigcap\overset{0}{W}_{p^1}(\Omega)}\|\nabla\upsilon^{\varepsilon}\|^2\leq$$

$$\leq le\delta\|\upsilon^{\varepsilon}\|^2_{W_p^2(\Omega)\bigcap\overset{0}{W}_{p^1}(\Omega)}+C_{\delta}\|\upsilon^{\varepsilon}\|^2_{\overset{0}{W}_{p^1}(\Omega)}. \tag{20}$$

We multiply equation (7) scalarly by $\upsilon^{\varepsilon}(t)$in space $L_2(\Omega)$.Then we estimate the integrals in absolute value from above and, applying the inequalities of imbedding theorems, we obtain the following estimate:

$$\begin{aligned}\int_{\Omega}|\nabla\upsilon^{\varepsilon}||\upsilon^{\varepsilon}||\upsilon^{\varepsilon}_t|dx&\leq C\|\nabla\upsilon^{\varepsilon}\|_{L_p(\Omega)}\max_{\Omega}|\upsilon^{\varepsilon}|\|\upsilon^{\varepsilon}_t\|_{L_p(\Omega)}\leq\\ &\leq C\|\upsilon^{\varepsilon}\|^{\frac12}_{L_p(\Omega)}\|\upsilon^{\varepsilon}\|^{\frac12}_{W_p^2(\Omega)\bigcap\overset{0}{W}_{p^1}(\Omega)}\|\upsilon^{\varepsilon}\|_{\overset{0}{W}_{p^1}(\Omega)}\|\upsilon^{\varepsilon}_t\|_{L_p(\Omega)}\leq\\ &\leq\delta\|\upsilon^{\varepsilon}_t\|^2_{L_p(\Omega)}+\delta\|\upsilon^{\varepsilon}\|^2_{W_{p^2}(\Omega)\bigcap\overset{0}{W}_{p^1}(\Omega)}+C_{\delta}\|\upsilon^{\varepsilon}\|^2_{\overset{0}{W}_{p^1}(\Omega)}.\end{aligned} \tag{21}$$

We finally get an estimate following the evaluation methodology in the works [8, 9]:

$$\|\upsilon^{\varepsilon}_t\|_{L_p(0,T,L_p(\Omega))}+\|\upsilon^{\varepsilon}\|_{L_p\left(0,T,W_p^2(\Omega)\right)}+\|\nabla p^{\varepsilon}\|_{L_p(0,T,L_p(\Omega))}\leq C<\infty, \tag{22}$$

where C does not depend on the value of the small ε.

We establish one more estimate of the global time character, the constant in which depends only on the data of the problem. This estimate in the future guarantees compactness in space $L_2(Q)$sequences of approximate solutions, which are built according to the Galerkin method.

Lemma. For any δ, such that the following condition holds $0<\delta<T$, the inequality holds:

$$\int_0^{T-\delta}\|\upsilon^{\varepsilon}(t+\delta)-\upsilon^{\varepsilon}(t)\|^2dt\leq C\delta^{\frac12}.$$

Evidence. We fix δ,t in such a way that inequality $0\leq t\leq T-\delta$. Consider equations (7)–(11) on the time interval $\tau\in(t,t+\delta)$.We multiply equation (7) scalarly by Φan arbitrary function in space$L_2(\Omega)$.

We arrive at the inequality after simple transformations:

$$\frac{d}{d\tau}\left(\rho^{\varepsilon}\upsilon^{\varepsilon},\Phi\right)_{L_2(\Omega)}=\left(\rho^{\varepsilon}\left(\upsilon^{\varepsilon}\cdot\nabla\right)\Phi,\upsilon^{\varepsilon}\right)_{L_2(\Omega)}+\frac12\left(\rho^{\varepsilon}div\upsilon^{\varepsilon}\cdot\upsilon^{\varepsilon},\Phi\right)_{L_2(\Omega)}+$$

$$+\left(\rho^{\varepsilon}f,\Phi\right)_{L_2(\Omega)}-\ell\left(\theta^{\varepsilon}\rho^{\varepsilon},\Phi\right)_{L_2(\Omega)}-\mu\left(\upsilon^{\varepsilon},\Phi\right)_{L_2(\Omega)}+\left(p^{\varepsilon},div\Phi\right)_{L_2(\Omega)},$$

where $\Phi=\upsilon(t+\delta)-\upsilon(t)$.

We integrate this identity with respect to the variable τ ranging from t before $t+\delta$, and then put $\Phi=\upsilon^{\varepsilon}(t+\delta)-\upsilon^{\varepsilon}(t)$.

We write expression $\rho^{\varepsilon}(t+\delta)\upsilon^{\varepsilon}(t+\delta)-\rho^{\varepsilon}(t)\upsilon^{\varepsilon}(t)$ in the following form $\rho^{\varepsilon}(t+\delta)\left(\upsilon^{\varepsilon}(t+\delta)-\upsilon^{\varepsilon}(t)\right)+$ $+\left(\rho^{\varepsilon}(t+\delta)-\rho^{\varepsilon}(t)\right)\upsilon^{\varepsilon}(t)$ and then we find the difference between $\rho^{\varepsilon}(t+\delta)-\rho^{\varepsilon}(t)$ by integrating equation (8) in the range from$t$before $t+\delta$.The resulting relation is integrated with respect to the variable $t$from 0 before$t+\delta$and for each term on the right-hand side, we use estimates from [10, 11], on the basis of which we derive the estimate of the lemma.

We now turn to the construction of approximate solutions by the Galerkin method.

Let $\{\varphi_j\}_{j=1}^{\infty}$ – orthonormal basis in space $L_2(\Omega)$ of $W_p^2(\Omega)\bigcap \overset{0}{\underset{p^1}{W}}(\Omega)$ and the following is true:

$$(\varphi_{jx},\omega_x)_{L_2(\Omega)}=\lambda_j(\varphi_j,\omega_j)_{L_2(\Omega)}.$$

Approximate solution $\upsilon^{N,\varepsilon}(t)$ looking for in the form:

$$\upsilon^{N,\varepsilon}(t)=\sum_{k=1}^{N}C_k^N(t)\,\varphi_k,$$

where $C_k^N(t)\in C^1[0,T]$.Density$\rho^{N,\varepsilon}(t)$ –there is a classical solution of the problem:

$$\frac{\partial\rho^{N,\varepsilon}(t)}{\partial t}+\left(\upsilon^{N,\varepsilon}(t)\cdot\nabla\right)\rho^{N,\varepsilon}(t)=0; \tag{23}$$

$$\rho^{N,\varepsilon}|_{t=0}=\rho_0^M(x),$$

where $\rho_0^M(x)$ –smooth initial function [12].

Sequence $\rho_0^M(x), M=1,2...$ converge to $\rho_0(x)$ in the norms $L_p(\Omega), W_p^1(\Omega), \rho_0^M(x)\in C^2(\Omega)$. Pressure $p^{N,\varepsilon}(t)$ – there is a classical solution of the problem:

$$div\upsilon^{N,\varepsilon}=\varepsilon p^{N,\varepsilon}; \tag{24}$$

$$\int_\Omega p^{N,\varepsilon}dx=0.$$

Temperature $\theta^{N,\varepsilon}(t)$ –is defined as the classical solution of the problem:

$$\rho^{N,\varepsilon}\left(\frac{\partial\theta^{N,\varepsilon}(t)}{\partial t}+\left(\upsilon^{N,\varepsilon}(t)\cdot\nabla\right)\theta^{N,\varepsilon}(t)\right)=div\left(\lambda\left(\theta^{N,\varepsilon}(t)\right)\nabla\theta^{N,\varepsilon}(t)\right)+\mu\sigma^{N,\varepsilon}; \tag{25}$$

$$\theta^{N,\varepsilon}|_{S=0}=\theta_0^M(x),\ \frac{\partial\theta^{N,\varepsilon}(t)}{\partial n}|_S=0, t\in[0,T],$$

where $\theta_0^M(x)$ – an initial smooth function satisfying the equation:

$$\frac{\partial\theta_0^M(t)}{\partial n}|_S=0, t\in[0,T].$$

Functions $C_k^N(t)$, $k=1,2,...,N$,are determined by a system of ordinary differential equations with coefficients that depend in an operator way on $\rho^{N,\varepsilon}(t)$,$p^{N,\varepsilon}(t)$:

$$\Bigg(\rho^{N,\varepsilon}(t)\left(\frac{\partial\upsilon^{N,\varepsilon}(t)}{\partial t}+\left(\upsilon^{N,\varepsilon}(t)\cdot\nabla\right)\upsilon^{N,\varepsilon}(t)+\frac{1}{2}\upsilon^{N,\varepsilon}(t)\,div\upsilon^{N,\varepsilon}\right)-\mu\Delta\upsilon^{N,\varepsilon}(t)+$$
$$+\nabla p^{N,\varepsilon}(t)-\ell\theta^{N,\varepsilon}(t)\,\rho^{N,\varepsilon}(t)+\rho^{N,\varepsilon}(t)\,f,\varphi_j\ )_{L_2(\Omega)}=0.$$

We can select subsequences for which we have based on the Schauder principle, using the obtained a priori estimates, from the sequences$\left\{\upsilon^{N,\varepsilon}\right\},\left\{\rho^{N,\varepsilon}\right\},\left\{p^{N,\varepsilon}\right\},\left\{\theta^{N,\varepsilon}\right\}$:

$\upsilon^{N,\varepsilon}\to\upsilon^\varepsilon$weaklyat$L_p\left(0,T,W_p^2(\Omega)\right)$;
$\theta^{N,\varepsilon}\to\theta^\varepsilon$weaklyat$W_p^{2,1}(Q)$;
$\rho^{N,\varepsilon}\to\rho^\varepsilon$ *weaklyat$W_p^{1,1}(Q)$;
$\upsilon^{N,\varepsilon}\to\upsilon^\varepsilon$strongat$L_p(0,T,L_p(\Omega))$;
$\theta^{N,\varepsilon}\to\theta^\varepsilon$strongat$L_p(0,T,L_p(\Omega))$;
$\upsilon_t^{N,\varepsilon}\to\upsilon_t^\varepsilon$weaklyat$L_p(0,T,L_p(\Omega))$;
$p^{N,\varepsilon}\to p^\varepsilon$weaklyat$L_p\left(0,T,W_p^1(\Omega)\right)$.

This completes the proof of Theorem 1.

The following is valid.

Theorem 2. Suppose that all the conditions of Theorem 1 are satisfied.Then the strong solution of problem (7)–(12) converges to the strong solution of problem (1)–(6) for $\varepsilon \to 0$.

Evidence. Because of the a priori estimates obtained earlier, we have:

$\upsilon^\varepsilon \to \upsilon$ weakly at $L_p\left(0, T, W_p^2(\Omega)\right)$;

$\theta^\varepsilon \to \theta$ weakly at $W_p^{2,1}(Q)$;

$\rho^\varepsilon \to \rho$ weakly at $W_p^{1,1}(Q)$; $\upsilon^\varepsilon \to \upsilon$ strong at $L_p(0, T, L_p(\Omega))$;

$\theta^\varepsilon \to \theta$ strong at $L_p(0, T, L_p(\Omega))$,

$\upsilon_t^\varepsilon \to \upsilon_t$weakly at $L_p(0, T, L_p(\Omega))$;

$p^\varepsilon \to p$weakly at $L_p\left(0, T, W_p^1(\Omega)\right)$.

Passing to the limit as $\varepsilon \to 0$ in the corresponding identities, we establish that the limit functions υ, p, ρ, $\theta-$ there is a strong solution of problem (1)–(6).

Theorem 2 is proved.

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Энергияны диссипациялауға арналған жолмен біртекті емес балқыламалардың температуралық моделін ε-жуықтауы

Үлкен сандармен тәжірибе және жуықтау есептеулер жасауға байланысты Навье-Стокс теңдеуі туралы жинақталған мәліметтерден табиғаттағы нақты құбылыстардың нақты балқымалар жүйесі мен тұтқыр балқымалардың математикалық моделі арасындағы сәйкестікті байқауға мүмкіндік береді. Бұл үшін көптеген себептер бар. Олардың бірі — сызықты емес Навье-Стокс теңдеуі. Сызықты емес теңдеулер үшін стационарлы емес есептерді қанағаттандыратын шешімдері $t \geq 0$ интервалында барлығында болмауы мүмкін екені белгілі. Соңғы уақыт аралығында шексіздікке ұмтылуы мүмкін немесе құлдырайды, яғни тұрақтылығын жоғалтады, теңдеуді қанағаттандырмайды және тармақтала бастайды. Егер осы шешім $t \geq 0$ болғанда, онда ол сыртқы әсер мен шекаралық шарттарды тұрақтандыру кезінде стационарлы есептің шешіміне ұмтылмауы математикалық тұрғыда дәлелденді. Алынған стационарлы емес есептің шешімі тегіс бастапқы тәртіпте және тегіс сыртқы әсерлерде уақыт өте тұрақтылығын азайта береді. Содан кейін тұрақсыз немесе турбулентті режимге көшеді. Шешімнің бір немесе бірнеше тармағын нақты іске асыру Навье-Стокс теңдеуінде ескерілмеген кез келген себептерге байланысты. Ұсынылған мақалада жақсы жуықтау арқылы сандық схема құрылды. ε-Жуықтауы арқылы дифференциалдық теңдеулердің бастапқы жүйелері регуляризацияланды. Сандық және аналитикалық түрде сығылмайтын тұтқыр ағым үшін шекаралық есептердің дұрыстығын зерттеуді қамтамасыз ететін Галеркин әдісі енгізілді. Әлсіз жуықтауы бар Навье-Стокс теңдеуі үшін бөлу схемасы құрылды. Гидродинамиканың сызықты емес теңдеуін Коши-Ковалевский типті теңдеулер жүйесіне келтіруге болатындай сығылмайтын балқыманың стационарлы және стационарлы емес моделдерін жуықтау қолданылды.

Кілт сөздер: энергияны диссипациялау, жүйені жақындату, Коши теңсіздігі, Галеркин әдісі, априорлық бағалау.

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ε-Аппроксимация температурной модели неоднородных расплавов с учетом диссипации энергии

Накопленные факты и сведения об уравнениях Навье-Стокса, наряду с большим числом экспериментов и приближенных расчетов, позволили выявить некоторые несоответствия между математической моделью вязкого расплава и реальными явлениями в природе реальных расплавленных систем. И этому есть много причин. Одна из них — нелинейность уравнений Навье-Стокса. А для нелинейных уравнений известно, что в нестационарных задачах удовлетворяющее им решение может существовать не на всем интервале $t \geq 0$. За конечный промежуток времени оно может либо уйти в бесконечность, либо рассыпаться, т.е. потерять регулярность и перестать удовлетворять уравнениям и начать ветвиться. Математически доказано, что если же это решение существует при $t \geq 0$, то оно может не стремиться к решению стационарной задачи при стабилизации краевых условий и внешних воздействий. Полученные решения нестационарной задачи даже при гладком начальном режиме и гладких внешних воздействиях могут со временем становиться менее регулярными, а затем вообще перейти в нерегулярные или турбулентные режимы. Фактическая реализация той или иной ветви решения зависит от посторонних причин, неучтенных в уравнениях Навье-Стокса. В предлагаемой статье построена численная схема, обладающая хорошей сходимостью. Построена регуляризация исходных систем дифференциальных уравнений путем ε-аппроксимации. Реализован метод Галеркина, обеспечивающий изучение корректности краевых задач для несжимаемого вязкого потока как численно, так и аналитически. Построена схема расщепления для уравнений Навье-Стокса со слабой аппроксимацией. Построена аппроксимация стационарной и нестационарной моделей несжимаемого расплава, что приводит нелинейные уравнения гидродинамики к системе уравнений типа Коши-Ковалевской.

Ключевые слова: диссипация энергии, аппроксимация системы, неравенство Коши, метод Галеркина, априорные оценки.

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On structures in hypergraphs of models of a theory

Hypergraphs of models of a theory are derived objects allowing toobtain an essential structural information about both giventheories and related semantic objects including graph ones. In the present paper we define and study structural properties of hypergraphs of modelsof a theory including lattice ones. Characterizations for thelattice properties of hypergraphs of models of a theory, as wellas for structures on sets of isomorphism types of models of atheory, are given.

Keywords: hypergraph of models, elementary theory, elementarily substructural set, lattice structure.

Hypergraphs of models of a theory are derived objects allowing to obtain an essential structural information about both given theories and related semantic objects including graph ones [1–9]. Studying of hypergraphs of models of a theory is closely related with a series of papers on description of lattices of substructures [10–22].

In the presented paper we define and study structural properties of hypergraphs of models of a theory including lattice ones. Characterizations for the lattice properties of hypergraphs of models of a theory as well as for structures on sets of isomorphism types of models of a theory are given.

Preliminaries

Recall that a *hypergraph* is a pair of sets (X, Y), where Y is some subset of the Boolean $\mathcal{P}(X)$ of the set X.

Let $\mathcal{M}$ be some model of a complete theory T. Following [5], we denote by $H(\mathcal{M})$ a family of all subsets N of the universe M of $\mathcal{M}$ that are universes of elementary submodels $\mathcal{N}$ of the model $\mathcal{M}$: $H(\mathcal{M}) = \{N \mid \mathcal{N} \preccurlyeq \mathcal{M}\}$. The pair $(M, H(\mathcal{M}))$ is called the *hypergraph of elementary submodels* of the model $\mathcal{M}$ and denoted by $\mathcal{H}(\mathcal{M})$.

Definition [8]. Let $\mathcal{M}$ be a model of a theory T with a hypergraph $\mathcal{H} = (M, H(\mathcal{M}))$ of elementary submodels, A be an infinite definable set in $\mathcal{M}$, of arity n: $A \subseteq M^n$. The set A is called *$\mathcal{H}$-free* if for any infinite set $A' \subseteq A$, $A' = A \cap Z^n$ for some $Z \in H(\mathcal{M})$ containing parameters for A. Two $\mathcal{H}$-free sets A and B of arities m and n respectively are called *$\mathcal{H}$-independent* if for any infinite $A' \subseteq A$ and $B' \subseteq B$ there is $Z \in H(\mathcal{M})$ containing parameters for A and B and such that $A' = A \cap Z^m$ and $B' = B \cap Z^n$.

Note the following properties [8].

1. Any two tuples of a $\mathcal{H}$-free set A, whose distinct tuples do not have common coordinates, have same type.

Indeed, if there are tuples $\bar{a}, \bar{b} \in A$ with $\mathrm{tp}(\bar{a}) \neq \mathrm{tp}(\bar{b})$ then for some formula $\varphi(\bar{x})$ the sets of solutions of that formula and of the formula $\neg\varphi(\bar{x})$ divide the set A into two nonempty parts A_1 and A_2, where at least one part, say A_1, is infinite. Taking A_1 for A' we have $A_1 = A \cap Z^n$ for appropriate $Z \in H(\mathcal{M})$ and n. Then by the condition for tuples in A we have $A_2 \cap Z^n = \emptyset$ that is impossible since Z is the universe of an elementary submodel of $\mathcal{M}$.

Thus the formula $\varphi(\bar{x})$, defining A, implies some complete type in $S^n(\emptyset)$, and if A is $\emptyset$-definable then $\varphi(\bar{x})$ is a principal formula.

In particular, if the set A is $\mathcal{H}$-free and $A \subseteq M$, then the formula, defining A, implies some complete type in $S^1(\emptyset)$.

2. If $A \subseteq M$ is a $\mathcal{H}$-free set, then A does not have nontrivial definable subsets, with parameters in A, i.e., subsets distinct to subsets defined by equalities and inequalities with elements in A.

Indeed, if $B \subset A$ is a nontrivial definable subset then B is defined by a tuple $\bar{a}$ of parameters in A, forming a *finite* set $A_0 \subset A$, and B is distinct to subsets of A_0 and to $A \setminus C$, where $C \subseteq A_0$. Then removing from A a set $B \setminus A_0$ or $(A \setminus B) \setminus A_0$, we obtain some $Z \in H(\mathcal{M})$ violating the satisfiability for B or its complement. It contradicts the condition that Z is the universe of an elementary submode of $\mathcal{M}$.

3. If A and B are two $\mathcal{H}$-independent sets, where $A \cup B$ does not have distinct tuples with common coordinates, then $A \cap B = \emptyset$.

Indeed, if $A \cap B$ contains a tuple $\bar{a}$, then, choosing infinite sets $A' \subseteq A$ and $B' \subseteq B$ with $\bar{a} \in A'$ and $\bar{a} \notin B'$, we obtain $\bar{a} \in A' = A \cap Z^n$ for appropriate $Z \in H(\mathcal{M})$ and n, as so $\bar{a} \in B \cap Z^n = B'$. This contradiction means that $A \cap B = \emptyset$.

Definition [6]. The *complete union* of hypergraphs (X_i, Y_i), $i \in I$, is the hypergraph $\left(\bigcup\limits_{i\in I} X_i, Y\right)$, where $Y = \left\{\bigcup\limits_{i\in I} Z_i \mid Z_i \in Y_i\right\}$. If the sets X_i are disjoint, the complete union is called *disjoint* too. If the set X_i form a $\subseteq$-chain, then the complete union is called *chain*.

By Property 3 we have the following theorem on decomposition of restrictions of hypergraphs $\mathcal{H}$, representable by unions of families of $\mathcal{H}$-independent sets.

Theorem 1.1 [8]. *A restriction of hypergraph $\mathcal{H} = (M, H(\mathcal{M}))$ to a union of a family of $\mathcal{H}$-free $\mathcal{H}$-independent sets $A_i \subseteq M$ is represented as a disjoint complete union of restrictions $\mathcal{H}_i$ of the hypergraph $\mathcal{H}$ to the sets A_i.*

Proof. Consider a family of $\mathcal{H}$-independent sets $A_i \subseteq M$. By Property 3 these sets are disjoint, and using the definition of $\mathcal{H}$-independence we immediately obtain that the union of restrictions $\mathcal{H}_i$ of $\mathcal{H}$ to the sets A_i is complete.

Recall that a subset A of a linearly ordered structure M is called *convex* if for any $a, b \in A$ and $c \in M$ whenever $a < c < b$ we have $c \in A$. A *weakly o-minimal structure* is a linearly ordered structure $M = \langle M, =, <, \ldots\rangle$ such that any definable (with parameters) subset of the structure M is a union of finitely many convex sets in M.

In the following definitions M is a weakly o-minimal structure, $A, B \subseteq M$, M be $|A|^+$-saturated, $p, q \in S_1(A)$ be non-algebraic types.

Definition. [23]. We say that p is not *weakly orthogonal* to q ($p \not\perp^w q$) if there exist an A-definable formula $H(x, y)$, $\alpha \in p(M)$ and $\beta_1, \beta_2 \in q(M)$ such that $\beta_1 \in H(M, \alpha)$ and $\beta_2 \notin H(M, \alpha)$.

Definition. [24]. We say that p is not *quite orthogonal* to q ($p \not\perp^q q$) if there exists an A-definable bijection $f : p(M) \to q(M)$. We say that a weakly o-minimal theory is *quite o-minimal* if the notions of weak and quite orthogonality of 1-types coincide.

In the work [25] the countable spectrum for quite o-minimal theories with non-maximal number of countable models has been described:

Theorem 1.2. Let T be a quite o-minimal theory with non-maximal number of countable models. Then T has exactly $3^k \cdot 6^s$ countable models, where k and s are natural numbers. Moreover, for any $k, s \in \omega$ there exists a quite o-minimal theory T having exactly $3^k \cdot 6^s$ countable models.

Realizations of these theories with a finite number of countable models are natural generalizations of Ehrenfeucht examples obtained by expansions of dense linear orderings by a countable set of constants, and they are called theories of *Ehrenfeucht type*. Moreover, these realizations are representative examples for hypergraphs of prime models [1, 3, 5]. We consider operators for hypergraphs allowing on one hand to describe the decomposition of hypergraphs of prime models for quite o-minimal theories with few countable models, and on the other hand pointing out constructions leading to the building of required hypergraphs by some simplest ones.

Having nontrivial structures like structures with some orders it is assumed that «complete» decompositions are considered modulo additional conditions guaranteing the elementarity for substructures with considered universes. So we use the *conditional* completeness taking unions with the properties of density, linearity etc.

Below we illustrate this conditional completeness for structures with dense linear orders.

Denote by $(M, H_{\rm dlo}(\mathcal{M}))$ the hypergraph of (prime) elementary submodels of a countable model $\mathcal{M}$ of the theory of dense linear order without endpoints.

Remark 1.3. The class of hypergraphs $(M, H_{\rm dlo}(\mathcal{M}))$ is closed under countable chain complete unions, modulo density and having an encompassing dense linear order without endpoints. Thus, any hypergraph $(M, H_{\rm dlo}(\mathcal{M}))$ is represented as a countable chain complete, modulo density, union of some its proper subhypergraphs. The notion of *weak o-minimality* was originally studied by D. Machpherson, D. Marker and C. Steinhorn in [26].

Any countable model of a theory of Ehrenfeucht type is a disjoint union of some intervals, which are ordered both themselves and between them, and of some singletons. Dense subsets of the intervals form universes of elementary substructures. So, in view of Remark 1.3, we have:

Theorem 1.4 [6]. *A hypergraph of prime models of a countable model of a theory of Ehrenfeucht type is represented as a disjoint complete, modulo density, union of some hypergraphs in the form $(M, H_{\rm dlo}(\mathcal{M}))$ as well as singleton hypergraphs of the form $(\{c\}, \{\{c\}\})$.*

Remark 1.5. Taking into consideration links between sets of realizations of 1-types, which are not weakly orthogonal, as well as definable equivalence relations, the construction for the proof of Theorem 1.4 admits a natural generalization for an arbitrary quite o-minimal theory with few countable models. Here conditional complete unions should be additionally *coordinated*, i.e., considering definable bijections between sets of realizations of 1-types, which are not quite orthogonal.

Elementarily substructural sets

Let $\mathcal{M}$ be a model of theory T, $(M, H(\mathcal{M}))$ be a hypergraph of elementary submodels of $\mathcal{M}$. The sets $N \in H(\mathcal{M})$ are called *elementarily submodel* or *elementarily substructural* in $\mathcal{M}$.

Elementarily substructural sets in $\mathcal{M}$ are characterized by the following well-known Tarski–Vaught Theorem, which is called the Tarski–Vaught test.

Theorem 2.1. Let $\mathcal{A}$ and $\mathcal{B}$ be structures in a language Σ, $\mathcal{A} \subseteq \mathcal{B}$. The following are equivalent:

(1) $\mathcal{A} \preccurlyeq \mathcal{B}$;

(2) *for any formula $\varphi(x_0, x_1, \ldots, x_n)$ in the language Σ and for any elements $a_1, \ldots, a_n \in A$, if $\mathcal{B} \models \exists x_0\, \varphi(x_0, a_1, \ldots, a_n)$ then there is an element $a_0 \in A$ such that $\mathcal{B} \models \varphi(a_0, a_1, \ldots, a_n)$.*

Corollary 2.2. A set $N \subseteq M$ is elementarily substructural in $\mathcal{M}$ if and only if for any formula $\varphi(x_0, x_1, \ldots, x_n)$ in the language $\Sigma(\mathcal{M})$ and for any elements $a_1, \ldots, a_n \in N$, if $\mathcal{M} \models \exists x_0\, \varphi(x_0, a_1, \ldots, a_n)$ then there is an element $a_0 \in N$ such that $\mathcal{M} \models \varphi(a_0, a_1, \ldots, a_n)$.

Proposition 2.3. Let A be a definable set in an ω_1-saturated model $\mathcal{M}$ of a countable complete theory T. Then exactly one of the following conditions is satisfied:

(1) *A is finite and contained in any elementarily substructural set in $\mathcal{M}$;*

(2) *A is infinite and has infinitely many distinct intersections with elementarily substructural sets in $\mathcal{M}$, and all these intersections are infinite; these intersections can be chosen forming an infinite chain/antichain by inclusion.*

Proof. If $|A| < \omega$ then A is contained in $\mathrm{acl}(\emptyset)$, and so it is contained in any elementary submodel of $\mathcal{M}$.

If $A = \varphi(\mathcal{M}, \bar{a})$ is infinite, we construct a countable submodel $\mathcal{N}_0 \prec \mathcal{M}$ containing parameters in $\bar{a}$. Since A is infinite, the set $A \cap N_0$ is countable. By compactness, since $\mathcal{M}$ is ω_1-saturated, the set $A \setminus N_0$ is infinite. Adding to N_0 new elements of A we construct a countable model $\mathcal{N}_1$ such that $\mathcal{N}_0 \prec \mathcal{N}_1 \prec \mathcal{M}$. Continuing the process we build an elementary chain of models $\mathcal{N}_k$, $k \in \omega$, such that $\mathcal{N}_k \prec \mathcal{M}$ and $A \cap N_k \subset A \cap N_{k+1}$, $k \in \omega$.

Constructing the required antichain of intersections $A \cap N$ with elementarily substructural sets N, it suffices to use [9, Theorem 2.10] allowing to separate disjoint finite sets, whose elements do not belong to $\mathrm{acl}(\emptyset)$.

The arguments for the proof of Proposition 2.3 stay valid for a countable saturated model $\mathcal{M}$. Thus, we have the following

Proposition 2.4. Let A be a definable set in a countable saturated model $\mathcal{M}$ of a small theory T. Then exactly one of the following conditions is satisfied:

(1) *A is finite and contained in any elementarily substructural set in $\mathcal{M}$;*

(2) *A is infinite and has infinitely many distinct intersections with elementarily substructural sets in $\mathcal{M}$, and all these intersections are infinite; these intersections can be chosen forming an infinite chain/antichain by inclusion.*

The following example illustrates that if $\mathcal{M}$ is not saturated then the conclusions of assertions 2.3 and 2.4 can fail.

Example 2.5. Let a set A is defined by a unary predicate P and includes infinitely many language constants c_i, $i \in I$. Then there is, in the language $\{P\} \cup \{c_i \mid i \in I\}$, a structure $\mathcal{M}$ having only finite set A_0 of elements in A, which are not interpreted by constants. Since elementarily substructural sets N take all constants, there are only finitely many possibilities for intersections $A \cap N$.

In view of aforesaid arguments it is interesting to describe possible cardinalities both for sets $H(\mathcal{M})$ and their restrictions $H(\mathcal{M}) \restriction A \rightleftharpoons \{A \cap N \mid N \in H(\mathcal{M})\}$ on definable sets $A \subseteq M$.

Since in Example 2.5 intersections $A \cap N$, taking all constants c_i, can include an arbitrary subset of A_0, then for this example we have $|H(\mathcal{M}) \restriction A| = 2^{|A_0|}$. The same formula holds for infinite sets A_0, but in such a case the set $H(\mathcal{M}) \restriction A$ is transformed from finite one directly to a set with continuum many elements.

Note that for $\mathcal{H}$-free sets $A \subseteq M$, *modulo* $\mathrm{acl}(\emptyset)$ (i.e., for sets A, whose each subset $B \subseteq A \setminus \mathrm{acl}(\emptyset)$ has a representation $B \cup (\mathrm{acl}(\emptyset) \cap A) = A \cap N$ for some $N \in H(\mathcal{M})$), the equality $|H(\mathcal{M}) \restriction A| = 2^{|A \setminus \mathrm{acl}(\emptyset)|}$ holds. Thus, we have the following *dichotomy theorem*.

Theorem 2.6. For any $\mathcal{H}$-free, modulo $\mathrm{acl}(\emptyset)$*, set $A \subseteq M$ its restriction to any elementary submodel $\mathcal{M}_0 \prec \mathcal{M}$ satisfies either $|H(\mathcal{M}_0) \restriction A| = 2^n$ for some $n \in \omega$, or $|H(\mathcal{M}_0) \restriction A| = 2^\lambda$ form some $\lambda \geq \omega$.*

Similar to Example 2.5, the following example illustrates the dichotomy for hypergraphs of elementary submodels.

Example 2.7. Consider the structure $\mathcal{M}$ of rational numbers, $\langle \mathbf{Q}, <, c_q \rangle_{q \in \mathbf{Q}}$, in which every element is interpreted by a constant. This structure does not have proper elementary substructures, therefore $|H(\mathcal{M})| = 1 = 2^0$. Extending $\mathcal{M}$ to a structure $\mathcal{M}_1$ by addition of n elements for pairwise distinct 1-types, defined by cuts, we have $|H(\mathcal{M}_1)| = 2^n$. If $\mathcal{M}$ is extended till a structure $\mathcal{M}_2$ by addition of at least two elements of fixed cut or of infinitely many elements for distinct cuts, then by density the summarized number of added elements occurs infinite and $|H(\mathcal{M}_2)| = 2^\lambda$ holds for some $\lambda \geq \omega$.

At the same time there are examples of hypergraphs of elementary submodels, for which the conclusion of Theorem 2.6 fails. For instance, as shown in [13], there are hypergraphs for the theory of arithmetic of natural numbers such that $|H(\mathcal{M})| = 5$ and the lattice of elementary submodels is isomorphic to the lattice P_5.

Lattice structures associated with hypergraphs of models of a theory

For given structure $\mathcal{M}$ we define the structure $L(\mathcal{M}) = \langle H(\mathcal{M}); \wedge, \vee \rangle$ by the following relations for $\mathcal{M}_1, \mathcal{M}_2 \prec \mathcal{M}$: $\mathcal{M}_1 \wedge \mathcal{M}_2 = \mathcal{M}_1 \cap \mathcal{M}_2$ and $\mathcal{M}_1 \vee \mathcal{M}_2 = \mathcal{M}(M_1 \cup M_2)$.

Consider the following question: when the structure $L(\mathcal{M})$ is a lattice?

Clearly, answering this question we have to characterize the conditions $\mathcal{M}_1 \cap \mathcal{M}_2 \prec \mathcal{M}$ and $\mathcal{M}(M_1 \cup M_2) \prec \mathcal{M}$. Assuming that $\mathcal{M}$ is infinite, the structures $\mathcal{M}_1 \cap \mathcal{M}_2$ should be infinite too, in particular, $M_1 \cap M_2 \neq \emptyset$. By [5, Theorem 3.2], assuming that $\mathcal{M}$ is λ-saturated, it can not contain separated sets A and B of cardinalities $< \lambda$, such that $\mathrm{acl}(A) \cap \mathrm{acl}(B) = \emptyset$.

By Theorem 2.1 we have the following theorems characterizing the elementarity of substructures.

Theorem 3.1. Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be elementary substructures of structure $\mathcal{M}$ in a language Σ, $M_1 \cap M_2 \neq \emptyset$. The following are equivalent:

(1) $(\mathcal{M}_1 \cap \mathcal{M}_2) \prec \mathcal{M}$;

(2) *for any formula $\varphi(x_0, x_1, \ldots, x_n)$ of the language Σ and for any elements $a_1, \ldots, a_n \in M_1 \cap M_2$ if $\mathcal{M} \models \exists x_0\, \varphi(x_0, a_1, \ldots, a_n)$ then there is an element $a_0 \in M_1 \cap M_2$ such that $\mathcal{M}_i \models \varphi(a_0, a_1, \ldots, a_n)$, $i = 1, 2$.*

Theorem 3.2. Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be elementary substructures of structure $\mathcal{M}$ in a language Σ. The following are equivalent:

(1) $\mathcal{M}(M_1 \cup M_2) \prec \mathcal{M}$;

(2) *for any formula $\varphi(x_0, x_1, \ldots, x_n)$ of the language Σ and for any elements $a_1, \ldots, a_n \in M_1 \cap M_2$ if $\mathcal{M} \models \exists x_0\, \varphi(x_0, a_1, \ldots, a_n)$ then there is an element $a_0 \in M(M_1 \cup M_2)$ such that $\mathcal{M}(M_1 \cup M_2) \models \varphi(a_0, a_1, \ldots, a_n)$.*

The following examples illustrate valuations of the conditions (2) in Theorems 3.1 and 3.2.

Example 3.3. Consider a structure $\mathcal{M}$ in a graph language $\{R^{(2)}\}$ with a symmetric irreflexive relation R and elements a_1, a_2, a_3, a_4 such that

$$R = \{[a_1, a_3], [a_1, a_4], [a_2, a_3], [a_2, a_4]\}.$$

The substructures $\mathcal{M}_1 \subset \mathcal{M}$ and $\mathcal{M}_2 \subset \mathcal{M}$ with the universes $\{a_1, a_2, a_3\}$ and $\{a_1, a_2, a_4\}$ respectively satisfy the formula $\varphi(a_1, a_2) \rightleftharpoons \exists x (R(a_1, x) \wedge R(a_2, x))$ whereas $\mathcal{M}_1 \cap \mathcal{M}_2$ does not satisfy that formula since appropriate elements for x belong to $M_1 \oplus M_2$.

Example 3.4. Consider a structure $\mathcal{M}$ of graph language $\{R^{(2)}\}$ with symmetric irreflexive relation R and with elements a_1, a_2, a_3 such that $R = \{[a_1, a_3], [a_2, a_3]\}$. The substructures $\mathcal{M}_1 \subset \mathcal{M}$ and $\mathcal{M}_2 \subset \mathcal{M}$ with the universes $\{a_1\}$ and $\{a_2\}$ form the substructure $\mathcal{M}(M_1 \cup M_2)$ with the universe $\{a_1, a_2\}$ and it does not satisfy the formula $\varphi(a_1, a_2)$ in Example 3.3. At the same time the structure $\mathcal{M}$ satisfies this formula.

Since in some cases elementary substructures of given structure $\mathcal{M}$ form the lattice with respect to the operations $\mathcal{M}_1 \wedge \mathcal{M}_2 = \mathcal{M}_1 \cap \mathcal{M}_2$ and $\mathcal{M}_1 \vee \mathcal{M}_2 = \mathcal{M}(M_1 \cup M_2)$, the study of hypergraphs $\mathcal{H}(\mathcal{M})$, for these cases, is reduced to study of the lattices $L(\mathcal{M})$. As Example in [13] shows, the lattices $L(\mathcal{M})$ can be non-distributive unlike the description in Theorem 2.6, where correspondent lattices are distributive, and for finite $H(\mathcal{M}_0)$ even form Boolean algebras.

In the given context hypergraphs/lattices with minimal, i.e. least structures play an important role. These structures can be obtained from an arbitrary structure by addition of constants interpreted by all elements of the structure. Besides, these minimal structures exist for finite sets $H(\mathcal{M})$.

In [27], the following theorem on dichotomy for minimal structures is proved.

Theorem 3.5. Let $\mathcal{M}_0$ be a minimal structure, $\mathcal{M}$ be its saturated elementary extension and $p \in S_1(\mathcal{M}_0)$ be unique non-algebraic 1*-type. Then exactly one of the following conditions holds:*

(I) *the structure* $(p(M), \mathrm{Sem}_p)$ *is a pregeometry, where* Sem_p *is the relation of semi-isolation on the set of realizations of the type* p*, i.e. the following conditions are satisfied:*

(S1) *Monotony: if* $A \subseteq B$ *then* $A \subseteq \mathrm{Sem}_p(A) \subseteq \mathrm{Sem}_p(B)$*;*

(S2) *Finite character:* $\mathrm{Sem}_p(A) = \bigcup\{\mathrm{Sem}_p(A_0) \mid A_0$ *is a finite subset of* $A\}$*;*

(S3) *Transitivity:* $\mathrm{Sem}_p(A) = \mathrm{Sem}_p(\mathrm{Sem}_p(A))$*;*

(S4) *Exchange property (Symmetry): if* $a \in \mathrm{Sem}_p(A \cup \{b\}) \setminus \mathrm{Sem}_p(A)$ *then* $b \in \mathrm{Sem}_p(A \cup \{a\})$*;*

(II) *for some finite* $A \subset M$ *there exists an infinite set* $C_0 \subseteq \mathrm{dcl}(A \cup M_0)$ *and a definable quasi-order* $\leq$ *on* $\mathcal{M}$ *such that* C_0 *orders a type over* A*:*

(D1) *for any* $c \in C_0$ *the set* $\{x \in C_0 \mid c \leq x\}$ *is a cofinite subset of* C_0*;*

(D2) C_0 *is an initial segment of* $\mathcal{M}$*: if* $c \in C_0$ *and* $m \leq c$*, then* $m \in C_0$*.*

Basic examples illustrating Theorem 3.5 are represented by ordered structures $\langle \omega, < \rangle$ and $\langle \omega + \omega^*, < \rangle$. The conclusion of Theorem 2.6 holds for both structures. Moreover, for $\mathcal{M}_1 \equiv \langle \omega, < \rangle$ and $\mathcal{M}_2 \equiv \langle \omega + \omega^*, < \rangle$ the structures $L(\mathcal{M}_1)$ and $L(\mathcal{M}_2)$ form atomic Boolean algebras, whose atoms are defined by equivalence classes, being closures of singletons, not in $\omega + \omega^*$, taking all predecessors and successors.

Return to Example 2.7. It is known that the intersection of convex sets is convex, whereas the intersection of dense orders can be not dense. For instance, any interval $[a, b]$ contains countable dense subsets X, Y such that $X \cap Y = \{a, b\}$. It means that for the structure $\mathcal{M}' \equiv \langle \mathbf{Q}, <, c_q \rangle_{q \in \mathbf{Q}}$ the structure $L(\mathcal{M}')$ forms a lattice, moreover, a Boolean algebra, if and only if each type in $S_1(\mathrm{Th}(\mathcal{M}'))$ has at most one realization in $\mathcal{M}'$. If $\mathcal{M}'$, with the lattice $L(\mathcal{M}')$, realizes λ non-principal 1-types, then $|L(\mathcal{M}')| = 2^\lambda$. Thus, the following proposition holds.

Proposition 3.6. For the structure $L(\mathcal{M}')$ *the following are equivalent:*

(1) $L(\mathcal{M}')$ *is a lattice;*

(2) $L(\mathcal{M}')$ *forms an atomic Boolean algebra;*

(3) *each type in* $S_1(\mathrm{Th}(\mathcal{M}'))$ *has at most one realization in* $\mathcal{M}'$*, and if* $\mathcal{M}'$ *realizes* λ *non-principal* 1*-types, then* $|L(\mathcal{M}')| = 2^\lambda$*.*

Proposition 3.6 admits natural modifications for a series of theories with minimal models, for instance, for models, obtained by replacement of elements in $\mathcal{M}'$ with finite antichains of fixed cardinality marked by unary predicates P_q instead of constants c_q. Note that admitting replacement of constants c_q by infinite antichains P_q the structure $L(\mathcal{M}')$ is not a lattice since P_q can be divided by some elementary substructures $\mathcal{M}'_1, \mathcal{M}'_2 \prec \mathcal{M}'$ into two disjoint parts, whence $\mathcal{M}'_1 \cap \mathcal{M}'_2 \not\prec \mathcal{M}'$.

Clearly, as above, in the general case if there are separable elements in definable sets $A \subseteq M$ of structure $\mathcal{M}$ then $L(\mathcal{M})$ is not closed under intersections, i.e., $L(\mathcal{M})$ is not even a lower semilattice. Thus, the following proposition holds.

Proposition 3.7. If $L(\mathcal{M})$ *is a lattice then* $\mathcal{M}$ *does not have definable sets* $A \subseteq M$ *containing elements separable each other, in particular,* $\mathcal{M}$ *does not contain* $\mathcal{H}$*-free sets* $A \subseteq M$*.*

In view of Proposition 3.7 it is natural, for given structure $\mathcal{M}$, along with $L(\mathcal{M})$ to consider for sets $X \subseteq M$ the following *relative* structures $L_X(\mathcal{M})$. Denote by $H_X(\mathcal{M}$ the family of all sets in $H(\mathcal{M}$ containing the set X. Then $L_X(\mathcal{M}) \rightleftharpoons \langle H_X(\mathcal{M}; \wedge, \vee \rangle$, where for structures $\mathcal{M}_1, \mathcal{M}_2 \prec \mathcal{M}$ containing X, $\mathcal{M}_1 \wedge \mathcal{M}_2 = \mathcal{M}_1 \cap \mathcal{M}_2$ and $\mathcal{M}_1 \vee \mathcal{M}_2 = \mathcal{M}(M_1 \cup M_2)$.

Note that if X is a universe of some elementary substructure of structure $\mathcal{M}$ then definable sets $A \subseteq M$ already do not contain elements separable by sets in $L_X(\mathcal{M})$. Then, in any case, $\mathcal{M}_1 \wedge \mathcal{M}_2$ is a substructure of $\mathcal{M}$ and the elementarity of that substructure is characterized by Theorem 3.1.

The following example illustrates that apart from the density there are other reasons preventing to consider $L(\mathcal{M})$ as a lattice.

Example 3.8 [28]. Let $\mathcal{M} = \langle M; <, P^1, U^2, c_i \rangle_{i \in \omega}$ be a linearly ordered structure such that $\mathcal{M}$ is a disjoint union of interpretations of unary predicates P and $\neg P$, where $\neg P(\mathcal{M}) < P(\mathcal{M})$. We identify interpretations of P and $\neg P$ with the set $\mathbb{Q}$ of rational numbers with the natural order.

The symbol U interprets the binary relation defined as follows: for any $a \in P(\mathcal{M}), b \in \neg P(\mathcal{M})$ $U(a, b) \Leftrightarrow$ $\Leftrightarrow b < a + \sqrt{2}$.

The constants c_i interpret an infinite strictly increasing sequence on $P(\mathcal{M})$ as follows: $c_i = i \in \mathbb{Q}$.

Clearly that $Th(\mathcal{M})$ is a weakly o-minimal theory. Let

$$p(x) := \{x > c_i \mid i \in \omega\} \cup \{P(x)\};$$

$$q(y) := \{\forall t (U(c_i, t) \to t < y) \mid i \in \omega\} \cup \{\neg P(y)\}.$$

Obviously, $p, q \in S_1(\emptyset)$ are nonisolated types and $p \not\leq^w q$. Since there are no $\emptyset$-definable bijections from $p(\mathcal{M}')$ onto $q(\mathcal{M}')$, where $\mathcal{M}'$ is a model of $Th(\mathcal{M})$ realizing some of these types then $Th(\mathcal{M})$ is not quite o-minimal.

As shown in [28], $Th(\mathcal{M})$ has exactly 4 pairwise non-isomorphic countable models: the prime model $\mathcal{M}$, i.e., with $p(\mathcal{M}) = \emptyset$ and $q(\mathcal{M}) = \emptyset$; the model $\mathcal{M}_1$ such that $p(\mathcal{M}_1)$ has the ordering type $[0, 1) \cap \mathbb{Q}$, $q(\mathcal{M}_1)$ has the ordering type $(0, 1) \cap \mathbb{Q}$; the model $\mathcal{M}_2$ such that $p(M_2)$ has the ordering type $(0, 1) \cap \mathbb{Q}$, $q(M_2)$ has the ordering type $[0, 1) \cap \mathbb{Q}$; and the countable saturated model $\mathcal{M}_3$.

Therefore $\mathcal{M}_1 \cap \mathcal{M}_2 \not\prec \mathcal{M}_3$. By this reason as well as by the possibility of violation of density in intersections, the structure $L(\mathcal{M}_3)$ does not form a lower semilattice.

Remark 3.9. Along with Example if we consider the known Ehrenfeucht's example with three models: a prime model $\mathcal{M}_0$, a weakly saturated model $\mathcal{M}_1$, and a countable saturated model $\mathcal{M}_2$, then the structure $L(\mathcal{M}_2)$ is not a lattice in view of presence of dense definable intervals but includes the three-element linearly ordered lattice consisting of the universes M_0, M_1, M_2.

Lattice structures on sets of isomorphism types of models of a theory

Following Example 3.8 and Remark 3.9 we consider a question on existence of natural lattices associated with hypergraphs $(M, H(\mathcal{M}))$ which a distinct to $L(\mathcal{M})$. Related lattices are lattices represented by Rudin–Keisler preorders RK(T) [1] for isomorphism types of prime models of a theory T, over finite sets, or their lattice fragments.

The description [29] of structures RK(T) for Ehrenfeucht quite o-minimal theories T implies that these structures, for the considered theories, form finite lattices LRK(T) consisting of $2^k \cdot 3^s$ elements and, in view of the main result of the paper [25], the number $I(T, \omega)$ of pairwise non-isomorphic countable models of T equals $3^k \cdot 6^s$, $k, s \in \omega$.

The Hasse diagrams illustrating these lattices LRK(T) are represented in Figures 1–9 for the following values k and s:

1) $k = 1$, $s = 0$;
2) $k = 0$, $s = 1$;
3) $k = 2$, $s = 0$;
4) $k = 3$, $s = 0$;
5) $k = 0$, $s = 2$;
6) $k = 0$, $s = 3$;
7) $k = 1$, $s = 1$;
8) $k = 2$, $s = 1$;
9) $k = 1$, $s = 2$.

Figure 1. $k = 1, s = 0$

Figure 2. $k = 0, s = 1$

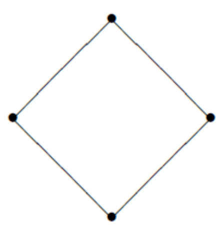

Figure 3. $k = 2, s = 0$

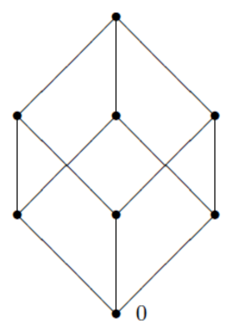

Figure 4. $k = 3, s = 0$

Theorem 4.1. Let T be an Ehrenfeucht quite o-minimal theory, $I(T,\omega) = 3^k \cdot 6^s$, $k, s \in \omega$. Then:

(1) $\mathrm{LRK}(T)$ *is a lattice;*

(2) $\mathrm{LRK}(T)$ *is a Boolean algebra* $\Leftrightarrow$ $k \geq 1$ *and* $s = 0$*; in such a case the Boolean lattice* $\mathrm{LRK}(T)$ *has a cardinality* 2^k*;*

(3) $\mathrm{LRK}(T)$ *is linearly ordered* $\Leftrightarrow$ $k + s \leq 1$.

Proof. Let $\Gamma = \Gamma_1 \cup \Gamma_2$ be a maximal independent set of nonisolated types in $S_1(T)$, where realizations of each type in Γ_1 generate three models, with prime one, and realizations of each type in Γ_2 generate six models, with prime one, $|\Gamma_1| = k$, $|\Gamma_2| = s$.

(1) We argue to show that $\mathrm{LRK}(T)$ is a lattice. Indeed, for isomorphism types $\widetilde{\mathcal{M}_1}$ and $\widetilde{\mathcal{M}_2}$ of prime model $\mathcal{M}_1$ and $\mathcal{M}_2$ over some finite sets A and B, respectively, we define sets $X, Y \subseteq \Gamma \times \{0,1\}$ defining these isomorphism types such that $X = \{(p,0) \mid \mathcal{M}_1 \models p(a)$ for some $a \in A$, and $|p(\mathcal{M}_1)| = 1$ or $p \in \Gamma_1\} \cup \{(p,1) \mid \mathcal{M}_1 \models p(a)$ for some $a \in A, |p(\mathcal{M}_1)| \geq \omega, p \in \Gamma_2\}$ and $Y = \{(q,0) \mid \mathcal{M}_2 \models q(b)$ for some $b \in B$, and $|q(\mathcal{M}_2)| = 1$ or $q \in \Gamma_1\} \cup \{(q,1) \mid \mathcal{M}_2 \models q(b)$ for some $b \in B, |q(\mathcal{M}_2)| \geq \omega, q \in \Gamma_2\}$. Then the isomorphism type for $\widetilde{\mathcal{M}_1} \wedge \widetilde{\mathcal{M}_2}$ corresponds to the set $U \subseteq \Gamma \times \{0,1\}$ consisting of all common pairs of X and Y, as well as all possible pairs $(p,0)$, if $(p,0) \in X$ and $(p,1) \in Y$, or $(p,1) \in X$ and $(p,0) \in Y$. And the isomorphism type for $\widetilde{\mathcal{M}_1} \vee \widetilde{\mathcal{M}_2}$ corresponds to the set $V \subseteq \Gamma \times \{0,1\}$ consisting of the following pairs:

i) all common pairs of X and Y,

ii) all pairs $(p,i) \in X$ such that $Y \cap \{(p,0),(p,1)\}\emptyset$,

iii) all pairs $(p,i) \in Y$ such that $X \cap \{(p,0),(p,1)\}\emptyset$,

iv) all pairs $(p,1)$ such that $(p,0) \in X$ and $(p,1) \in Y$, or $(p,1) \in X$ and $(p,0) \in Y$.

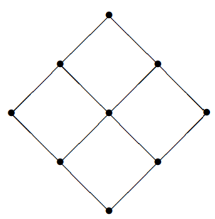

Figure 5. $k = 0, s = 2$

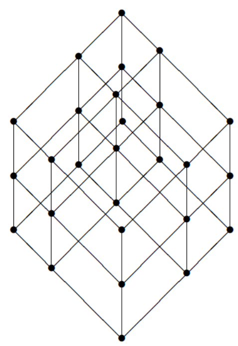

Figure 6. $k = 0, s = 3$

The defined correspondence witnesses that $\mathrm{LRK}(T)$ is a lattice.

(2) If $s \neq 0$ then $\mathrm{LRK}(T)$ is not a Boolean algebra by Stone Theorem, since the cardinality of each finite Boolean algebra equals 2^n for some $n \in \omega$ whereas $|\mathrm{LRK}(T)| = 2^k \cdot 3^s$. If $s = 0$ then $\mathrm{LRK}(T)$ is a Boolean algebra of a cardinality 2^k such that for isomorphism types $\widetilde{\mathcal{M}_1}$ and $\widetilde{\mathcal{M}_2}$ of prime models $\mathcal{M}_1$ and $\mathcal{M}_2$ over some finite sets A and B, respectively, and for sets $X, Y \subseteq \Gamma$ such that $X = \{p(x) \in \Gamma \mid \mathcal{M}_1 \models p(a)$ for some $a \in A\}$ and $Y = \{q(x) \in \Gamma \mid \mathcal{M}_2 \models q(b)$ for some $b \in B\}$, the isomorphism type $\widetilde{\mathcal{M}_1} \wedge \widetilde{\mathcal{M}_2}$ corresponds to the set $X \cap Y$, and the isomorphism type $\widetilde{\mathcal{M}_1} \vee \widetilde{\mathcal{M}_2}$ corresponds to the set $X \cup Y$.

(3) If $k + s \leq 1$ then $\mathrm{LRK}(T)$ is linearly ordered as shown in Figures 1 and 2. If $k + s > 1$ then $|\Gamma| > 1$ and for distinct types $p, q \in \Gamma$ the isomorphism types of models $\mathcal{M}_p$ and $\mathcal{M}_q$ are incomparable in $\mathrm{LRK}(T)$.

The description for distributions of disjoint unions of Ehrenfeucht theories and the arguments for the proof of Theorem 4.1 allow to formulate the following theorem modifying Theorem 4.1.

Theorem 4.2. Let T be a disjoint union of theories T_1 and T_2 in disjoint languages and having finite numbers $I(T_1,\omega)$ and $I(T_2,\omega)$ of countable models. Then:

(1) $\mathrm{LRK}(T)$ *is a* (*Boolean*) *lattice* $\Leftrightarrow$ $\mathrm{LRK}(T_1)$ *and* $\mathrm{LRK}(T_2)$ *are* (*Boolean*) *lattices;*

(2) $\mathrm{LRK}(T)$ *is linearly ordered* $\Leftrightarrow$ $\mathrm{LRK}(T_1)$ *and* $\mathrm{LRK}(T_2)$ *are linearly ordered, and*

$$\min\{I(T_1,\omega), I(T_2,\omega)\} = 1.$$

Proof. (1) If $\mathrm{LRK}(T)$ is a (Boolean) lattice, then $\mathrm{LRK}(T_1)$ and $\mathrm{LRK}(T_2)$ are (Boolean) lattices, since $\mathrm{LRK}(T_1)$ and $\mathrm{LRK}(T_2)$ are isomorphic to sublattices L_1 and L_2 of the lattice $\mathrm{LRK}(T)$, and elements/complements of elements in LRK(T) are represented as pairs of elements/complements of elements in L_1 and L_2. If LRK(T_1) and LRK(T_2) are (Boolean) lattices, then LRK(T) is a (Boolean) lattice again in view of aforesaid representation.

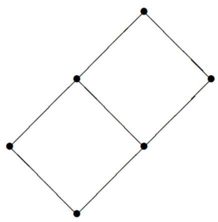

Figure 7. $k = 1, s = 1$

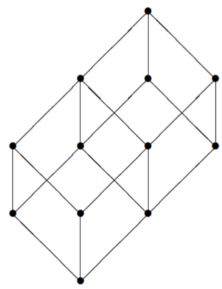

Figure 8. $k = 2, s = 1$

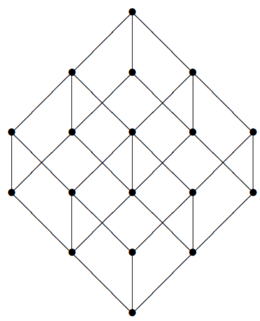

Figure 9. $k = 1, s = 2$

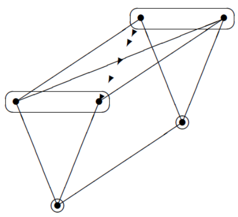

Figure 10. 6-Element diagram

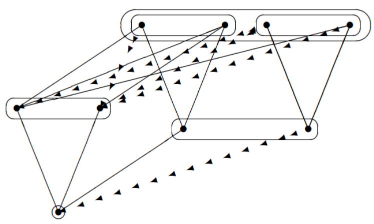

Figure 11. 9-Element diagram

(2) If LRK(T) is linearly ordered then LRK(T_1) and LRK(T_2) are linearly ordered, being isomorphic to substructures of LRK(T). Here T_1 or T_2 should be ω-categorical, since otherwise prime models over pairs (p_1, q_1) and (p_2, q_2) occur LRK(T)-incomparable, where $p_1, p_2 \in S_1(T_1)$, $q_1, q_2 \in S_1(T_2)$, p_1, q_2 are isolated, p_2, q_1 are nonisolated.

If structures LRK(T_1) and LRK(T_2) linearly ordered, and $\min\{I(T_1, \omega), I(T_2, \omega)\} = 1$, then LRK($T$) is linearly ordered, since LRK(T) $\simeq$ LRK(T_1) for $I(T_2, \omega) = 1$, and LRK(T) $\simeq$ LRK(T_2) for $I(T_1, \omega) = 1$.

In Figures 10 and 11 we illustrate Theorem by structures LRK(T) in [30], for disjoint unions of theories, which are not lattices.

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Теория модельдерінің гиперграфтарындағы құрылымдар туралы

Теория модельдерінің гиперграфиясы теорияларды, сонымен бірге графикалық объектілерді қоса, семантикалық объектілер туралы маңызды құрылымдық ақпаратты алуға мүмкіндік беретін объектілерге жатады. Осы мақалада құрылымдық, оның ішінде торлы, модельдік теорияның гиперграфтардың қасиеттері анықталып зерттелді. Теория үлгілерінің гиперграфты торын сипаттау, сондай-ақ теория түрлерінің изоморфизм түрлеріндегі құрылымдар берілген.

Кілт сөздер: модельдер гиперграфы, элементарлық теория, элементарлық ішкі құрылымдық жиын, тор құрылымы.

Б.Ш. Кулпешов, С.В. Судоплатов

О структурах в гиперграфах моделей теории

Гиперграфы моделей теории относятся к производным объектам, позволяющим получать существенную структурную информацию как о самих теориях, так и о сопутствующих семантических объектах, включая графовые объекты. В статье определены и исследованы структурные, в том числе решеточные свойства гиперграфов моделей теории. Дана характеризация решеточности гиперграфов моделей теории, а также структур на множествах типов изоморфизма теории.

Ключевые слова: гиперграф моделей, элементарная теория, элементарно подструктурное множество, решеточная структура.

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Forking and independence for fragments of Jonsson sets

The concept of independence plays a very important role in Model Theory for classification of a fixed complete theory. In this paper, we study the Jonsson theories, which, generally speaking, are not complete. For such theories, the concept of forking is introduced axiomatically in the framework of the study of the Jonsson subsets of the semantic model of this theory. Equivalence of forking by Shelah, by Laskar-Poizat and an axiomatically given forking for existential types over subsets of the semantic model of the Jonsson theory is given. Further, as and for complete theories, independence is defined through the notion of non-forking.

Keywords: Jonsson theory, semantic model, existential type, Jonsson set, a fragment of the Jonsson set, forking, independence.

One of the most important concepts of modern Model Theory is the concept of forking. With the help of this concept, we can evaluate the dependence of the properties of an element on each other in a first-order language. It should be noted that this concept was introduced by S. Shelah [1] to solve a very important problem of the spectrum of an arbitrary complete theory. Over time, experts in the theory of models, evaluating the depth and significance of the concept of forking, began to seek new approaches for its simpler explanation. One of the well-known sources in this direction is the well-known work of French mathematicians D.Laskar and B.Poizat [2], in which the concept of forking was redefined in the framework of a certain order. Later, other mathematicians observed that it is possible to consider the abstract properties of the independence of the model elements from each other and to associate this with the properties of the first order of the types of these elements for the subject of non-forking. In particular, as an example, we can cite the following monograph by D. Baldwin [3], where he considered a system of axioms that defines an abstract property of independence.

The study of Jonsson theories is inherently back to the tasks of the so-called «eastern» Model Theory, founded by Abraham Robinson, who lived on the eastern coast of the United States, unlike Alfred Tarski, who lived on the west coast of the United States. And accordingly, the tasks that were determined at the time by A.Tarsky's theoretical-model problems became the basis for the so-called «Western» Model Theory. All the main differences between these two trunk directions of Model Theory of that time can be found in the well-known book by J. Barwise [4].

The Jonsson theories, generally speaking, are not complete and the morphisms that serve them, as a rule, are isomorphic embeddings and homomorphisms. At the same time, the semantic aspect of these theories, in view of certain theoretical-model circumstances, reflects the class of existential-closed models of the Jonsson theory under consideration. In [5], homomorphisms in positive Model Theory were defined. In [6], a variant of the study of the Jonsson theories was proposed in the framework of the positive Model Theory. In an earlier work, A.R. Yeshkeyev [7] considered positive analogs of Jonsson theories and their particular cases - the Robinson theories.

Let's give the basic definitions necessary to understand the content of this article.

Definition 1 [4]. The theory T is called Jonsson if:

1) T has an infinite model;

2) T is inductive, i.e. T is equivalent to the set $\forall\exists$-propositions;

3) T has the joint embedding property (JEP), that is, any two models $\mathfrak{A} \models T$ and $\mathfrak{B} \models T$ are isomorphically embedded in a certain model $\mathfrak{C} \models T$;

4) T has the property of amalgamation (AP), that is, if for any $\mathfrak{A}, \mathfrak{B}, \mathfrak{C} \models T$ such that $f_1 : \mathfrak{A} \to \mathfrak{B}$, $f_2 : \mathfrak{A} \to \mathfrak{C}$ are isomorphic embeddings, exist $\mathfrak{D} \models T$ and isomorphic embeddings $g_1 : \mathfrak{B} \to \mathfrak{D}$, $g_2 : \mathfrak{C} \to \mathfrak{D}$ such that $g_1 f_1 = g_2 f_2$.

Definition 2 [8]. Let $\kappa \geqslant \omega$. The model $\mathfrak{M}$ of theory T is said to be κ-universal for T if every model T of cardinality is strictly less than κ is isomorphically embedded in $\mathfrak{M}$.

Definition 3 [8]. Let $\kappa \geqslant \omega$. The model $\mathfrak{M}$ of theory T is said to be κ-homogeneous for T if for any two models $\mathfrak{A}$ and $\mathfrak{A}_1$ of T, which are submodels of $\mathfrak{M}$, the cardinality is strictly less than κ, and the isomorphism

$f : \mathfrak{A} \to \mathfrak{A}_1$, for each extension $\mathfrak{B}$ of the model $\mathfrak{A}$, which is a submodel of $\mathfrak{M}$ and a model T of cardinality strictly less than κ, there exists an extension $\mathfrak{B}_1$ of the model $\mathfrak{A}_1$, which is a submodel of $\mathfrak{M}$, and an isomorphism $g : \mathfrak{B} \to \mathfrak{B}_1$ that extends f.

A homogeneous-universal model for T is a κ-homogeneous-universal model for T of cardinality κ, where $\kappa \geqslant \omega$.

Definition 4 [8]. The semantic model C of Jonsson theory T is the ω^+-homogeneous-universal model of theory T.

Definition 5 [8]. The Jonsson theory T is said to be perfect if its semantic model C is saturated.

The central concept of this paper is the notion of a fragment of the Jonsson set that was defined in [9] and some of its model-theoretic properties were considered in [10–12]. In this paper we carry over the main results from [13, 14], and, as can be seen from the following definition, the concept of the Jonsson set is very well coordinated with the concept of a basis of a linear space. We note that linear spaces are a particular case of modules, and the theory of modules is a Jonsson theory.

In definition 6 we changed the point a), in contrast to the definition of the Jonsson set in [15]. In the original definition there was a requirement of the existential definability of this set, now we require simply definability.

Definition 6 [8]. The set X is called the Jonsson set in the theory T if it satisfies the following properties:

a) X is a definable subset of C, where C is the semantic model of the theory T;

b) $dcl(X)$ is the carrier of some existentially closed submodel of C, where $dcl(X)$ is the definable closure of the set X.

Definition 7 [5]. We say that all $\forall\exists$-consequences of an arbitrary theory create a Jonsson fragment of this theory if the deductive closure of these $\forall\exists$-consequences is a Jonsson theory.

Consider the countable language L, complete for existential sentences the perfect Jonsson theory T in the language L and its semantic model C. Let X be the Jonsson set in T and M an existentially closed submodel of the semantic model C, where $dcl(X) = M$. Then let $Th_{\forall\exists}(M) = Fr(X)$, where $Fr(X)$ is the Jonsson fragment of the Jonsson set X.

Since the concept of forking is central to stability theory, it is natural to want to study it from different points of view. For this purpose, we first describe forking axiomatically. We recall the definition of forking.

Definition 8. a) It is said that formula $\varphi(\bar{x}, \bar{b})$ divide over A, if there exists a sequence $\langle \bar{b}_n : n < \omega \rangle$ and a number $k < \omega$, satisfying the following conditions: 1) $\bar{b}_n \equiv_A \bar{b}$, $n < \omega$; 2) $\{\varphi(\bar{x}, \bar{b}_n) : n < \omega\}$ k-inconsistent.

b) It is said that the type p (not necessarily complete) forks over A, if there exists a finite set Σ of formulas that are divisible over A such that $p \vdash \vee\{\varphi : \varphi \in \Sigma\}$;

Let T be Jonsson theory, $S^J(X)$ be the set of all existential complete n-types over X, that are compatible with T, for each finite n.

Definition 9. We say that the Jonsson theory T is J-λ-stable if for any T-existentially closed model A, for any subset X of the set A, $|X| \leqslant \lambda \Rightarrow |S^J(X)| \leqslant \lambda$. We will call the Jonsson theory J-stable if it is J-λ-stable for some λ.

Let $\mathcal{A}$ be the class of all subsets of the semantic model $\mathfrak{M}$, $\mathcal{P}$ be the class of all existential complete types, $JNF \subseteq \mathcal{P} \times \mathcal{A}$ is some binary relation. We write in the form of axioms some conditions imposed on *JNF*.

Axiom 1. If $(p, A) \in JNF$, $f : A \to B$ are isomorphic embeddings, then $(f(p), f(A)) \in JNF$.

Axiom 2. If $(p, A) \in JNF$, $q \subseteq p$, then $(q, A) \in JNF$.

Axiom 3. If $A \subseteq B \subseteq C$, $p \in S^J(C)$, then $(p, A) \in JNF \Leftrightarrow (p, B) \in JNF$ & $(p \upharpoonright B, A) \in JNF$.

Axiom 4. If $A \subseteq B$, $dom(p) \subseteq B$, $(p, A) \in JNF$, then $\exists q \in S^J(B)$ $(p \subseteq q$ & $(q, A) \in JNF)$.

Axiom 5. There exists a cardinal μ such that if $A \subseteq B \subseteq C$, $p \in S^J(B)$, $(p, A) \in JNF$, then $\left|\{q \in S^J(C) : p \subseteq q \text{ \& } (q, A) \in JNF\}\right| < \mu$.

Axiom 6. There exists a cardinal $\varkappa$ such that $\forall p \in \mathcal{P}$, $A \in \mathcal{A}$ if $(p, A) \in JNF$, then $\exists A_1 \subseteq A$, $(|A_1| < \varkappa$ & $(p, A_1) \in JNF)$.

Axiom 7. If $p \in S^J(A)$, then $(p, A) \in JNF$.

Let F be the fragment of some Jonsson set D, where D is a subset of the semantic model $\mathfrak{M}$ of some Jonsson theory T, i.e. $F = Th_{\forall\exists}(dcl(D))$, $dcl(D) = M' \in E_T$.

Theorem 1. The following conditions are equivalent:

1. In the theory F, the relation JNF satisfies axioms 1–7.

2. T^* is stable for any $p \in \mathcal{P}$, $A \in \mathcal{A}$ $((p, A) \in JNF \Leftrightarrow p$ does not fork over $A)$, where $T^* = Th(M')$.

Proof. It follows from Theorem 10 [14].

Consider the strengthening of Lemma 19.7 from [13]. For this we give the following known definitions.

Definition 10. If $A \subseteq M \cap N$, $p \in S(M)$, $q \in S(N)$, then $p \geqslant_A q$ means that $\forall \varphi(\bar{x}, \bar{y}) \in L(A)$ $(\exists \bar{m} \in M\ p \ni \varphi(\bar{x}, \bar{m}) \Rightarrow \exists \bar{n} \in N\ q \ni \varphi(\bar{x}, \bar{n}))$; $p \sim_A q$ means that $p \geqslant_A q$ & $q \geqslant_A p$.

$$[p]^A \rightleftharpoons \{q : \exists N \models T,\ q \in S(N),\ p \sim_A q\}.$$

It is easy to understand that the relation $\geqslant_A$ induces an analogous relation between classes, which is a partial order relation. If $A = \varnothing$, then the index A for $\geqslant_A$ and $\sim_A$ will be omitted. Each equivalence class in $\sim_A$ is uniquely determined by the set of formulas from $L(A)$, representable in each type of this class.

Definition 11. The formula $\varphi(\bar{x}, \bar{y}) \in L(A)$ is said to be representable in $p \in S(M)$, $A \subseteq M$ если $\exists \bar{m} \in M$ $(p \vdash \varphi(\bar{x}, \bar{m}))$. Obviously, in this way, the number of equivalence classes in $\sim_A$ is at most $2^{|L(A)|} = 2^{|L| \cdot |A|}$.

The equivalence classes with respect to $\sim_A$ will be denoted by ξ^A. If $p \in S(A)$, then Ω_p denotes the partially ordered set $\langle \{\xi^A : \exists M \supseteq A,\ p_1 \in S(M),\ p \subseteq p_1,\ p_1 \in \xi^A\}; \geqslant \rangle$.

The following results are known, their proof can be extracted from [13].

Lemma 1. In Ω_p there is a maximal element.

Lemma 2. If T is stable, $p \in S(M)$, $M \subseteq B$, $p' \in S(B)$ is the successor of p, then $p \sim_M p'$.

Lemma 3. If T is stable, $p \in S(M)$, $M \subseteq B$, $q \in S(B)$, $p \subseteq q$ and $p \sim_M q$, then q is the successor of p.

Lemma 4. If $A \subseteq M \cap N$, $p \in S(M)$, $q \in S(N)$, $p \upharpoonright A = q \upharpoonright A$ and p, q do not fork over A, then $p \sim_A q$.

Lemma 5. If T is stable, $p \in S(A)$, then Ω_p has a unique maximal (i.e., greatest) element.

The following definition belongs to A.E.Yeshkeyev.

If T is a J-stable, existentially complete Jonsson theory, $p \in S^J(A)$, then $\beta^J(p)$ is the largest element of Ω_p.

Now we can introduce the following relation *JNFLP* (Jonsson non-forking by Lascar-Poizat) on $\mathcal{P} \times \mathcal{A}$.

Definition 12. Let T be a J-stable, existentially complete Jonsson theory.

1. If $p \in S^J(B)$, $A \subseteq B$, then $(p, A) \in JNFLP \Leftrightarrow \beta^J(p) = \beta^J(p \upharpoonright A)$.

2. If p is an arbitrary existential type, then $(p, A) \in JNFLP \Leftrightarrow$ there exists a $p' \in S^J(A \cup dom(p))$ such that $p \subseteq p'$ and $(p', A) \in JNFLP$.

Theorem 2. In the J-stable existentially complete Johnson theory, the relation *JNFLP* satisfies axioms 1–7.

The axioms 1, 2, 3, 4, 7 are trivially verified. Axiom 6 is satisfied for $\varkappa = |L|^+$. Suppose the contrary. Let $p \in S^J(A)$ and $\forall A_1 \subseteq A$, if $|A_1| < \varkappa$, then $(p, A_1) \notin JNFLP$. Obviously, $|A| \geqslant \varkappa = |L|^+$. There exists a sequence $\langle A_\alpha : \alpha < |L|^+ \rangle$ such that $|A_\alpha| \leqslant |L|$, $A_\alpha \subseteq A_\beta$ for $\alpha < \beta < |L|^+$ and $(p \upharpoonright A_{\alpha+1}, A_\alpha) \notin JNFLP$. Let $M \supseteq \bigcup\limits_\alpha A_\alpha$ be an arbitrary existentially closed submodel of the semantic model of the theory T of cardinality $|T|$, $p_\alpha \supseteq p \upharpoonright A_\alpha$ such that $p_\alpha \in S^J(M)$ and $[p_\alpha]^{A_\alpha}$ is the largest element in $\Omega_{(p \upharpoonright A_\alpha)}$. Then $\langle \{p_\alpha : \alpha < |L|^+\}; \geqslant \rangle$ is strictly decreasing sequence. Hence, there exist the formulas $\varphi_\alpha(\bar{x}, \bar{y}_\alpha) \in L$, $\alpha < |L|^+$ such that $\varphi_\alpha(\bar{x}, \bar{y}_\alpha)$ is representable in p_α, but is not representable in $p_{\alpha+1}$. It is clear that for $\alpha \neq \gamma$ $\varphi_\alpha(\bar{x}, \bar{y}_\alpha) \neq \varphi_\gamma(\bar{x}, \bar{y}_\gamma)$ since there is no power set $> |L|$ of formulas of the language L. Contradiction.

Axiom 5 is satisfied for $\mu = \left(2^{|T|}\right)^+$. In fact, let $p \in S^J(B)$, $(p, A) \in JNFLP$, $A \subseteq B \subseteq C$. By axiom 6, there exists is $A_0 \subseteq A$ such that $|A_0| \leqslant |L|$, $(p, A_0) \in JNFLP$.

1 case: Let C be an existentially closed submodel of the semantic model $\mathfrak{M}$ of the theory T. $C \models T$. Let $A_0 \subseteq M_0 \preccurlyeq_{E_1} C$. If $p' \in S^J(C)$, $p \subseteq p'$, $(p', B) \in JNFLP$, then $(p', A_0) \in JNFLP$. Therefore, $(p', M_0) \in JNFLP$. Hence p' is the successor of $p' \upharpoonright M_0$. There are no more such types than $|S^J(M_0)| \leqslant 2^{|T|}$.

2 case: $C \not\models T$. Then we take $N \in E_T$ such that $N \supseteq C$. $|\{q \in S^J(C) : p \subseteq q \ \&\ (q, A) \in JNFLP\}| \leqslant$ $\leqslant |\{q \in S^J(N) : p \subseteq q \ \&\ (q, A) \in JNFLP\}| \leqslant 2^{|T|}$.

The following theorem is an extension of Theorem 19.8 of [13] and is the main result of this paper.

Theorem 3. If F is J-stable, then the concepts of *JNF* and *JNFLP* are the same.

The proof follows from Theorem 1 and Theorem 2.

Next, we define the concept of independence. Non-forking extensions will in some sense be «free», i.e. independent. So in what follows we will talk about the concept of forking when we are dealing with some type in the Jonsson theory, which satisfies the relation *JNF*. We will follow the following definition.

Definition 13. We say that $\bar{a}$ does not depend on B over A if $tp(\bar{a}/A)$ does not fork over $A \bigcup B$. We denote this fact by $\bar{a} \bot_A B$.

In particular, one can note that the concept of independence for Jonsson sets has many good properties: monotonicity, transitivity, finite basis, symmetry, etc., similarly to complete theories.

Forking, as in Theorem 1, can be used to give the notion of independence in J-ω-stable theories [8].

Summarizing, we note that in [14] was obtained a result, where for the Jonsson theories the binary relation *JNF* was determined and it was proved that the notion *JNF* in the class of J-stable theories coincides with the concept of non-forking in stable theories in the sense of S. Shelah. In this paper we obtain the following result: for a fixed fragment of a certain Jonsson subset of the semantic model of some fixed J-stable existentially complete Jonsson theory, we prove both equivalences of the binary relations *JNF* and *JNFLP*. Moreover, for *JNF* in this class of theories, we have obtained a more detailed version of Theorem 10 [14]. Namely, we get the assertion that the binary relation *JNF* is also equivalent to the condition obtained in [13] with respect to some definable closure of the Jonsson subset of the semantic model of the Jonsson theory under consideration. The results obtained with these binary relations provide an additional opportunity to characterize the behavior of existential types in the framework of the study of the examined fragment of the Jonsson subset of the semantic model of this Jonsson theory.

All concepts that are not defined here can be extracted from [8].

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О.И. Ульбрихт

Йонсондық жиындардың фрагменттері үшін форкинг пен тәуелсіздік

Тәуелсіздік ұғымы бекітілген толық теорияның модельдерді классификациялау теориясында өте маңызды рөлін атқарады. Мақалада йонсондық теориялар қарастырылды, олар, жалпы айтқанда, толық емес болып табылады. Мұндай теориялар үшін форкинг ұғымы берілген теорияның семантикалық моделінің йонсондық ішкі жиындарының зерттеу аясында аксиоматикалық түрде енгізіледі. Шелах, Ласкар-Пуаза форкингі және йонсондық теорияның семантикалық моделінің йонсондық ішкі жиындарының экзистенциалды түрлері үшін аксиоматикалық түрде берілген форкингінің эквиваленттілігі келтірілді. Әрі қарай, толық теориялардағыдай, тәуелсіздік форкинг ете алмайтындылылық ұғымы арқылы анықталады.

Кілт сөздер: йонсондық теория, семантикалық модель, экзистенциалды түр, йонсондық жиын, йонсондық жиынның фрагменті, форкинг, тәуелсіздік.

О.И. Ульбрихт

Форкинг и независимость для фрагментов йонсоновских множеств

Понятие независимости играет очень важную роль в теории классификации моделей фиксированной полной теории. В статье изучены йонсоновские теории, которые, вообще говоря, не полны. Для таких теорий аксиоматически вводится понятие форкинга в рамках изучения йонсоновских подмножеств семантической модели данной теории. Приведена эквивалентность форкинга по Шелаху, Ласкара-Пуаза и аксиоматически заданного форкинга для экзистенциальных типов над подмножествами семантической модели йонсоновской теории. Далее, как и для полных теорий, определяется независимость через понятие неформкуемости.

Ключевые слова: йонсоновская теория, семантическая модель, экзистенциальный тип, йонсоновское множество, фрагмент йонсоновского множества, форкинг, независимость.

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Syntactic similarity of definable closures of Jonsson sets

In the framework of the classification of the Jonsson theories concept of interpretability and admissibility in the language of the semantic triple of the Jonsson theory was considered. A description of the syntactic and semantic similarity of perfect fragments of Jonsson subsets of the semantic model of the existential-prime convex Jonsson theory was obtained. Some model-theoretic properties for Jonsson's theories are considered. Such theories, as group theory, the theory of Abelian groups, the theory of Boolean algebra, the theory of ordered sets, the theory of polygons, and many others satisfies Jonsson's properties.

Keywords: Jonsson theory, perfect Jonsson theory, semantic model, Jonsson set, fragment of Jonsson set, syntactic and semantic similarity, existentially prime model.

The study of Jonsson theories is one of the interesting problems of the classical model theory. In the works [1, 2] you can find the main aspects of this type of research. One of the important concepts of model theory is the concept of definability (interpretability) of one algebraic system in another. It is said that the algebraic system $\mathfrak{B} =< B, R_i, i \in I >$ is definable on $\mathfrak{A} =< A, P_j, j \in J >$, if exists such formular relations $\Phi_i\ i \in I$ in language $\mathfrak{A}$ that $< A;\ \Phi_i,\ i \in I >$ is an isomorphic $< B;\ R_i,\ i \in I >$. In the course of the development of model theory, this notion was generalized, and the most general (at present) definition can be formulated as follows. If $\mathfrak{A}$ algebraic system, $n < \omega$, $B \subseteq A^n$, λ is the cardinal then B is called τ_λ-subset if exists such n-type $p(x_1, ..., x_n)$ over $\emptyset$ of language of system $\mathfrak{A}$, such $|p(x_1, ..., x_n)| < \lambda$ and B consists of all n of A^n, realising $p(x_1, ..., x_n)$ in $\mathfrak{A}$. Obviously, τ_λ-subsets are invariant relatively automorphisms. Therefore, τ_λ it can be considered as a way of isolating a certain class of invariant subsets of algebraic systems. If $\mathfrak{A}$, $\mathfrak{B}$ - algebraic systems $G = Aut($ A), then we say that $\mathfrak{B}$ is $\tau_\lambda-$interpreted in $\mathfrak{A}$, if $\mathfrak{B}$, is $\tau_\lambda-$interpreted in pure pair (A, G). If $\lambda = \omega$ then usually instead of τ_λ – interpretability says formally (or elementarily) interpreted (definable). The problem of interpretability can be considered through other similar concepts, for example, syntactic and semantic similarity.

Definition 1. A theory T is called Jonsson if:

1) the theory T has an infinite model;
2) the theory T is inductive;
3) the theory T has the joint embedding property JEP;
4) the theory T has the amalgamation property AP.

Definition 2. The set X is said to be Jonsson in theory T, if it satisfied to following properties:

1) X is definable subset of C, where C is semantic model of theory T;
2) $dcl(X)$ is support of some existentially closed submodel C, where $dcl(X)$ is definable closure of X.

Definition 3. We say that all $\forall\exists$ consequences of an arbitrary theory form the Jonsson fragment if the deductive closure of these $\forall\exists$ consequences is the Jonsson theory.

Let T is an arbitrary Jonsson theory, then $E(T) = \cup_{n<\omega} E_n(T)$, where $E_n(T)$ is a lattice of $\exists$ formulas with n free variables, T^* is a center of Jonsson theory T, i.e. $T^* = Th(C)$, where C is semantic model of Jonsson theory T in the sense of [3].

Definition 4 [4]. Let T_1 and T_2 are Jonsson theories. We say, that T_1 and T_2 are Jonsson's syntactically similar, if exists a bijection $f : E(T_1) \longrightarrow E(T_2)$ such that:

1) restriction f to $E_n(T_1)$ is an isomorphism of lattices $E_n(T_1)$ and $E_n(T_2)$, $n < \omega$;
2) $f(\exists v_{n+1}\varphi) = \exists v_{n+1} f(\varphi)$, $\varphi \in E_{n+1}(T)$, $n < \omega$;
3) $f(v_1 = v_2) = (v_1 = v_2)$.

We would like to give some examples of syntactic similarity of certain algebraic examples. For this, we recall the basic definitions associated with these examples following denotions from B. Poizat [5].

A Boolean ring is an associative ring with identity, in which $x^2 = x$ for any x is called a Boolean ring; then, we have also $(\mathrm{x + y})^2 = x^2 + xy + yx + y^2 = x + xy + yx + y$ and besides $(x + y)^2 = x + y$; therefore

$xy + yx = 0$ for an arbitrary x, y; $x^2 + x^2 = 0$ means, $x + x = 0$, for any x or $x = -x$; hence the Boolean ring has characteristic 2 and, since $xy = -yx = yx$, it is commutative.

To axiomatize this concept, we introduce a language containing two symbols of constants 0 and 1, two symbols of binary relations + and ·

We write down some universal axioms, expressing, that A is the Boolean ring, without forgetting thus $0 \neq 1$. In the Boolean ring we will define two binary operations $\wedge$ and $\vee$, and unary operation $\neg$ as follows: $x \wedge y = x \cdot y$; $x \vee y = x + y + xy$; $\neg x = 1 + x$.

It is easy to verify, that the following are true for all x, y and z:

– (de Morgan's laws or duality): $\neg(\neg x) = x$, $\neg(x \wedge y) = \neg x \vee \neg y \neg (x \vee y) = \neg x \wedge \neg y$;

– $x \vee x = x \wedge x = x$;

– (associativity $\wedge$): $(x \wedge y) \wedge z = x \wedge (y \wedge z)$;

– (associativity $\vee$): $(x \vee y) \vee z = x \vee (y \vee z)$;

– (distributivity $\wedge$ over $\vee$):$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$;

– (distributivity $\vee$ over $\wedge$): $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$;

– (commutativity$\wedge$ over $\vee$): $x \wedge y = y \wedge x$,$x \vee y = y \vee x$;

– $x \wedge \neg x = 0, x \vee \neg x = 1$;

– $x \wedge 0 = 0, x \vee 0 = \mathrm{x}, x \wedge 1 = x, x \vee 1 = 1$;

– $0 \neq 1, \neg 0 = 1, \neg 1 = 0$.

A structure in language $0, 1, \neg, \wedge, \vee$ satisfying to these universal axioms is called a Boolean algebra.

Fact 1 [5]. In each Boolean ring one can interpret a certain Boolean algebra.

Proof. With the Boolean ring A we have connected some Boolean algebra $b(A)$; the converse is also true: $x \cdot y = x \wedge y$, $x + y = (x \vee y) \wedge (\neg x \vee \neg y)$, then we receive the Boolean ring $a(B)$; and besides $a(b(A)) = A$, $b(a(B)) = B$. Thus we see, that up to a language, the Boolean ring and Boolean algebras have the same structures, the Boolean ring canonically is transformed into a Boolean algebra and vice versa, transformations in both directions are carried out using quantifier free formulas.

The following example connects Boolean algebras with Abelian groups. In work [6], conditions were found for the cosemanticness of Abelian groups.

Fact 2 [7]. In each Boolean algebra one can interpret an Abelian group.

Proof. In Boolean algebra A we suppose $a + b = (a \wedge b') \vee (a' \wedge b)$.

$[A, +]$ is Abelian group, in which each not unit element has an order 2.

The element 0 is group unit in G, and each element x is reciprocal to itself: $x + x = 0$ for all $\mathrm{x} \in \mathrm{A}$.

We state the obtained results.

Let's denote through T_{BA}, T_{BR}, T_{AG} accordingly theories in their signatures (they are different) of Boolean algebras, Boolean rings and Abelian groups.

Lemma 1. T_{BA}, T_{BR}, T_{AG} are examples of Jonsson theories.

Proof. T_{BA} and T_{BR} from [4], T_{AG} from [6].

Theorem 1. Theories T_{BA} and T_{BR} are syntactically similar, and mutually interpreted among themselves, as for complete theories and for Jonsson theories.

Proof. Follows from the fact 1.

Theorem 2. Theory T_{BA} is interpreted in theory T_{AG}, as for complete theories and for the Jonsson theories.

Proof. Follows from Fact 2 and Theorem 1.

Let L be a countable first-order language and T is some inductive theory in this language, E_T and AP_T are denoting correspondingly the following classes of this theory: class of all existentially closed models and class of all algebraically prime models.

Definition 5. The inductive theory T called existential-prime (EP), if it has a algebraically prime model and $AP_T \cap E_T \neq \emptyset$.

Definition 6. The theory T is called convex (C) if for any model $\mathfrak{A}$ and any family $\{\mathfrak{B}_i | i \in I\}$ of its substructures, which are models of the theory T, the intersection $\bigcap_{i \in I} \mathfrak{B}_i$ is a model theory T. It is assumed that this intersection is not empty. If this intersection is never empty, then the theory is called the strongly convex (SC).

An inductive theory is called an existentially prime strongly convex theory if it satisfies the above definitions simultaneously and is denoted by $EPSC$.

Let X be the Jonsson set in the theory T and M is existentially closed submodel semantic model $\mathfrak{C}$, where $dcl(X) = M$. Then let $Th_{\forall\exists}(M) = Fr(X)$, $Fr(X)$ is Jonsson fragment of Jonsson set X. Let A_1 and A_2

are Jonsson subset of the semantic model the some of Jonsson $EPSC$-theory. Where $Fr(A_1)$ and $Fr(A_2)$ are fragments of Jonsson sets A_1 and A_2.

In the work [8] was obtained the result on syntactically similarity in the frame of EPPCJ theories in some enrichment. The class of EPPCJ theories is the subclass of class of all Jonsson theories. Now we are considering the following result for Jonsson theoreis without any enrichment. We have the following result.

Let T be $\exists$ -complete perfect Jonsson theory and $Fr(A_1)$ and $Fr(A_2)$ be fragments of A_1 and A_2 correspondingly, where A_1 and A_2 are Jonsson subset of the semantic model for theory T.

Even if given theory is $\exists$ -complete perfect Jonsson theory, its fragments can be not perfect.So we will be demand perfectness for fragments of the following theorem.

Theorem 3. Let $Fr(A_1)$ and $Fr(A_2)$ are $\exists$-complete perfect Jonsson theories. Then following conditions are equivalent:

1) $Fr(A_1)$ and $Fr(A_2)$ are J-syntactically similar as Jonsson theories [9];

2) $(Fr(A_1))^*$ and $(Fr(A_2))^*$ are syntactically similar as the complete theories [9], where $(Fr(A_1))^*$ and $(Fr(A_2))^*$ respectively be the centers of fragments of considered sets A_1, A_2.

Proof. We also need the following facts.

Fact 3 [10]. For any complete for the existential sentences Jonsson's theory T the following conditions are equivalent:

1) T is perfect;

2) T^* model-complete.

Fact 4 [10]. For any complete for the existential sentences Jonsson's theory T the following conditions are equivalent: are equivalent:

1) T^* model-complete;

2) For each $n < \omega$, $E_n(T)$ is a Boolean algebra, where $E_n(T)$ is a lattice of existential formulas with n free variables.

We note that by the perfectness of $Fr(A_1)$ and $Fr(A_2)$ implies that $(Fr(A_1))^*$ and $(Fr(A_2))^*$ are J Jonsson's theory.

We will show 1) $\Rightarrow$ 2). We have that for every $n < \omega$, $E_n(Fr(A_1))$ is an isomorphic to $E_n(Fr(A_2))$. Let this is an isomorphism of f_{1n}.By the hypothesis of the theorem and facts 3, 4 for every $n < \omega$, $E_n(Fr(A_1))$ and $E_n(Fr(A_2))$ are a Boolean algebras. But due to the perfection $Fr(A_1)$ and $Fr(A_2)$ follow that $(Fr(A_1))^*$ and $(Fr(A_2))^*$ are model-complete by virtue of fact 3, and so for each $n < \omega$, for any formula $\varphi(\bar{x})$ of $F_n((Fr(A_1))^*)$ by Corollary 1 there is a formula $\psi(\bar{x})$ of $E_n((Fr(A_1))^*)$ so that in $(Fr(A_1))^* \models \varphi \leftrightarrow \psi$. Because the theory of $Fr(A_1)$ is complete for existential sentences and $E_n(Fr(A_1)) \subseteq E_n((Fr(A_1))^*)$ (as $Fr(A_1) \subseteq (Fr(A_1))^*$), follow that $E_n(Fr(A_1)) = E_n((Fr(A_1))^*)$. Due to the fact that theory of $Fr(A_2)$ is complete for existential proposals and $E_n(Fr(A_2)) \subseteq E_n((Fr(A_2))^*)$ (as $Fr(A_2) \subseteq (Fr(A_2))^*$), follow that $E_n(Fr(A_2)) = E_n((Fr(A_2))^*)$. For each $n < \omega$, for each $\varphi_1(\bar{x})$ of $F_n((Fr(A_1))^*)$, we define the following mapping between the $F_n((Fr(A_1))^*)$ and $F_n((Fr(A_2))^*)$: $f_{2n}(\varphi_1(\bar{x})) = f_{1n}(\psi_1(\bar{x}))$, where $(Fr(A_1))^* \models \psi_1 \leftrightarrow \varphi_1$, $\psi_1 \in E_n(Fr(A_1))$. It is easy to understand, that by virtue of properties of f_{1n} and above what has been said, f_{2n} is a bijection, an isomorphism between $F_n((Fr(A_1))^*)$ and $F_n((Fr(A_2))^*)$. Consequently, $(Fr(A_1))^*$ and $(Fr(A_2))^*$ are syntactically similar (in the sense of [6]).

We show 2) $\Rightarrow$ 1). Is trivial, since $F_n((Fr(A_1))^*)$ an isomorphic to $F_n((Fr(A_2))^*)$ for each $n < \omega$, and by the hypothesis of the theorem and the facts 3, 4 this an isomorphism extends to all subalgebras.

All concepts that are not defined in this article can be extracted from [4].

Lemma 2. Any two cosemantic Jonsson's theories are J — semantically similar.

Proof. Follows from the definitions.

Lemma 3. If two perfect $\exists$ — complete of Jonsson's theories are J — syntactically similar, then they are J — semantically similar.

Proof. It follows from [9, Prop. 1] and above what was said.

Definition 7. A property (or a notion) of theories (or models, or elements of models) is called semantic if and if it is invariant relative to semantic similarity.

Let us recall the definition of polygon.

Definition 8. By polygon over monoid S we mean a structure with only unary functions $\langle A; f_{\alpha:\alpha\in S}\rangle$ such that:

(i) $f_e(a) = a, \forall a \in A$, where e is the unit of S;

(ii)$f_{\alpha\beta}(a) = f_\alpha(f_\beta(a)), \forall\alpha, \beta \in S, \forall a \in A$.

And now we can formulate the main result of this job.

Theorem 4. For each ∃ - complete perfect Jonsson J theory there exists a J syntactically similar ∃ - complete perfect Jonsson's J theory of polygons, such that its center is model complete.

Proof. It follows from theorems [11, Th.7, Th.8] and [9, Th.4, Th.5].

As we know from [9] the following Proposition 1 is true for any complete theory, so we will be intrested such properties from this Proposition 1 in the frame of Jonsson theories and to research the notion of semantic similarity of Jonsson's theories. Recall the content of the Proposition 1.

Proposition 1 [9]. The following properties and notions are semantic:

(1) type;

(2) forking;

(3) λ-stability;

(4) Lascar rank;

(5) Strong type;

(6) Morley sequence;

(7) Orthogonality, regularity of types;

(8) $I(\aleph_\alpha, T)$ — the spectrum function.

Finally we can note that all above properties from Proposition 1 will be semantic also in the frame of Jonsson theory. The proof trivial follows from Proposition 1 using Jonsson analogues of Proposition' main notions.

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Г.А. Уркен

Йонсондық жиындардың анықталған тұйықтамалардың синтаксистік ұқсастылығы

Йонсондық теорияның аясында интерпретациялау және рұқсатылық ұғымы йонсондық теорияның семантикалық үштік тілінде қарастырылды. Экзистенционалды-жай дөңес йонсондық теорияның семантикалық моделінің кемел йонсондық ішкі жиындар фрагментінің синтаксистік және семантикалық ұқсастылық сипаттамасы алынды. Йонсондық теорияның кейбір теория-модельдік қасиеттері зерттелді. Йонсондық теорияның қасиеттерді группалар теориясы абелдік группалар теориясы, бульдік алгебра теориясы, реттелген группалар теориясы, полигондар теориясы және тағы басқа теориялар қанағаттандырады.

Кілт сөздер: йонсондық теория, кемел йонсондық теория, семантикалық модель, йонсондық жиын, йонсондық жиынның фрагменті, семантикалық және синтаксистік ұқсастылық, экзистенционалдыжай модель.

Г.А. Уркен

Синтаксическое подобие определимых замыканий йонсоновских множеств

В рамках классификации йонсоновских теорий рассмотрено понятие интерпретируемости и допустимости на языке семантической тройки йонсоновской теории. Получено описание синтаксического и семантического подобий совершенных фрагментов йонсоновских подмножеств семантической модели экзистенциально-простой выпуклой йонсоновской теории. Рассмотрены некоторые теоретико-модельные свойства для йонсоновских теорий. Йонсоновским свойствам удовлетворяют такие теории, как теория групп, теория абелевых групп, теория булевой алгебры, теория упорядоченных множеств, теория полигонов и другие.

Ключевые слова: йонсоновская теория, совершенная йонсоновская теория, семантическая модель, йонсоновское множество, фрагмент йонсоновского множества, семантическое и синтаксическое подобие, экзистенциально-простая модель.

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МЕХАНИКА
MECHANICS

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Automation research of thermal and physical characteristics of particulate-filled epoxy composites

Thermal-physical characteristics of filled epoxy composites were investigated by means of the developed device which allowed to do computer-aided research within defined temperature range. Forward and reverse run of relative linear elongation factor of the sample ϵ(T,q)=Δl/l at heating rate 1,5-2 K/min. and cooling rate 2,5-3 K/min. for the composites with particulate filler has been studied. Laplacian operator to scalar field *eu* and *ed* was used, which enables to derive stationarity or quasi-stationarity 2D areas of relaxation of macromolecular stress. Automation research will allow to determine the parameters of temperatures and filler content corresponding to minimal stress values. In the first approximation these are the following areas: 30<q<50, 385<T<395 K and 60<q<90, 340<T<380 K. Relaxation processes at the above-mentioned parameters are the most intensive in the composite. Such composites have the lowest residual stress.

Keywords: composite, epoxy matrix, thermal coefficient of linear expansion, relaxation processes, automation research.

Introduction

Nowadays, composite polymer materials are used not only as constructional materials but as coatings of various functions as well [1]. For the latter an important indicator of operational characteristics of composition coatings is thermal coefficient of linear expansion (TCLE). In most epoxy binders-based materials particulate fillers of various nature, shape and size are used [2]. Mechanism of particulate fillers influence on heat characteristics of polymer composites hasn't been investigated properly. A particulate filler when being put into epoxy composite causes the structural change due to the external surface layer formation between the filler hard surface and the binder [3]. It results in change of thermal-physical characteristics of epoxy binders-based heterogeneous systems, including thermal coefficient of linear expansion, thermal resistance and heat conductivity. While the material is being formed the filler hard surface (it depends how active it is with the binder) makes a big difference in structure and volume of external surface layers. When a composite material is being heated the particulate filler restrains its expansion that is important in operation under thermal cyclic load conditions. The relationship between physical nature of a filler in a composite material and TCLE value has been proved [4]. Taking into consideration the difference in thermal-physical characteristics of matrix and filler and activity coefficient of particulate fillers, it's possible to control the whole heat- physical characteristics of composites to change the volume and structure of surface layers. According to modern beliefs of some authors, boundary inlays may redistribute the stress in the system «matrix-filler» [5, 6]. Stress state formation of meta-polymer systems on the phase division boundary both during and after the composite formation affects heat- physical characteristics of the material. High heat load in such layers may cause micro cracks in the material [5]. While using particles of the same dispersion the probability of cracks spread under performance conditions is increasing. Stress on the crack top can be lower due to plasticizers use. As we know, matrix-filler relation increase provides the improvement of physical-mechanical materials, including the case when dispersed magnetic fillers are added [6].

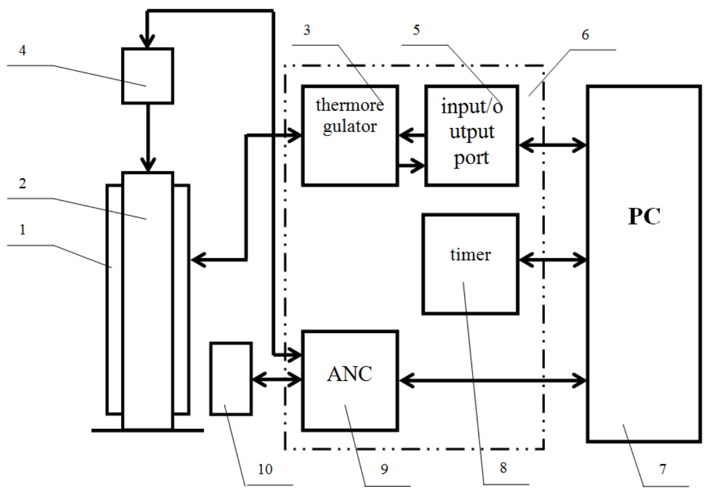
Thus, reasonable choice of plasticizers and structurally active filler (ferromagnetic, paramagnetic, diamagnetic materials) is one of the most efficient ways to solve the problem of properties of polymer composite materials being used as coatings. The use of epoxy binders of high adhesive strength to metal base, physical-mechanical characteristics and workability at coatings forming on long surfaces of complex contour is a promising direction for the development of epoxy composites of various functions operating under thermal scraping conditions.

Results and discussion

Low molecular epoxide resin ЕД-16 (ГОСТ 10687-76) was used as a binder for polymer composites formation. Aliphatic resin ДЕГ-1 (ТУ 6-05-1645-73) was added into the binder as a plasticizer to improve physical-chemical and technological properties of the matrix. Polyethylene polyamine (ТУ 6-02-594-73) was used as a hardener enabling to form the material at room temperature. Dispersed ferromagnetic (red mud), paramagnetic (TiC) and diamagnetic (Al_2O_3) powders were used at polymer systems filling. Polymer composites formation was taking place at temperature 293±2 K for 24 hours followed by further thermal processing at 433±2 K for 2 hours. After that the samples were kept for 48 hours at temperature 293±2 K. Then the study of thermal properties of composite materials was conducted.

Dilatometer investigation of polymer composites has demonstrated that TCLE value of composite material depends on the filler concentration and nature. It was determined that at temperature rise the relaxation of residual stress in materials is taking place due to the change of macromolecules conformation set in the composite during surface layers' formation near the filler hard surface. Relaxation behavior of the composite at thermal expansion corresponds to hysteresis form at heating and cooling cycle. In this way redistribution of internal stress has been determined which takes place in the system «epoxide matrix-particulate filler» that is in accord with the results of work [7, 8].

TCLE was studied by means of the developed device which allowed to do computer-aided research within defined temperature range (Fig. 1). Thermal-physical characteristics of epoxy composites were determined according to the sample length change at temperature variation under stationary conditions (GOST 15173-70). The device consists of heater (1). In the heating area of the sample (2) the temperature was regulated by temperature controller (3), which provided the defined heating rate. The sample stretching was recorded by motion sensor (4). The temperature in the heater was measured by thermocouple (5) and was sent by analog-to-number converter (ANC) to PC (7). Heating control signal was transmitted to the thermal regulator by input/output port (8). The operation of the whole input/output board of discrete signals (9) was synchronized by the timer (10).



1 – heater; *2* – sample; *3* – sample stretching meter; *4* – displacement transducer; *5* – thermocouple; *6* – analog-to-number converter; *7* – personal computer; *8* – input/output port; *9* – timer; *10* – input/output board of discrete signals

Figure 1. Device design for the TCLE samples study

According to modern views on physics and chemistry of polymers, active dispersed particles adding whilst epoxy composites formation resulted in rigid surface layers building on the phase boundary. The parameters of such layers depend on the number of physical nodes, i.e. physical joining level [9, 10]. If temperature rises higher the vitrification temperature, physical nodes are ruined [11]. It should be admitted that in this case macromolecules which are near the hard surface of the filler may form migrating physical nodes. Flexibility increase of the latter takes place due to the change of macromolecules conformation set of the binder. According to Sperling research the above-mentioned nodes failure has been observed between transversal links of macromolecular chains. Such nodes are restored when being cooled at the temperature lower vitrification matrix temperature. It should be noted that material generation is taking place thanks to physical joints formation with catalytically-active centers on the dispersed fillers surface due to macromolecules recombination in the binder material. Thus, polymer cross-linking yield is increasing [12].

Forward and reverse run of relative linear elongation factor of the sample $\epsilon(T,q)=\Delta l/l$ at heating rate 1,5-2 □K/min and cooling rate 2,5-3 □K/min for the composites with particulate filler TiC has been studied. Time variation of temperature T most probably can be considered steady (quasi-steady). That's why variable T is time like. Factor q is a priori considered the one that is subjected to steady trend, in case of dispersions equal distribution on the matrix volume.

Relaxation behavior of thermal expansion was observed at continuous cyclic heating and cooling of composite material. As a result of discrete measurements the table dependencies of relative linear elongation of the sample were observed (see Table). Such dependencies in reality due to well-known statements of mathematical analysis are of two-dimensional continuous smooth character for the set of all possible degrees of composite q filling with dispersions (Fig. 2). For each one-dimensional cross-section for q=const they are in the form of hysteresis loops. An irreversible component of the sample relative elongation is determined to be available. The reason of this phenomena is viscoelasticity of composites causing internal residual stress in the system which avoided relaxation due to their significant deviations from steady features of transient processes at composites formation.

Table

Dilatometer investigation of forward ($\nearrow$) and reversed ($\swarrow$) run of the samples relative elongation of composites filled with titanium carbide dispersions (parameter q is a value of filling per 100 resin mass fraction)

q=30 mass fraction				**q**=50 mass fraction			
$T^0K\nearrow$	$\varepsilon\nearrow$	$T^0K\swarrow$	$\varepsilon\swarrow$	$T^0K\nearrow$	$\varepsilon\nearrow$	$T^0K\swarrow$	$\varepsilon\swarrow$
317	0,05	397	0,95	323	0,12	423	0,94
324	0,18	386	0,68	333	0,20	410	0,83
327	0,44	378	0,54	343	0,24	397	0,75
335	0,39	370	0,39	352	0,27	386	0,64
346	0,52	352	0,30	366	0,30	375	0,52
360	0,70	340	0,27	381	0,39	359	0,34
366	0,81	327	0,24	395	0,54	352	0,27
377	0,90	314	0,20	407	0,68	344	0,15
396	0,95	308	0,12	423	0,94	336	0,02

q=80 mass fraction				**q**=100 mass fraction			
$T^0K\nearrow$	$\varepsilon\nearrow$	$T^0K\swarrow$	$\varepsilon\swarrow$	$T^0K\nearrow$	$\varepsilon\nearrow$	$T^0K\swarrow$	$\varepsilon\swarrow$
347	0,01	423	0,67	353	0,01	423	0,60
354	0,13	416	0,56	362	0,14	418	0,52
357	0,17	408	0,42	370	0,22	413	0,45
365	0,30	399	0,32	377	0,29	406	0,32
376	0,43	381	0,24	381	0,31	397	0,23
389	0,51	370	0,21	393	0,40	382	0,16
396	0,55	357	0,17	400	0,45	363	0,14
408	0,61	344	0,12	415	0,54	347	0,11
423	0,67	324	0,06	423	0,60	324	0,08

Sloping zones of trend response (Fig. 2) of two-dimensional dependencies graph ϵ(T,q) change indicates the stationarity areas of residual stress relation. The necessary condition of extreme value of the function f=A(x, y) first derivative in a random point is that the second derivative is equal to zero in the same point, in this case the second derivative must have different signs on the opposite sides of the point.

For the two-dimensional option the analogue of the second derivative A(x, y) is Laplacian (1) — scalar operator like

$$\nabla^2(f) = \frac{\partial^2 A(x,y)}{\partial^2 + x^2} + \frac{\partial^2 A(x,y)}{\partial^2 + y^2}. \qquad (1)$$

We use Laplacian operator to scalar field eu and ed (Fig. 2), which enables to derive stationarity or quasi-stationarity 2D areas [13], which are characterized by constant trend (mathematical expectation) and deviation (dispersion). Hysteresis dependence characterizes the rate of relaxation of macromolecular stress [14, 15]. This relaxation rate provides the residual stress minimization. The most optimal process of the composite hardening must have some features of stationarity to provide the least residual internal stress. Using numerical calculation methods of Laplacian operator (1) (listing 1) for discrete fields of measurements eu and ed of tests results (Fig. 2), we'll pay attention to the areas where the values of calculation are close to zero (Fig. 3). We must admit that the calculated field of Laplacian values lu and ld is smaller than the initial field due to specific discrete calculations. We build the field of vector product lu and ld (Fig. 3) to guarantee the stationarity of forward and reverse run. The obtained results prove the relaxation processes stationarity of forward and reverse run of dilatometer measurements for the composites filled with titanium carbide dispersions [16, 17]. Non stationary areas are found to be the ones close to those which satisfy the ratio: {340<T<350, 30<q<55} та {400<T<420, 30<q<40}.

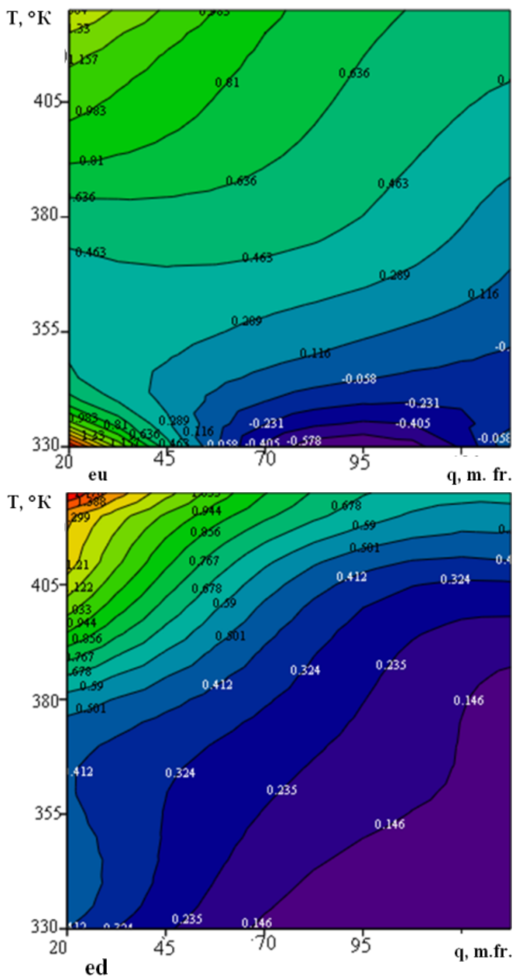


Figure 2. Two-dimensional dependencies graph ϵ(T,q) for forward (eu) and reverse (ed) run of dilatometer measurements of relative linear elongation of the sample

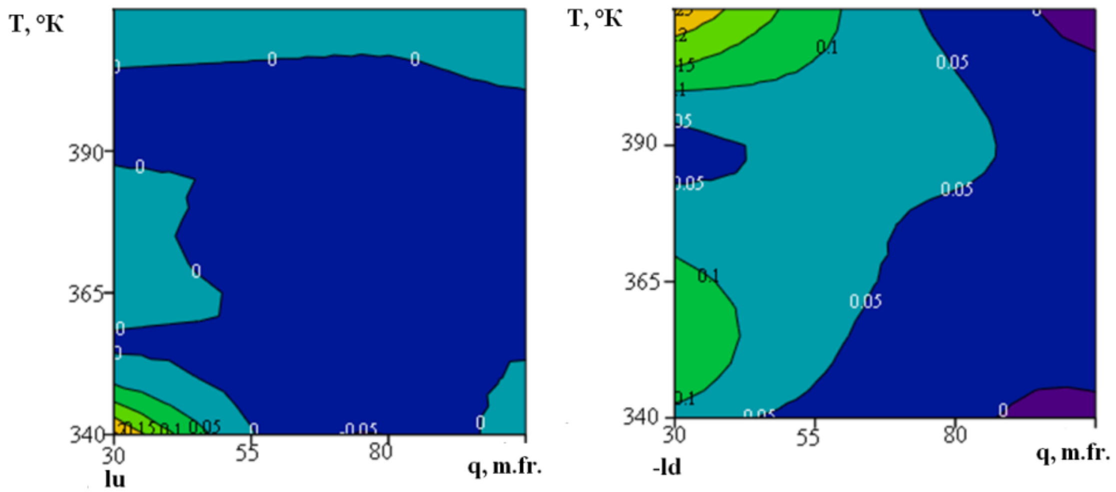


Figure 3. Result of Laplacian operator to the fields of forward (lu) and reverse (ld) run of dilatometer investigation

Listing 1. Calculation with the help of MathCAD-14 program of two-dimensional Laplacian operator for the field of dilatometer measurements of dependencies ϵ(T,q) for forward (eu) and reverse (ed) run:

$$j := 1..18 \quad i := 1..18$$

$$edq_{i,j} := 0.25 \cdot (ed_{i+1,j+1} - ed_{i-1,j+1} + ed_{i+1,j-1} - ed_{i-1,j-1});$$

$$edt_{i,j} = 0.25 \cdot (ed_{i+1,j+1} - ed_{i+1,j-1} + ed_{i-1,j+1} - ed_{i-1,j-1});$$

$$euq_{i,j} = 0.25 \cdot (eu_{i+1,j+1} - eu_{i-1,j+1} + eu_{i+1,j-1} - eu_{i-1,j-1});$$

$$eut_{i,j} = 0.25 \cdot (eu_{i+1,j+1} - eu_{i+1,j-1} + eu_{i-1,j+1} - eu_{i-1,j-1});$$

$$cols(euq) = 19 \quad rows(euq) = 19;$$

$$edq := submatrix(edq, 1, 18, 1, 18) \quad edt := submatrix(edt, 1, 18, 1, 18);$$

$$euq := submatrix(euq, 1, 18, 1, 18) \quad eut := submatrix(eut, 1, 18, 1, 18);$$

$$cols(euq) = 18 \quad rows(euq) = 18;$$

$$j := 1..16 \quad i := 1..16;$$

$$euq2_{i,j} = 0.25 \cdot (euq_{i+1,j+1} - euq_{i-1,j+1} + euq_{i+1,j-1} - euq_{i-1,j-1});$$

$$eut2_{i,j} = 0.25 \cdot (eut_{i+1,j+1} - eut_{i+1,j-1} + eut_{i-1,j+1} - eut_{i-1,j-1});$$

$$euq2_{i,j} = 0.25 \cdot (euq_{i+1,j+1} - euq_{i-1,j+1} + euq_{i+1,j-1} - euq_{i-1,j-1});$$

$$edt2_{i,j} = 0.25 \cdot (ed_{i+1,j+1} - ed_{i+1,j-1} + ed_{i-1,j+1} - ed_{i-1,j-1});$$

$$cols(euq2) = 17 \quad rows(euq2) = 17;$$

$$euq2 := submatrix(euq\ 2, 1, 16, 1, 16) \quad eut2 := submatrix(eut\ 2, 1, 16, 1, 16);$$

$$edq2 := submatrix(edq\ 2, 1, 16, 1, 16) \quad edt2 := submatrix(edt\ 2, 1, 16, 1, 16);$$

$$cols(euq2) = 16 \quad rows(euq2) = 16;$$

$$lu := euq2 + eut2 \quad ld := edq2 + edt2.$$

By means of the program MathCAD-14 we determine the stationarity fields of hysteresis loops passing (forward and reverse) forming the operators of vector product lu*ld for sampling investigation (at least one of them) and the sum of simultaneous study (the both). In the sum lu-ld the sign «-» appears due to the temperature countdown (Fig. 4).

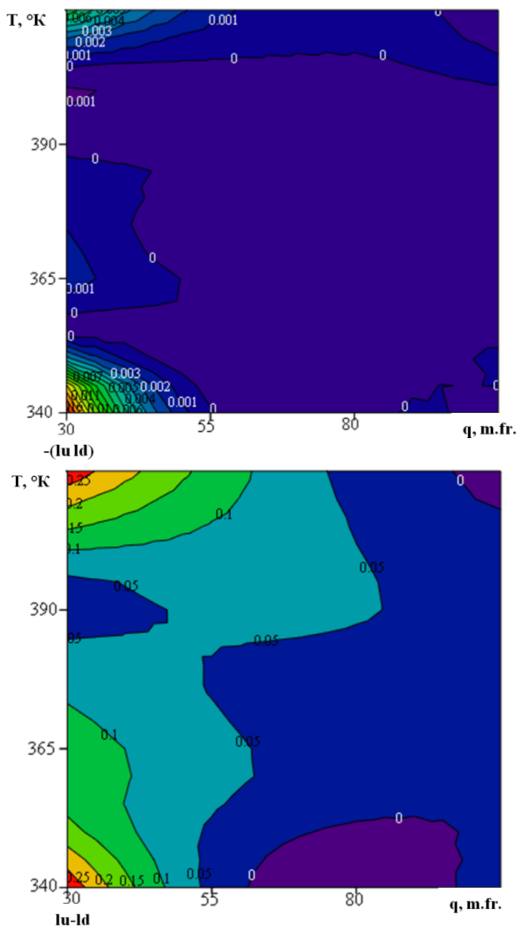


Figure 4. Fields of characteristics of sampling lu ld and simultaneous lu-ld stationarity performance of forward and reverse run of relaxation processes of epoxy composite filled with titanium carbide

Conclusions

Automation research will allow to determine the parameters of temperatures and filler content corresponding to minimal stress values. In the first approximation these are the following areas: $30<q<50$, $385<T<395$ K та $60<q<90$, $340<T<380$ K. Relaxation processes at the above-mentioned parameters are the most intensive in the composite. Such composites have the lowest residual stress. To make more accurate recommendations one must turn to the theory of higher orders operators and the theory of pattern recognition that will be the matter of further research [18]. The research method of composites dilatometer properties under discussion can be used for another composition of composite materials.

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А.Г. Микитишин, Р.З. Золотий, О.В. Тотоско

Дисперсті толтырылған эпоксикомпозиттердің жылу-физикалық сипаттамаларын зерттеуді автоматизациялау

Автоматтандырылған режимде берілген температуралардың диапазонында зерттеулер жүргізуге мүмкіндік беретін, әзірленген құрылғы көмегімен толтырылған эпоксикомпозиттердің жылу-физикалық сипаттамалары зерттелген. Дисперсті толтырылған композиттер үшін (T, q) = Δ1/1 салыстырмалы сызықтық ұзарту үлгісінің 1,5-2 К/мин қызу және салқындату жылдамдықтарымен 2,5-3 К/мин тікелей және кері әрекетінің тәуелділігі қарастырылған. Лаплас операторы мен *eu* және *ed* скаляр өрісін қолдану арқылы макромолекулалық кернеулердің релаксациялық процестерінің стационарлы немесе квазистационарлық аймақтарын бөлуге болады. Зерттеу автоматтандыру кернеудің минималды мәндеріне сәйкес келетін температураның және толтырғыштың параметрлерін анықтауға мүкіндік береді. Бірінші жуықтауда бұл жолақтар $30 < q < 50$, $385 < T < 395$ K және $60 < q < 90$,

340 < T < 380 K. Релаксация процестері осы көрсеткіштерде композитте қарқынды жүреді. Мұндай композиттер төмен қалдықты кернеулі болады.

Кілт сөздер: композит, эпоксидті матрица, сызықтық кеңейтудің термалды коэффициенті, зерттеуді автоматизациялау.

И.Г. Добротвор, Д.П. Стухляк, А.В. Букетов,
А.Г. Микитишин, Р.З. Золотий, О.В. Тотоско

Автоматизация исследований теплофизических характеристик дисперсно-наполненных эпоксикомпозитов

Исследованы теплофизические характеристики наполненных эпоксикомпозитов с помощью разработанного устройства, которое позволяет в автоматизированном режиме проводить исследования в заданном диапазоне температур. Изучены зависимости прямого и обратного хода показателя относительного линейного удлинения образца ϵ (T, q) = Δl / l со скоростью нагрева 1,5–2 K/мин и скоростью охлаждения 2,5–3 K/мин для композитов с дисперсным наполнителем. Использование оператора Лапласа и скалярного поля *eu* и *ed* позволило выделить плоские области стационарности или квазистационарности процессов релаксации макромолекулярных напряжений. Автоматизация исследований позволит определить параметры температур и содержания наполнителя, соответствующие минимальным значениям напряжения. В первом приближении это зоны 30 <q <50, 385 <T <395 K и 60 <q <90, 340 <T < 380 K. Релаксационные процессы при данных параметрах проходят в композите наиболее интенсивно. Такие композиты имеют низкие остаточные напряжения.

Ключевые слова: композит, эпоксидная матрица, термический коэффициент линейного расширения, автоматизация исследований.

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Stress-strain state horizontal coal seam of finite length

As the result of work, analytical expressions for calculation of tension in coal stratum of terminating length, which is under the influence of overlying breeds and is between two drifts are received. The decision is presented in the form of the sum of a polynom and a convergent series. For determination of coefficients of a row it is not required to solve the infinite systems of the algebraic equations. It promotes fast numerical finding of the required sizes with an accuracy, sufficient for practice. The given decision can be used for interpretation of data of geomechanical monitoring at combinegouge of coal in the conditions of real time and for monitoring of tension in the developed coal layer for the purpose of prediction of a possibility of mountain emission that still is very relevant problem of safety of mining operations.

Keywords: stress, plane problem of the theory of elasticity, biharmonic equation, coal bed, model of coal layer.

1 Introduction

The paper deals with the plane problem of the deformation of a horizontal coal bed of finite length that is under the influence of overlying rocks and lies between two drifts. An analytical solution in the form of a series is constructed. It takes a small amount of time to calculate it. To calculate ita small amount of time is needed. The method is specifically designed for monitoring of geomechanical fields during the development of coal seams in real time.

Conveyor technologies for coal extraction are increasingly used for underground mining. In this connection, the security problems associated with the increase in the likelihood of occurrence of mountain impacts and sudden gas releases are exacerbated [1]. In mines and mining camps the passivesystemsof monitoring of environment are used. They record microseismic emission (ITU), which occurs due to deformation of the rock mass caused by a quasistatic change in stresses during cleaning operations [2, 3]. Interpretation of the received data on the spot is carried out locally by statistical methods [4, 5]. On the other hand, there are significant correlations between the characteristics of the ITU and the integral parameters of the stress-strain state of rocks [6]. However, their use in practice is difficult, because the currently available numerical methods for calculating stress and strain fields in geo-environments [7–9] and, in particular, of carbonaceous massifs [10], despite the universality and existence of many commercial codes (ANSYS, ABACUS, FLAC, etc.), it is very difficult to use for the rapid estimation of geomechanical fields with the purpose of making decisions over times of the order of tens of seconds, which is required when forecasting technogenic dynamic events [2]. Therefore, the development of analytical methods for calculating geomechanical fields remains relevant. These methods allow obtaining the required solution for the minimum time required in practice.

The mathematically stated problem reduces to solving a two-dimensional homogeneous biharmonic equation. Numerous methods have been proposed to solve it. It is necessary to mention solutions in the form of polynomials, solutions of Fileon and Ribier. However, these solutions are not suitable for any kind of boundary conditions. There is an approach of the so-called nonclosed solution, when the solution is represented as the sums of several series, when the coefficient of one series is expressed in terms of all the coefficients of the second series, i.e. an infinite system of linear algebraic equations is obtained for finding coefficients. If it is possible to prove its completely regularity, then it is possible to use the simple reduction method. However, this approach is associated with a great deal of computation. It is possible to find a solution with the help of fundamental beam functions A.N. Krylov, but among them there are hyperbolic sines and cosines, which can lead to large errors during computationin large domains. To solve the biharmonic equation, there are other approaches (see, for example, [11–14]), which compare solutions obtained by various methods (they can be found in the work) [15].

In our opinion, the most promising approach seems to be S.A. Khalilov's approach. They were offered and studied special basis functions [16, 17], which allow, first, to obtain a solution in the form of a series; secondly, simple actions are performed to search for the coefficients of this series; it is not necessary to solve infinite systems of linear algebraic equations. This approach was proposed to solve the applied problems of aircraft building (see, for example [15–24]). It was shown that the numerical solution of the biharmonic equation, constructed with the help of proposed functions, is calculated very accurately. The maximum deviation is localized near the corner points of the domain (see, for example, [15–20], etc.) and is small (about 1.2 %), the largest error is achieved for a square area, the more the rectangular domain differs from the square, the less the calculation error.

We used the experience of S.A. Khalilov and his disciples for the construction of an algorithm for the rapid calculation of stresses in a coal-rock massif. This algorithm will be used in the future to interpret geological information in order to predict the possibility of mining, which is still a very urgent task of ensuring the safety of mining operations. The first attempts in this direction were made in [25, 26].

2 Formulation of the task

It is necessary to calculate stress fields in a coal seam of finite length lying between two drifts. Due to the long length of the reservoir compared to its thickness and distance between the drifts, we assume that the model of a flat deformed state is applicable [27]. In this case, the Navier equilibrium equation is written in the following form:

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xz}}{\partial z} = 0, \qquad \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \sigma_z}{\partial z} = 0. \tag{1}$$

The stress state in the reservoir is described by the equation of continuity of deformation of Saint-Venant

$$\frac{\partial^2 \varepsilon_x}{\partial z^2} + \frac{\partial^2 \varepsilon_z}{\partial x^2} = \frac{\partial^2 \gamma_{xz}}{\partial x \partial z} \tag{2}$$

and Hooke's law

$$\varepsilon_x = \frac{1}{E'}\left(\sigma_x - \nu'\sigma_z\right), \quad \varepsilon_z = \frac{1}{E'}\left(\sigma_z - \nu'\sigma_x\right), \quad \gamma_{xz} = \frac{2(1+\nu')}{E'}\tau_{xz}. \tag{3}$$

Here E' and ν' are the Young's modulus and the Paussson coefficient for the plane problem of the deformed state [27].

We assume that the boundary conditions hold

$$\sigma_z|_{z=\pm l_z} = \left\{ \begin{array}{l} f_1(x) \\ f_2(x) \end{array} \right. ; \qquad \tau_{xz}|_{z=\pm l_z} = \left\{ \begin{array}{l} g_1(x) \\ g_2(x) \end{array} \right. \tag{4}$$

and the matching conditions at the corner points: $g_j(\pm l_x) = 0$ $(j = 1, 2)$.

In the simple case, we can assume: $f_1(x) = f_2(x)$, $g_1(x) = -g_2(x)$.

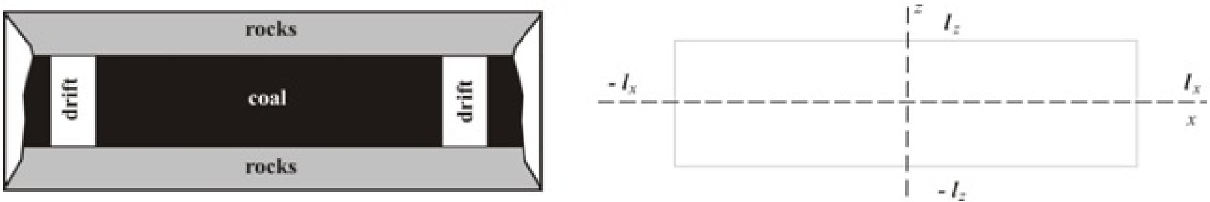

Figure. Coal layer model and coordinate system adopted in the paper

We consider that the considered system is balanced (the coal seam, being under the influence of forces on it, is stationary) (see. Fig.). In this case, the moment of forces and the sum of the forces acting on the layer must be zero. That is, the following equations must be satisfied:

$$\int_{-l_x}^{l_x} \left[x\sigma_z(x,l_z) - l_z\tau_{xz}(x,l_z)\right]dx + \int_{l_x}^{-l_x} \left[x\sigma_z(x,-l_z) + l_z\tau_{xz}(x,-l_z)\right]dx = 0; \tag{5}$$

$$\int_{-l_x}^{l_x} \sigma_z(x,l_z)dx = \int_{-l_x}^{l_x} \sigma_z(x,-l_z)dx, \qquad \int_{-l_x}^{l_x} \tau_{xz}(x,l_z)dx = \int_{-l_x}^{l_x} \tau_{xz}(x,-l_z)dx$$

or, taking into account (4),

$$\frac{1}{2}\left(\int_{-l_x}^{l_x} x\, f_1(x)\, dx - \int_{-l_x}^{l_x} x\, f_2(x)\, dx\right) = l_z \int_{-l_x}^{l_x} g_1(x)\, dx; \tag{6}$$

$$\int_{-l_x}^{l_x} f_1(x)\, dx = \int_{-l_x}^{l_x} f_2(x)\, dx, \qquad \int_{-l_x}^{l_x} g_1(x)\, dx = \int_{-l_x}^{l_x} g_2(x)\, dx.$$

Following the well-known approach for calculating stresses (see, for example, [27]), we introduce the Erie function and obtain the differential equation to which it satisfies.

A consequence of relations (2)–(3) is the equation

$$\frac{\partial^2}{\partial z^2}(\sigma_x - \nu'\sigma_z) + \frac{\partial^2}{\partial x^2}(\sigma_z - \nu'\sigma_x) = 2(1+\nu')\frac{\partial^2}{\partial x \partial z}\tau_{xz}. \tag{7}$$

It follows from (1)–(3), (7) that

$$\Delta\,(\sigma_x + \sigma_z) = 0. \tag{8}$$

We introduce the Erie function φ so that equations (1) are satisfied automatically

$$\sigma_x = \frac{\partial^2\varphi}{\partial z^2}, \quad \sigma_z = \frac{\partial^2\varphi}{\partial x^2}, \quad \tau_{xz} = -\frac{\partial^2\varphi}{\partial x \partial z}, \tag{9}$$

then from (8) for the Erie φ function the biharmonic equation will take place:

$$\Delta^2\varphi = 0. \tag{10}$$

To solve equation (10), from (4) and (9) follow the boundary conditions:

$$\begin{aligned} &\varphi_{xx}|_{z=\pm l_z} = \begin{cases} f_1(x) \\ f_2(x) \end{cases}, \quad \varphi_{xz}|_{z=\pm l_z} = \begin{cases} g_1(x) \\ g_2(x) \end{cases}, \\ &\varphi_{zz}|_{x=\pm l_x} = 0, \qquad\qquad \varphi_{xz}|_{x=\pm l_x} = 0. \end{aligned} \tag{11}$$

Thus, in order to find the stresses σ_x, σ_z, and τ_{xz} in the domain $[-l_x, l_x] \times [-l_z, l_z]$ it is necessary to find the solution of the problem (10)–(11), and then use the representations (9).

To apply the mathematical technique (as in the case of using the solutions of Fileon or Ribiere), it is necessary that for the basis functions $H_m(x)$ on the interval $[-l_x, l_x]$ the following conditions exist at the ends: $H_m(\pm l_x) = 0$, $H'_m(\pm l_x) = 0$. The functions $\sin(\pi x/l_x)$ or $\cos(\pi x/l_x)$ do not satisfy such conditions.

3 Selection of basic functions

S.A. Khalilov proposed to use the basis functions $H_m(x)$ [16, 17] for the solution of the biharmonic equation of the following form:

$$H_m(x) = P^4_{m+4}(x), \quad m = 0, 1, 2, \ldots,$$

where $P^4_{m+4}(x)$ are normalized adjoint Legendre polynomials. The system of functions $\{H_m(x)\}_{m=0}^{\infty}$ is complete and orthonormal on the interval $[-1, 1]$.

A continuous function $s(x)$ with boundary values $s(\pm 1) = 0$, $s'(\pm 1) = 0$ can be decomposed into the Fourier series in the system of functions $\{H_m(x)\}_{n=0}^{\infty}$, the series converges absolutely and uniformly.

The presentation takes place [16, 17]

$$H_m(x) = (1-x^2)^2 \sum_{k=0}^{[m/2]} W_{mk}\, x^{m-2k}, \quad W_{mk} = \frac{(-1)^k}{2^{m+3}}\sqrt{\frac{m!(2m+9)}{2(m+8)!}}\frac{(2m-2k+7)!}{(m-k+3)!k!(m-2k)!},$$

([$\cdot$] is the integer part of a number), the recurrence formula

$$H_m(x) = \xi_m x H_{m-1}(x) - \zeta_m H_{m-2}(x), \quad m = 1, 2, ..., \quad H_{-1}(x) = 0, \quad H_0(x) = W_{00}(1 - x^2)^2,$$

$$\xi_m = \sqrt{\frac{(2m+9)(2m+7)}{m(m+8)}}, \quad \zeta_m = \sqrt{\frac{(m-1)(m+7)(2m+9)}{m(m+8)(2m+5)}}$$

and the following equalities [17,18]:

$$\|H'_n\|^2_{[-1,1]} = \frac{1}{15}(2n+9)(n^2+9n+5);$$

$$\|H''_n\|^2_{[-1,1]} = \frac{1}{4}(2n+9)\left((n+2)(n+7)\left[1 + \frac{1}{60}n(n+2)(n+7)(n+9)\right] - \right.$$
$$\left. - n(n+9)\left[3 + \frac{1}{84}(n-1)(n+4)(n+5)(n+10)\right]\right).$$

S.A. Khalilov and his co-authors have been shown and shown on numerical examples [15–20] that the functions $H'_m(x)$ and $H''_m(x)$ are quasi-orthogonal in the sense of the following conditions:

$$\frac{\langle\, H_n^{(k)}(x), H_m^{(k)}(x)\,\rangle}{\|H_n^{(k)}(x)\|\;\|H_m^{(k)}(x)\|} = \theta, \quad |\theta| \approx 0, \quad m \neq n, \quad k = 1, 2,$$

This remarkable property of these functions made it possible to apply the Bubnov-Galerkin procedure to the search for the solution of the biharmonic equation and greatly simplify it.

In this paper, the functions $X_m(x; L) = (1/\sqrt{L})H_m(x/L)$ will be used. These functions are orthonormal on the interval $[-L, L]$. For their derivatives, the equalities

$$\|X'_m\|^2_{[-L,L]} = L^{-2}\|H'_m\|^2_{[-1,1]}, \qquad \|X''_m\|^2_{[-L,L]} = L^{-4}\|H''_m\|^2_{[-1,1]}.$$

Below we need expansions of functions in a series in the derivatives of the functions $X_m(x, L)$.

The expansion of the continuous function $s(x)$ ($s(\pm L) = 0$) in a series in the functions $X'_m(x; L)$ has the form

$$s(x) = \frac{3}{4L}\int\limits_{-L}^{L} s(y)dy \cdot \left(1 - \frac{x^2}{L^2}\right) + \sum_{n=0}^{\infty} c_m^S X'_m(x; L).$$

The expansion of the continuous function $s(x)$ ($|s(x)| < \infty$) in a series in the functions $H''_k(x; L)$ has the form

$$s(x) = -\frac{3}{2L^3}\int\limits_{-L}^{L} y\, s(y)dy \cdot x + \frac{1}{4L}\int\limits_{-L}^{L} s(y)dy + \sum_{m=0}^{\infty} c_m^\Phi X''_m(x; L).$$

The decomposition data were obtained in [28].

4 Construction of the solution of the biharmonic equation

First of all, we introduce some notation. Let the functions $f_j\,(x)$ и $g_j\,(x) (j = 1, 2)$ be representable in the form of series

$$f_j(x) \;=\; f^j_{-2}\cdot x + f^j_{-1} + \sum_{m=0}^{\infty} f^j_m X''_m(x; l_x), \quad f^j_{-1} = \frac{1}{4l_x}\int\limits_{-l_x}^{l_x} f_j(s)ds, \quad f^j_{-2} = -\frac{3}{2l_x^3}\int\limits_{-l_x}^{l_x} s f_j(s)ds;$$

$$g_j(x) \;=\; g^j_{-1}\left(1 - \frac{x^2}{l_x^2}\right) + \sum_{m=0}^{\infty} g^j_m X'_m(x; l_x), \quad g^j_{-1} = \frac{3}{4l_x}\int\limits_{-l_x}^{l_x} g_j(s)ds.$$

From conditions (6) it follows that:

$$f^1_{-1} = f^2_{-1} \equiv F, \quad g^1_{-1} = g^2_{-1} = G, \quad \frac{1}{2}(f^1_{-2} - f^2_{-2}) = -\frac{2l_z}{l_x^2}G.$$

We denote by

$$\frac{1}{2}(f^1_{-2} + f^2_{-2}) = H.$$

We seek the solution of the biharmonic equation in the form

$$\varphi(x,z) = \varphi_1(x,z) + \varphi_2(x,z).$$

The function $\varphi_1(x,z)$ is a solution of the biharmonic equation (10) and satisfies the boundary conditions

$$\left.\frac{\partial^2\varphi_1}{\partial x^2}\right|_{z=\pm l_z} = \begin{Bmatrix} f^1_{-2} \\ f^2_{-2} \end{Bmatrix} \cdot x + F, \quad \left.\frac{\partial^2\varphi_1}{\partial x \partial z}\right|_{z=\pm l_z} = -G\left(1 - \frac{x^2}{l_x^2}\right);$$

$$\left.\frac{\partial^2\varphi_1}{\partial z^2}\right|_{z=\pm l_x} = 0, \quad \left.\frac{\partial^2\varphi_1}{\partial x \partial z}\right|_{z=\pm l_z} = 0.$$

Using the method of undetermined coefficients, it is not difficult to obtain $\varphi_1(x,z)$ as a polynomial:

$$\varphi_1(x,z) = \frac{H}{6}x^3 + \frac{F}{2}x^2 - \frac{G}{3l_x^2}x^3 z + Gxz.$$

Obviously, the function $\varphi_1(x,z)$ is found up to linear functions of x and z, since the above-mentioned boundary conditions do not ensure the uniqueness of the solution of the biharmonic equation.

The function $\varphi_2(x,z)$ is a solution of the biharmonic equation (10) and satisfies the boundary conditions

$$\left.\frac{\partial^2\varphi_2}{\partial x^2}\right|_{z=\pm l_z} = \sum_{m=0}^{\infty} \begin{Bmatrix} f^1_m \\ f^2_m \end{Bmatrix} X''_m(x; l_x), \quad \left.\frac{\partial^2\varphi_2}{\partial x \partial z}\right|_{z=\pm l_z} = -\sum_{m=0}^{\infty} \begin{Bmatrix} g^1_m \\ g^2_m \end{Bmatrix} X'_m(x; l_x);$$

$$\left.\frac{\partial^2\varphi_2}{\partial z^2}\right|_{x=\pm l_x} = 0, \quad \left.\frac{\partial^2\varphi_2}{\partial x \partial z}\right|_{x=\pm l_x} = 0.$$

These boundary conditions automatically satisfy the relations (6) by the properties of the functions $X_m(x; l_x)$ $(\forall m)$.

An approximate analytic solution $\varphi_2(x,z)$ will be sought in the form

$$\varphi_2(x,z) = \sum_{m=0}^{\infty} R_m(z) X_m(x; l_x).$$

The Bubnov-Galerkin procedure applied to the solution of the homogeneous biharmonic equation (10) leads to an infinite system of ordinary differential equations

$$\sum_{m=0}^{\infty} \left[R_m \langle X''_m, X''_s\rangle - 2R''_m \langle X'_m, X'_s\rangle + R''''_m \delta_{ms}\right] = 0, \qquad s = 0,1,2..., \tag{12}$$

here δ_{ms} — is the Kronecker symbol.

We use the property of quasiorthogonality of the first and second derivatives of the functions $X_m(x; l_x)$, from (12) we obtain the problems

$$R''''_m - 2c_m^2 R''_m + d_m^4 R_m = 0, \quad R_m(\pm l_x) = \begin{cases} f^1_m \\ f^2_m \end{cases}, \quad R'_m(\pm l_x) = \begin{cases} g^1_m \\ g^2_m \end{cases}, \qquad m = 0,1,2,..., \tag{13}$$

where $c_m^2 = \|X'_m(\cdot\,; l_x)\|^2$ and $d_m^4 = \|X''_m(\cdot\,; l_x)\|^2$.

Since $d_m > c_m$ for all m [17], the four roots of the characteristic equation can be found in the form: $\pm b_m e^{\pm i\vartheta_m}$, where $2\vartheta_m = \operatorname{arctg}\sqrt{d_m^4/c_m^4 - 1}$. Consequently, the solutions of problems (13) have the form

$$\begin{aligned} R_m(z) &= \hat{L}_m \sin(\vartheta_m z)\,\mathrm{sh}(d_m z) + \hat{K}_m \cos(\vartheta_m z)\,\mathrm{ch}(d_m z) + \\ &+ \tilde{L}_m \sin(\vartheta_m z)\,\mathrm{ch}(d_m z) + \tilde{K}_m \cos(\vartheta_m z)\,\mathrm{sh}(d_m z); \end{aligned}$$

$$\hat{L}_m = \frac{\vartheta_m \sin(\vartheta_m l_z)\,\mathrm{ch}(d_m l_z) - d_m \cos(\vartheta_m l_z)\,\mathrm{sh}(d_m l_z)}{\Delta_1}(f_m^1 + f_m^2) - \frac{\cos(\vartheta_m l_z)\,\mathrm{ch}(d_m l_z)}{\Delta_1}(g_m^1 - g_m^2);$$

$$\hat{K}_m = \frac{\vartheta_m \cos(\vartheta_m l_z)\,\mathrm{sh}(d_m l_z) + d_m \sin(\vartheta_m l_z)\,\mathrm{ch}(d_m l_z)}{\Delta_1}(f_m^1 + f_m^2) + \frac{\sin(\vartheta_m l_z)\,\mathrm{sh}(d_m l_z)}{\Delta_1}(g_m^1 - g_m^2);$$

$$\tilde{L}_m = \frac{\vartheta_m \sin(\vartheta_m l_z)\,\mathrm{sh}(d_m l_z) - d_m \cos(\vartheta_m l_z)\,\mathrm{ch}(d_m l_z)}{\Delta_2}(f_m^1 - f_m^2) - \frac{\cos(\vartheta_m l_z)\,\mathrm{sh}(d_m l_z)}{\Delta_2}(g_m^1 + g_m^2);$$

$$\tilde{K}_m = \frac{\vartheta_m \cos(\vartheta_m l_z)\,\mathrm{ch}(d_m l_z) + d_m \sin(\vartheta_m l_z)\,\mathrm{sh}(d_m l_z)}{\Delta_2}(f_m^1 - f_m^2) + \frac{\sin(\vartheta_m l_z)\,\mathrm{ch}(d_m l_z)}{\Delta_2}(g_m^1 + g_m^2);$$

$$\Delta_1 = \vartheta_m\,\mathrm{sh}(2d_m l_z) + d_m \sin(2\vartheta_m l_z), \qquad \Delta_2 = \vartheta_m\,\mathrm{sh}(2d_m l_z) - d_m \sin(2\vartheta_m l_z).$$

5 The resulting expressions for stresses

Summarizing the above expressions, we arrive at the expression

$$\varphi(x,z) = \frac{H}{6}x^3 + \frac{F}{2}x^2 - \frac{G}{3l_x^2}x^3 z + Gxz + \sum_{m=0}^{\infty} R_m(z) X_m(x; l_x),$$

where all the necessary quantities are obtained above. From the definition (9) for the stresses, we obtain the following equalities:

$$\sigma_x(x,z) = \sum_{m=0}^{\infty} R_m''(z) X_m(x; l_x);$$

$$\sigma_z(x,z) = Hx + F - 2\frac{G}{l_x^2}xz + \sum_{m=0}^{\infty} R_m(z) X_m''(x; l_x);$$

$$\tau_{xz}(x,z) = -G\left(1 - \frac{x^2}{l_x^2}\right) - \sum_{m=0}^{\infty} R_m'(z) X_m'(x; l_x).$$

Conclusion

Analytic expressions are constructed for the approximate calculation of stresses in a coal seam of finite length that are under the influence of overlying rocks and lie between two drifts. For acceptable accuracy, 20-40 series members are required for the expansion of the functions f_j and g_j, so the stress distribution used to interpret the geomechanical monitoring data for the combine harvesting of stocks can be determined in real time.

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М.К. Кудайбергенов, А.Л. Карчевский, К.Т. Искаков

Көлденең жатқан көмір қабатының шектік ұзындығындағы кернеулі-деформацияланған күйі

Мақалада жоғары жатқан жыныстардың әсерінде тұрған және екі штрек арасында жатқан шектік ұзындығы бар көмір қабаттарындағы кернеулерді есептеуге арналған аналитикалық өрнектер алынған. Бұл шешімдер алгебралық теңдеулердің шексіз жүйелерін шешуді талап етпейтін полином қосындылары мен жинақты қатар түрінде көрсетілген. Осы әдіс тәжірибедегі жеткілікті дәлдікпен, санды түрде тез арада қажетті шамаларды табуға ықпалын тигізеді. Комбайндық тәсілмен көмір қазу кезінде геомеханикалық мониторинг деректерін интерпретациялауда және мүмкін болатын тау соққысын болжау үшін өңделіп жатқан көмір қабатындағы кернеулерге мониторинг жүргізуде бұл шешімді қолдануымызға болады. Тау-кен жұмыстарын жүргізу кезінде қауіпсіздікті қамтамасыз ету үшін осы күндегі өзекті мәселелердің бірі болып табылады.

Кілт сөздер: серпімділік теориясының жазықтық есебі, бигармоникалық теңдеулер, кернеу, көмір қабаты.

М.К. Кудайбергенов, А.Л. Карчевский, К.Т. Искаков

Напряженно-деформированное состояние горизонтального угольного пласта конечной длины

В статье получены аналитические выражения для вычисления напряжений в угольном пласте конечной длины, который находится под действием лежащих выше пород и между двумя штреками. Решение представлено в виде суммы полинома и сходящегося ряда. Для определения коэффициентов ряда не требуется решать бесконечных систем алгебраических уравнений. Это способствует быстрому численному нахождению требуемых величин с достаточной для практики точностью. Данное решение может быть использовано для интерпретации данных геомеханического мониторинга при комбайновой выемке угля в режиме реального времени и напряжений в разрабатываемом угольном пласте с целью прогнозирования возможности горного выброса, что до сих пор является весьма актуальной задачей обеспечения безопасности горных работ.

Ключевые слова: плоская задача теории упругости, бигармоническое уравнение, напряжение, угольный пласт.

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Mathematical modeling of the roller-rope coupling operation in misalignment conditions

The article deals with the results of construction and mathematical modeling of rope-roller coupling in condition of radial misalignment. As a result of modeling, the operation of coupling is revealed in the presence of misalignment of the shafts connected by it. Formulas are obtained that allow to determine the elongation of the rope depending on the magnitude of misalignment, as well as the design parameters of the coupling. The results of the investigations also allow determining the radial load from the coupling on the shafts at any angle of rotation of the coupling. Numerical simulation by the obtained formulas made it possible to establish that coupling is characterized by high compensating ability and low radial stiffness. So, with a radial displacement of 0.5 mm (for a prototype coupling with elastic star allow only 0.25 mm), the rope-roller coupling creates a load of only 5 N on the shafts, with a rope elongation of 5.5 μm. The dependencies obtained are approved during the design of the coupling, and the results of the calculations on them are checked by comparison with the results of the construction and showed a coincidence. The obtained results can be used in the design of rope-roller couplings.

Keywords: mathematical model, misalignment, coupling, rope, load, tightening.

Introduction

For the declared prospects of any design, a specific input can be obtained by performing an experimental or calculated study of its parameters and characteristics. For couplings, the most important indicator is the compensating capacity, which is characterized by the force of the coupling on the shafts - the radial load, which connects the radial stiffness of the coupling and the radial misalignmentΔ_r. Therefore, the estimation of the radial load on the shafts from the coupling during its operation under misalignment conditions is an actual task. Determination of the load from the coupling to the shafts, in turn, is impossible without determining the deformation of the elastic link of the coupling.

Statement of the research task

Researching coupling shown on Figure 1. It contains two half-couplings - the external (1) and the internal (2), which are joined by a segment of the rope (3), which may be several, both ends (4) and (5) of the rope (3) are fixed in the pins (6), which are installed in the external half-coupling (1), and the internal half-coupling (2) contains the rollers (8) installed with the possibility of turning on the axes (7), between which there is a rope (3) that covers the rollers (8).

The coupling works as follows. When rotating by the actuator of the external half-coupling (1), with it, the pins (6) are rotated causing the tension of the rope (3). The tensioned rope (3), covering the rollers (8), is actually presses on them and on the axis (7), creating a torque on the driven internal half-coupling (2) and rotating it. In the event of misalignment of the half couplings (1) and (2), the rollers 8 of the driven half-coupling (2) shall be rolled up along the rope (3), due to which the compensation of misalignment will occur. In the case of in-line arrangement of the half-coupling the rollers (8) will not rotate along the rope (3).

The rope 3 of the coupling is blown by air when the coupling is rotated, and when the rope is destroyed, half-couplings have the possibility of a non-impact relative rotation due to the difference in their outer diameters. The coupling parts have simple configuration and do not require precise machining, they can be made on a lathe, even in ship conditions. Replacing the rope (3) is not a problem - just unscrew the two screws that fix its ends (4) and in (5) in the pins (6).

Determination of the deformation of the elastic link of the rope-roller coupling for calculating the radial load from the coupling to the shafts when operating under misalignment is the purpose of this article.

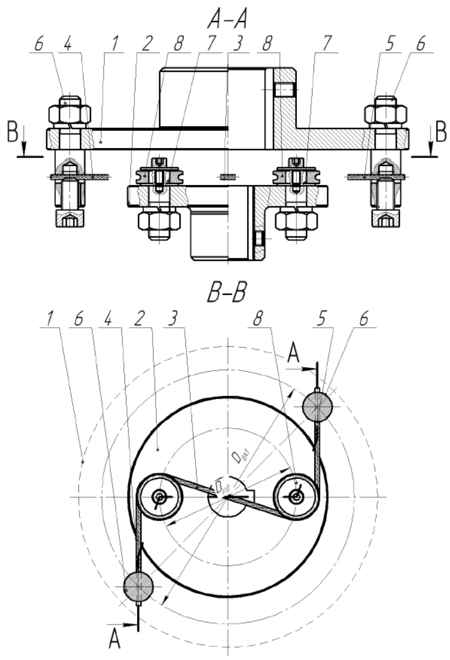

Figure 1. General scheme of roller-rope coupling

Statement of the main material

Lets consider the operation of the coupling in the conditions of radial misalignment Δ_r. In this the centre of driven half-coupling in which fixed the centre of the rollers (Fig. 2) will shift by Δ_r from point O to point O_1. In this while coupling rotating range of distances are changing — AB, SA, arc SR by changing the articles за рахунок зміни кутів $\beta_1, \beta_2, \beta_4 - \beta_7$ and the angle of the mounting displacement of the half-couplings ξ.

Points on the calculating scheme, which are relate to roller and rope and lays over radial displacement plane (plane in which both axes of displaced shafts lay) are denoted by an index 1, and those points, which lays under the plane of radial displacement – by an index 2. Respectively angles are denoted by one and two dashes. In this way rope section $O_1R_1S_1A_1$,

which situated over the radial displacement plane, became longer than section $O_1R_2S_2A_2$ under those plane by increasing the length of section S_1A_1 in comparison with з S_2A_2 and arc R_1S_1 in comparison with R_2S_2(increasing the roller reach angle $\beta_2^{/}$ in comparison with angle $\beta_2^{//}$).

For rope elongation determination and respectively load on shafts from coupling in present of radial misalignment Δ_r it necessary to determinate rope length in that conditions. This task is reducing to determination of mentioned sections $O_1R_1S_1A_1$ and $O_1R_2S_2A_2$ length and respectively angles $\beta_2^{/} = \beta_5^{/} + \beta_6^{/} + \beta_7^{/}$ and $\beta_2^{//} = \beta_5^{//} + \beta_6^{//} + \beta_7^{//}$.

To find this angles we need to determinate also reduced mounting displacement angles $\xi_1^{/}$ and $\xi_2^{/}$. It is necessary to determinate all parameters depending of coupling rotation angle relatively of radial displacement plane.

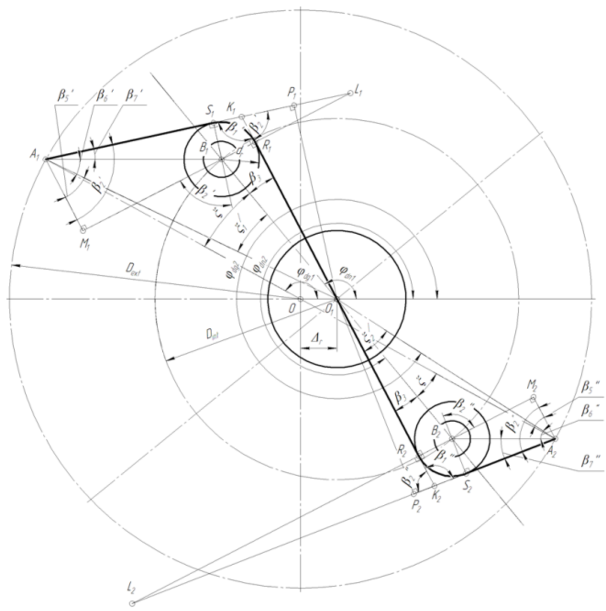

Figure 2. Scheme to geometric calculation coupling

On the firs stage we will show connection with rotation angles $\varphi_{dg}$of driving half-coupling and φ_{dn} of driven half-coupling when operation of radial displacement conditions. For this let's consider the scheme shown on Figure 3.

Let's consider coupling position in which sections AB are parallel to radial displacement plane (Fig. 3). Triangles OA_1C_1 and $O_1B_1C_3$are giving the opportunity to obtain formulas for angle coordinates the roller centre point B and the point A of rope end anchoring in this position.

$$\cos\chi = \frac{O_1C_3}{O_1B_1} = \frac{h}{0,5D_{int}}, \tag{1}$$

where h — triangle OA_1C_1 height.

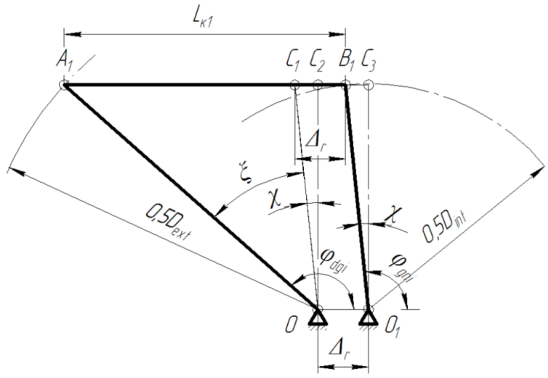

Figure 3. Scheme to calculation of angle coordinate coupling elements

Withal, writing formula for triangle OA_1C_1 like a half of area parallelogram, built on sides OA_1 and OC_1:

$$h = \frac{2S_{OA_1C_1}}{A_1C_1} = \frac{2\times 0,5\times OA_1\times OC_1\times \sin\xi}{L_{\text{к}0}} = \frac{0,25D_{ext}D_{int}\sin\xi}{L_{\text{к}0}}, \tag{2}$$

where $L_{\text{к}0}$ — the original rope length without radial displacement influence.

With considering (1) and (2) we obtain:

$$\cos\chi = \frac{0,5D_{\text{зв}}\sin\xi}{L_{\text{к}0}}. \tag{3}$$

Than expressions for obtaining angle coordinates in that position will have the form:

$$\varphi_{dg1} = \frac{\pi}{2} + \chi + \xi = \frac{\pi}{2} + arc\cos\frac{0,5D_{ext}\sin\xi}{L_{AB}} + \xi; \quad (4)$$

$$\varphi_{dn1} = \frac{\pi}{2} + \chi = \frac{\pi}{2} + arc\cos\frac{0,5D_{ext}\sin\xi}{L_{AB}}; \quad (5)$$

$$\varphi_{dgi} - \varphi_{dni} = \xi. \quad (6)$$

The problem of finding section AB length in each position we shall solve by method closed vector circuits of prof. V.A. Zinoviev [1, 2]. For obtaining further calculation lets imagine kinematic scheme of replacement mechanism in the form of a closed vector circuit AOO_1B (Fig. 4), for which we can write vector equation:

$$\vec{V}_1 + \vec{V}_2 = \vec{V}_3 + \vec{V}_4, \quad (7)$$

where $\vec{V}_1 = OA$ — driving half coupling radius; $\vec{V}_2 = L$ — distance between A and B points; $\vec{V}_3 = \Delta_r$ — radial misalignment; $\vec{V}_4 = O_1B$ — driven half coupling radius.

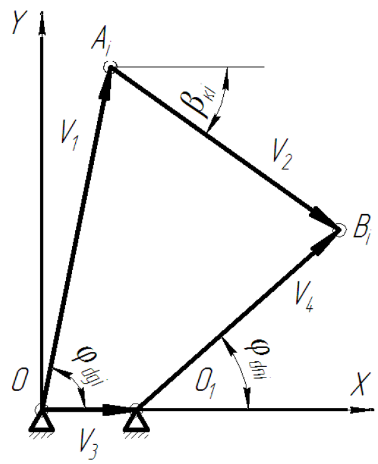

Figure 4. Scheme of closed vector circuit replacement mechanism

The resulting equation (7) I projections on coordinate axes will have the form of a equation system (8):

$$\begin{cases} X: V_1\cos\varphi_{dgi} + V_2\cos\beta_{\kappa i} = V_3\cos 0 + V_4\cos\varphi_{dni}; \\ Y: V_1\sin\varphi_{dgi} + V_2\sin\beta_{\kappa i} = V_3\sin 0 + V_4\sin\varphi_{dni}. \end{cases} \quad (8)$$

After the transformation we obtain:

$$\begin{cases} V_2\cos\beta_\kappa i = V_3 + V_4\cos\varphi_{dni} - V_1\cos\varphi_{dgi} \\ V_2\sin\beta_\kappa i = V_4\sin\varphi_{dni} - V_1\sin\varphi_{dgi} \end{cases}. \quad (9)$$

Square both equations (9) and add them, where after transformation we obtain:

$$V_2^2(\cos^2\varphi_{\kappa i} + \sin^2\varphi_{\kappa i}) = (V_3 + V_4\cos\varphi_{dni} - V_1\cos\varphi_{dgi})^2 + (V_4\sin\varphi_{dni} - V_1\sin\varphi_{dgi})^2; \quad (10)$$

$$V_2 = \sqrt{(V_3 + V_4\cos\varphi_{dni} - V_1\cos\varphi_{dgi})^2 + (V_4\sin\varphi_{dni} - V_1\sin\varphi_{dgi})^2}. \quad (11)$$

Whence distance AB in any coupling position will be:

$$L_{A_iB_i} = \sqrt{[\Delta_r + 0,5D_{ext}(\cos\varphi_{dni} - \cos\varphi_{dgi})]^2 + [0,5D_{ext}(\sin\varphi_{dni} - \sin\varphi_{dgi})]^2}. \quad (12)$$

Returning to the Figure 3, from triangle OA_1O_1 we can wrote:

$$A_1O_1 = \sqrt{OO_1^2 + OA_1^2 - 2 \times OO_1 \times OA_1\cos\varphi_{dg1}};$$

$$L_{A_1O_1} = \sqrt{\Delta_r^2 + 0,25D_{ext}^2 - \Delta_r D_{ext}\cos\varphi_{dg1}}; \qquad (13)$$

$$L_{A_iO_i} = \sqrt{\Delta_r^2 + 0,25D_{ext}^2 - \Delta_r D_{ext}\cos\varphi_{dgi}}.$$

From triangle $O_1A_1B_1$ by the cosine theorem we have:

$$A_1B_1^2 = A_1O_1^2 + O_1B_1^2 - 2\times A_1O_1 \times O_1B_1 \cos\xi_1^{/}, \qquad (14)$$

whence we obtain:

$$\xi_1^{/} = \arccos\left[\frac{A_1O_1^2 + O_1B_1^2 - A_1B_1^2}{2\times A_1O_1\times O_1B_1}\right];$$

$$\xi_1^{/} = \arccos\left[\frac{L_{A_1O_1}^2 + 0,25D_{вн}^2 - L_{A_1B_1}^2}{L_{A_1O_1}D_{вн}}\right]; \qquad (15)$$

or in general form

$$\xi_i^{/} = \arccos\left[\frac{L_{A_iO_i}^2 + 0,25D_{int}^2 - L_{A_iB_i}^2}{L_{A_iO_i}D_{int}}\right].$$

Next we can use obtained earlier equations (16), (17), (19)–(21) substituting in place of angle ξ reduced mounting displacement angle $\xi_i^{/}$.

$$\beta_2^i = \xi_i^{/} + \beta_3 + \beta_6^i + \beta_7^i = $$
$$= \xi_i^{/} + \arcsin\frac{d_r}{D_{\text{int}}} + \arcsin\left(\frac{0,5D_{\text{int}}}{L_{A_iB_i}}\sin\xi_i^{/}\right) + \arcsin\left(\frac{0,5d_r}{L_{A_iB_i}}\right), \qquad (16)$$

$$\beta_1^V = 180 - \beta_2^V, \qquad (17)$$

where d_r — roller diameter.

Respectively rope length in each position of coupling will be:

$$L_{r\Delta} = L_{r1} + L_{r2}, \qquad (18)$$

where $L_{r1} = S_1A_1 + R_1S_1 + O_1R_1$ — the length of a rope section under radial displacement plane;
$L_{r2} = S_2A_2 + R_2S_2 + O_2R_2$ — the length of a rope section over radial displacement plane, where

$$S_iA_i = A_iB_i\cos\beta_7 = L_{A_iB_i}\cos\left(\arcsin\frac{0,5d_r}{L_{A_i}B_i}\right); \qquad (19)$$

$$R_iS_i = \frac{\pi d_r}{360}\beta_2^i; \qquad (20)$$

$$O_1R_1 = O_2R_2 = 0,5D_{\text{int}}\cos\beta_3 = 0,5D_{\text{int}}\cos\left(\arcsin\frac{d_r}{D_{\text{int}}}\right). \qquad (21)$$

Rope elongation ΔL_r in relatively with rope length $L_r$in coupling without radial displacement:

$$\Delta L_r = L_{r\Delta} - L_r; \qquad (22)$$

$$L_r = 2\left(0,5D_{\text{int}}\cos\beta_3 + \frac{\pi d_r}{360}\beta_2 + L_{AB}\cos\beta_7\right). \qquad (23)$$

Radial load on shafts for rope elongation will be:

$$F_{rad} = A_\Sigma E_r\frac{\Delta L_r}{L_r}, \qquad (24)$$

where A_Σ — total area of wires cross section in rope; $E_p = 1,0\times10^5$ MPa — rope tensile modulus [3].

For illustration of obtained results modeling of real coupling operation done. The parameters of coupling were following: $D_{ext} = 120$ mm, $D_{int} = 65$ mm, $d_r = 20$ mm, $\xi = 45°$. Rope length in absence radial displacement, calculated with obtained earlier equation is $L_r = 184,37$ mm. Force calculation, made with methodic, developed earlier, show that strength condition corresponds rope with construction $6 \times 7(1+6)$ GOST 3069 with diameter $d_r = 2,2$ mm with total area of wires cross section in $A_{\sum} = 1,64$ mm^2. Radial misalignment for clarity $\Delta_r = 0,5$ mm — twice as much as allowed radial displacement for prototype coupling ROTEX SIZE 28 $[\Delta_r] = 0,25$ mm. Angle of rotation Jp. varied in range 0...360° across 30°, respectively angles of location coupling elements were calculate of formulas:

$$\varphi_{dg1i} = \varphi_i; \tag{25}$$

$$\varphi_{dn1i} = \varphi_i - \xi; \tag{26}$$

$$\varphi_{dg2i} = \varphi_{dg1i} + \pi; \tag{27}$$

$$\varphi_{dn2i} = \varphi_{dn2i} + \pi. \tag{28}$$

Whence with equation (12) determinate in each position distances $L_{A_1B_1}$and $L_{A_2B_2}$, with equation (13) distances $L_{A_1O_1}$ and $L_{A_2O_2}$, that with equation (15) reduced mounting displacement angles $\xi_1^{/}$ та $\xi_2^{/}$, and also angles $\beta_2^{/}$, $\beta_2^{//}$, $\beta_1^{/}$, $\beta_1^{//}$, and respectively rope length L_r for coaxial (23) and desaxial $L_{r\Delta}$ (18) coupling, rope elongation ΔL_r (22), and radial load F_{rad} from coupling on shafts (24).

Results of modeling are illustrated by Figure 5. From this figure we can make following conclusions. Rope elongation and radial load changing occurs under the asymmetric constant-sign law. In this maximal rope elongation is 0.0055 mm which is 9 times less than radial misalignment Δ_r in modeling. Calculated radial load in this conditions is only 5 N, which illustrating high compensating ability of new proposed coupling. Therefore in modeling misalignment of 0.5 mm radial stiffness of new coupling will be nearly 10 N/mm. The other types of couplings with elastic elements have much higher radial stiffness and making higher radial loads on shafts connected with coupling.

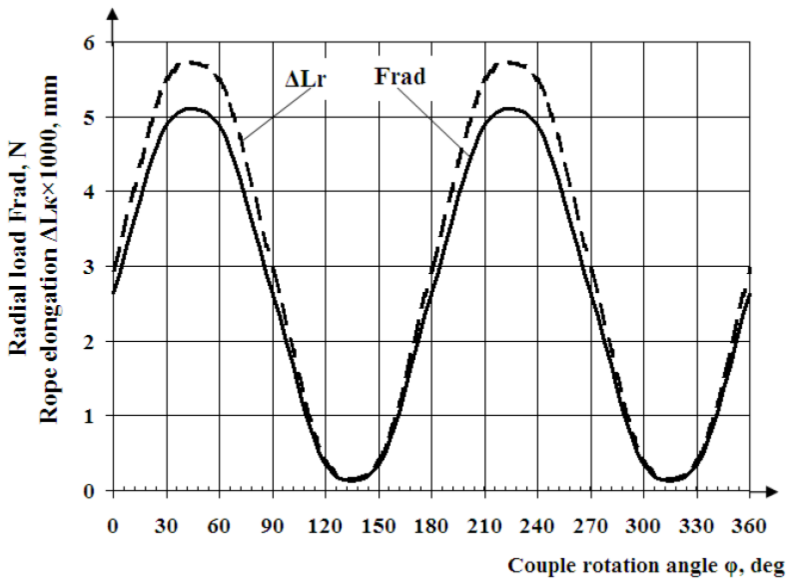

Figure 5. Graph of rope elongation and radial load changing from couple angle of coupling rotation

Done analytical researches allows to make following conclusions:

1. Mathematic model of roller-rope coupling in condition of misalignment made. It allows to calculate rope elongation and radial load on shafts from rope deformation depending on angle of coupling rotation.

2. Rope elongation and radial load changing occurs under the asymmetric constant-sign law and reaches a maximum in position when rope axe R_1R_2 approaching to radial displacement vector OO_1.

3. Roller-rope coupling construction provides minimal rope elongation and respectively radial load from coupling on shafts. Modeled coupling, for example, in radial misalignment of 0.5 mm have maximum elongation nearly 0.0055 mm and making radial load of only 5 N.

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Сәйкессіз шартты арқанды-роликті муфтаның жұмысын математикалық моделдеу

Мақалада радиалды сәйкессіздік жағдайында арқанды-роликті муфтаның конструкциясы, сонымен қатар оның математикалық моделінің жұмысы ұсынылған. Модельдеу нәтижесінде байланыстыратын біліктердің сәйкессіз болғандағы муфта жұмысының процесі анықталды. Сәйкессіздік шамасына қарай арқанның ұзаруын, сондай-ақ муфтаның коструктивті параметрлерін анықтайтын формулалар алынған. Зерттеу нәтижелері муфтаны кез келген бұрышқа бұрағанда муфтадан біліктерге радиалды жүктемені анықтауға мүмкіндік береді. Алынған формулалардың сандық модельдеуі муфтаның жоғары компенсациялық қасиетімен және төмен радиалды қаттылығымен сипатталатынын анықтауға мүмкіндік берді. Радиалды 0,5 мм жылжуда (резеңке жұлдызшасы бар муфта-прототипі үшін 0,25 мм болуы келісілген) арқанның ұзаруы 5,5 мкм болғанда арқанды-роликті муфта біліктерге 5 Н жүктеме жасайды. Алынған тәуелділіктер муфтаны жобалауда апробацияланған, ал есептеу нәтижелері оларды салыстыру арқылы тексеріледі және сәйкестік көрсетті. Алынған нәтижелерді арқанды-роликті муфталарды жобалауда пайдалануға болады.

Кілт сөздер: математикалық модель, сәйкессіздік, муфта, арқан, жүктер, кернеу.

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Математическое моделирование работы канатно-роликовой муфты в условиях несоосности

В статье представлена конструкция канатно-роликовой муфты, а также разработана математическая модель ее работы в условиях радиальной несоосности. В результате моделирования раскрыт процесс работы муфты при наличии несоосности соединяемых ею валов. Получены формулы, позволяющие определить удлинение каната в зависимости от величины несоосности, а также конструктивных параметров муфты. Результаты исследований позволяют также определить радиальную нагрузку от муфты на валы при любом значении угла поворота муфты. Численное моделирование по полученным формулам дало возможность установить, что муфта характеризуется высокой компенсирующей способностью и низкой радиальной жесткостью. Так, при радиальном смещении 0,5 мм (для муфты-прототипа с резиновой звездочкой допускается 0,25 мм) канатно-роликовая муфта создает на валы нагрузку всего 5 Н при удлинении каната в 5,5 мкм. Полученные зависимости апробированы при проектировании муфты, а результаты расчета по ним сравнены с результатами построения и совпали. Результаты могут быть использованы при проектировании канатно-роликовых муфт.

Ключевые слова: математическая модель, несоосность, муфта, канат, нагрузки, натяжение.

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On the calculation of the rectangular finite element of the plate

The article is devoted to the study of the thin plate bending by the finite element method. The application of the finite element method to solving the problem of the plate bending leads to the necessity of studying the rectangular finite element of the plate. All deformation and statics characteristics of the plate are functions of the displacement in the direction of the normal to the middle surface of the plate, which is determined by the deflection function. In the article, the formation of the plate deflection function in explicit form is carried out. The ways for finding the deflection function by division of the variables in the equilibrium equation of the plate, through an incomplete fourth-degree polynomial and in the form of Hermite polynomials are presented. The article is focused mainly on mechanics, engineers and scientific employees of technical specialties.

Keywords: finite element method, rectangular finite element, deflection function, angular displacements, Hermite polynomials

Introduction. Thin-walled structures are encountered in many branches of technology, such as civil, mechanical, aeronautical, marine, and chemical engineering. Such a widespread use of plate and shell structures arises from their intrinsic properties. When suitably designed, even very thin plates, and especially shells, can support large loads. Thus, they are utilized in structures such as aerospace vehicles in which light weight is essential.

One of the elements of thin-walled structures is a plate, which has an independent and wide application. Plates represent principal elements of aerospace structures, including fuselages of planes and missiles, control surfaces, bulkheads, helicopter blades, and others. In construction, the plates are widely used in the form of decking and panels, reinforced concrete slabs for coating industrial and residential buildings, slabs for foundations of massive structures and etc.

Mathematical models of calculating plates, closely related to the study of applied problems, have acquired special relevance in connection with the expanding volume of their applications in various fields of science and technology. The multiple applications, shapes, and materials found in plate structures dictate the necessity of a comprehensive approach to their analysis reflected in relevant theories and methodologies. Therefore, questions related to theoretical studies of the work of plates remain significant and relevant [1].

At calculation of plates by analytical methods in the most general formulation (with arbitrary contour supports (including elastic supports), with different types of loading) one has to face big mathematical difficulties, and in the majority of cases it is not possible to receive the analytical solution. It is possible to solve such a problem using a very efficient finite element method, which for plates is a numerical approximate method, but gives a sufficiently high accuracy of solutions.

The finite element method usually abbreviated as FEM is a numerical technique to obtain approximate solution to physical problems. FEM was originally developed to study stresses in complex aircraft structures; it has since been extended and applied to the broad field of continuum mechanics, including fluid mechanics and heat transfer and also mechanics of deformable solids and structural mechanics. Because of its capability to handle complex problems and its flexibility as an analysis tool, FEM has gained a prominent role in engineering analysis and design.

The name of this method to some extent predetermines its essence: when using the finite element method, the calculated design is mentally divided into separate elements, the stress-strain state of which is previously studied in detail and can be considered known. It is supposed that the elements are connected to each other at a finite number of points, called nodes. At these points, forces characterizing the interaction of individual elements, or displacements, through which, ultimately, the stresses and displacements of each element are calculated, are determined. Thus, the problem is discretized and reduced to solving a system of algebraic equations with respect to unknown forces or node displacements. FEM is characterized by a variational formulation, a discretization strategy, one or more solution algorithms and post-processing procedures [2].

According to the method of obtaining the basic resolving equations, the finite element method has four main types: the direct method, the variational method, the weighted residual method and the energy balance method.

Depending on what values are taken as unknown, there are three classical approaches used in FEM: the force method, the displacement method and the mixed method. We note that due to a number of advantages, the approach based on the idea of the displacement method is the most widespread in the FEM [3].

Replacement of the original construction by a set of discrete elements makes it possible to simplify the calculation of various construction objects: rod systems, thin-walled and massive structures and real structures in which rods, plates, shells, arrays are combined. This circumstance makes the finite element method very universal and explains its increased popularity.

Moreover, the advantage of the finite element method is a comparatively simple implementation on a PC with the help of a software package. At the same time, it is easy to set any boundary conditions of the plate on the contour, including elastic ones, and various types of load.

Depending on the type of the considered construction, the type of the finite element is determined. So for rod systems rods with different supports at the ends, representing the nodes of the element, can be taken as a finite element. Thin-walled spatial systems consisting of plates and shells are divided into triangular, rectangular or elements of any other shape with nodes at angular points. Next, we focus on the consideration of rectangular finite elements and their application in the calculations of plates.

Bending of thin plates. We consider the problem of calculating thin rigid plates. Their thickness should not exceed the $\frac{1}{5}$ of smallest side of the plate, and the deflection in bending should not exceed the thickness h (Fig. 1, a). On the basis of Kirchhoff-Lyava's hypotheses about the smallness of the normal stresses, perpendicular to the middle surface of a plate, and the smallness of direct normals to the same surface the technical theory of a bending of thin plates is constructed [4].

The assumptions derived from accepted hypotheses can be formulated as follows:

1. Normal stresses σ_z and also tangential stresses τ_{xz}, τ_{yz} are negligibly small in comparison with stresses which are considered as the main: σ_x, σ_y, τ_{xy} (Fig. 1, b). Therefore, we accept $\sigma_z = \tau_{xz} = \tau_{yz} = 0$.

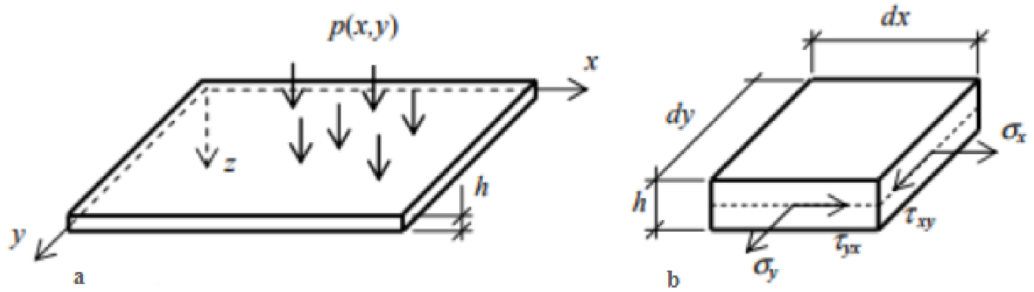

Figure 1. The thin plate

2. The displacements in the direction of the axis z are constant along the thickness of the plate and are equal to the deflections of the middle surface, which does not deform in its plane. At the same time the external load must be perpendicular to the plate surface, that is, to the xy coordinate plane.

These assumptions simplify the mathematical model of plate bending, reducing it to a two-dimensional problem. All deformation and statics quantities of the plate are functions of only one unknown, namely the displacement in the direction of the normal to the middle surface of the plate.

However, unlike a plane problem, as a classical two-dimensional problem, the deflections of a plate are described by a fourth order differential equation, but not of the second order, deformations are derivatives of displacements of the second, but not of the first order. Thus, in the expression for the potential energy functional, second-order derivatives also appear. When using the finite element method, this leads to some difficulties related to the approximation of the deflection function $w(x, y)$.

Firstly, the approximating polynomials used in the bending of plates are much more complicated than for the plane problem of the theory of elasticity, besides, their form is not unique for a particular finite element. For example, for a deflection function, a fourth-degree polynomial with twelve indefinite coefficients can contain different terms, which in turn leads to different stiffness matrices for the elements.

Secondly, the conditions of compatibility between adjacent elements must be fulfilled not only for the function of deflections, but also for its first derivatives. Nevertheless, in practice incompatible finite elements, in which continuity on the boundaries between elements is performed only for deflections, are often applied. For example, such an element is a four-node rectangular finite element with twelve degrees of freedom, which provides acceptable accuracy of the solution and is used to calculation of plates having a rectangular shape [3].

The deflection function of a finite rectangular element. We select a rectangular finite element $0 < x_1 < a$, $0 < x_2 < b$ from the plate and consider it in the local coordinate system $(x_1, x_2\ )$.

We number knots of a rectangular final element consistently (clockwise, starting from the upper left node) and introduce new coordinates x, y so that $x = \frac{x_1}{a}$, $y = \frac{x_2}{b}$. The deformed state is completely determined by nodal displacements. In each node i $(i = 1, 2, 3, 4)$ of the finite element, there are three displacements: w_i is a deflection, φ_{xi} is an angle of rotation along the axis x, φ_{yi} is an angle of rotation along the axis y (Fig. 2).

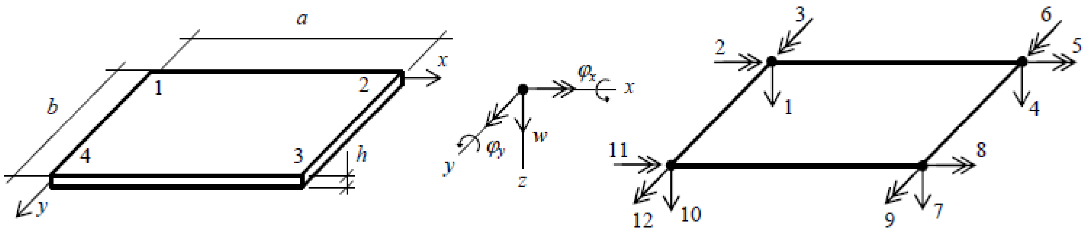

Figure 2. Displacements of the finite element in the nodes

Angular displacements are defined in terms of partial derivatives of the deflection function as follows

$$\varphi_x = \frac{\partial w}{\partial x}, \quad \varphi_y = \frac{\partial w}{\partial y}. \tag{1}$$

To determine the deflection function of the finite element, we use the equation of plate equilibrium in the absence of a transverse external load

$$\nabla^2 \nabla^2 w = 0. \tag{2}$$

We search the solution of equation (2) in the form

$$w(x, y) = X(x)Y(y). \tag{3}$$

Substituting (3) into (2) we obtain an equation which is divided into three independent equations

$$X^{IV} = 0, \quad X''Y'' = 0, \quad Y^{IV} = 0. \tag{4}$$

The first and the third equations are the bending equations for mutually perpendicular beams. The solutions of the first and the third equations (4) in the coordinate functions have the form

$$X(x) = f_1(x)z_i + f_2(x)\theta_i + f_3(x)z_j + f_4(x)\theta_j;$$

$$Y(y) = g_1(y)z_k + g_2(y)\theta_k + g_3(y)z_l + g_4(y)\theta_l, \tag{5}$$

where i, j, k, l — are beam nodes; the values z_m $(m = i, j, k, l)$ are vertical displacements; $\frac{\theta_i}{a}$, $\frac{\theta_j}{a}$, $\frac{\theta_k}{b}$, $\frac{\theta_l}{b}$ — are angular displacements of the given beam. The beam coordinate functions for transverse bending have the values

$$f_1(x) = 2x^3 - 3x^2 + 1, \quad f_2(x) = x^3 - 2x^2 + x,$$

$$f_3(x) = -2x^3 + 3x^2, \quad f_4(x) = x^3 - x^2. \tag{6}$$

The functions $g_n(y)$ $(n = 1, 2, 3, 4)$ are obtained from the expressions (6) by replacing x by y

$$g_1(y) = 2y^3 - 3y^2 + 1, \quad g_2(y) = y^3 - 2y^2 + y;$$

$$g_3(y) = -2y^3 + 3y^2, \quad g_4(y) = y^3 - y^2. \tag{7}$$

Substituting (5) into (3), we write down the deflection function of the finite element in the following form

$$w(x, y) = \sum_{i,j=1}^{4} c_{ij}\xi_{ij}(x, y), \tag{8}$$

where arbitrary constants c_{ij} and functions $\xi_{ij}(x, y)$ have the form

$$
\begin{array}{llll}
c_{11}=z_iz_k, & c_{12}=z_i\theta_k, & c_{13}=z_iz_l, & c_{14}=z_i\theta_l,\\
c_{21}=z_k\theta_i, & c_{22}=\theta_i\theta_k, & c_{23}=z_l\theta_i, & c_{24}=\theta_i\theta_l,\\
c_{31}=z_jz_k, & c_{32}=z_j\theta_k, & c_{33}=z_j\theta_l, & c_{34}=z_jz_l,\\
c_{41}=z_k\theta_j, & c_{42}=\theta_j\theta_k, & c_{43}=z_l\theta_j, & c_{44}=\theta_j\theta_l;
\end{array} \tag{9}
$$

$$
\xi_{i,j}(x,y)=f_i(x)g_j(y),\quad i,j=1,2,3,4. \tag{10}
$$

At the nodes of the finite element, we have the following boundary conditions

$$
w(\alpha_i,\beta_i)=w_i,\quad \varphi_x(\alpha_i,\beta_i)=\lambda_i,\quad \varphi_y(\alpha_i,\beta_i)=\mu_i,\quad i=1,2,3,4, \tag{11}
$$

$$
\alpha_1=\alpha_4=0,\quad \alpha_2=\alpha_3=1,\quad \beta_1=\beta_2=0,\quad \beta_3=\beta_4=1.
$$

The values w_1-w_4 — are the linear displacements of the finite element; $\lambda_1-\lambda_4$ — are angular displacements of nodes of a finite element along the axis x; $\mu_1-\mu_4$ — are the angular displacements of the nodes of the element along the axis y.

Defining the derivatives (1) of the function (8) and substituting them and the function (8) in turn into the boundary conditions (11), we obtain the values of the arbitrary constants (9) in the form

$$
c_{m_i,n_i}=w_i,\quad c_{p_i,n_i}=\lambda_i a,\quad c_{m_i,p_i}=\mu_i b,\quad i=1,2,3,4; \tag{12}
$$

$$
m_1=m_4=n_1=n_2=1,\quad p_1=p_4=2,\quad m_2=m_3=n_3=n_4=3,\quad p_2=p_4=4.
$$

Thus, twelve of the sixteen arbitrary constants from the boundary conditions (11) are calculated.

To determine the remaining four arbitrary constants, we use the second equation in (4) and introduce the denotation

$$
\eta(x,y)=X''(x)Y''(y). \tag{13}
$$

For the function $\eta(x,y)$, from (4) we have the following boundary conditions at the nodes of the finite element

$$
\eta(\alpha_i,\beta_i)=0,\quad i=1,2,3,4. \tag{14}
$$

Having determined the second derivatives of (5) and computing their products by (13), and then substituting in (14) we obtain the following system of equations.

We calculate the second derivatives of (5) and their products by (13). Then substituting these expressions into (14), we obtain the following system of equations

$$
\begin{cases}
4c_{22}+2c_{24}+2c_{42}+a_{44}=-d_1;\\
2c_{22}+c_{24}+4c_{42}+2_{44}=d_2;\\
2c_{22}+4c_{24}+c_{42}+2a_{44}=d_3;\\
c_{22}+2c_{24}+2c_{42}+4a_{44}=-d_4;
\end{cases} \tag{15}
$$

where the free terms expressed in terms of the known coefficients (12) take the following values

$$
\begin{aligned}
d_1&=9c_{11}+6c_{12}-9c_{13}+3c_{14}+6c_{21}-6c_{23}-9c_{31}+6c_{32}+9c_{33}-3c_{34}+3c_{41}-3c_{43};\\
d_2&=-9c_{11}-6c_{12}+9c_{13}-3c_{14}-3c_{21}+3c_{23}+9c_{31}+6c_{32}-9c_{33}+3c_{34}-6c_{41}+6c_{43};\\
d_3&=-9c_{11}-3c_{12}+9c_{13}-6c_{14}-6c_{21}+6c_{23}+9c_{31}+3c_{32}-9c_{33}+6c_{34}-3c_{41}+3c_{43};\\
d_4&=9c_{11}+3c_{12}-9c_{13}+6c_{14}+3c_{21}-3c_{23}-9c_{31}+3c_{32}+9c_{33}-6c_{34}+6c_{41}-6c_{43}.
\end{aligned} \tag{16}
$$

Solving the system of equations (15) and considering (16), we find the remaining arbitrary constants

$$
\begin{cases}
c_{22}=-c_{11}-c_{12}+c_{13}-c_{21}+c_{23}+c_{31}+c_{32}-c_{33};\\
c_{24}=-c_{11}+c_{13}-c_{14}-c_{21}+c_{23}+c_{31}-c_{33}+c_{34};\\
c_{42}=-c_{11}-c_{12}+c_{13}+c_{31}+c_{32}-c_{33}-c_{41}+c_{43};\\
c_{44}=-c_{11}+c_{13}-c_{14}+c_{31}-c_{33}+c_{34}-c_{41}+c_{43}.
\end{cases} \tag{17}
$$

Taking into account the values of the beam functions (6), (7) and the values of the arbitrary constants (12), (17), we write the deflection function of the finite element (8) with regard to (10) in the following form

$$w(x,y) = \vec{r}^T \vec{s}; \tag{18}$$

$$\vec{r}^T = [r_1 \ r_2 \ r_3 \ r_4 \ r_5 \ r_6 \ r_7 \ r_8 \ r_9 \ r_{10} \ r_{11} \ r_{12}];$$

$$\vec{s} = [w_1 \ \lambda_1 \ \mu_1 \ w_2 \ \lambda_2 \ \mu_2 \ w_3 \ \lambda_3 \ \mu_3 \ w_4 \ \lambda_4 \ \mu_4].$$

Here $\vec{r}^T$ is the transposed vector of the coordinate functions of the plate; $\vec{s}$ is the vector of nodal displacements of a rectangular finite element.

The coordinate functions of the plate have the form

$$r_1(x,y) = f_1(x)(1-y) + (1-x)g_1(y) - (1-x)(1-y), \quad r_2(x,y) = af_2(x)(1-y), \quad r_3(x,y) = b(1-x)g_2(y);$$

$$r_4(x,y) = f_3(x)(1-y) - xg_3(y) + xy, \quad r_5(x,y) = af_4(x)(1-y), \quad r_6(x,y) = bxg_2(y); \tag{19}$$

$$r_7(x,y) = f_3(x)y + xg_3(y) + x, \quad r_8(x,y) = af_4(x)y, \quad r_9(x,y) = bxg_4(y);$$

$$r_{10}(x,y) = (1-x)g_3(y) - f_3(x)y + xy, \quad r_{11}(x,y) = af_2(x)y, \quad r_{12}(x,y) = b(1-x)g_4(y),$$

where $1-x$, x, $1-y$, y are the coordinate functions of the linear element (the rod) when the torsion is made in the direction of coordinate axes.

The same form (18) for the deflection function can be obtained if the solution of the equilibrium equation for plates (2) is sought in a more particular form

$$w(x,y) = \sum_{i,j=1}^{4} c_{ij} f_i(x) g_j(y), \tag{20}$$

where c_{ij} are arbitrary constants to be determined; $f_i(x)$ and $g_j(y)$ are the beam coordinate functions for bending along the axes directions and , respectively. Arbitrary constants are defined in a manner similar to that described above. A complete calculation on finding the deflection function in the form (20) is presented in [5].

The proposed classical method of determining the deflection function allows to obtain this function in explicit form and to give a physical meaning for it. The analysis of the formulas (19) shows that the coordinate functions of the finite element of the plate are equal to the product of the coordinate function of the beam when it is bent to the coordinate function of the rod during torsion. Each coordinate function describes a finite element deformation caused by a single nodal displacement value. The final form of the defined deflection function depends on the coordinate beam functions in bending and on the coordinate functions of the rod (linear element) in torsion.

Thus, the method based on the General solution of the biharmonic equation (2) allows us to obtain the deflection function of a rectangular finite element in explicit form and to give it a vivid physical meaning: the deformation of the finite element of the plate is representable through deformations of the beam and the rod.

In the traditional approach, the deflection function is given as an incomplete fourth-degree polynomial. The following expression of the deflection function

$$w(x,y) = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 x^2 + \alpha_5 xy + \alpha_6 y^2 + \alpha_7 x^3 + \alpha_8 x^2 y + \alpha_9 xy^2 + \alpha_{10} y^3 + \alpha_{11} x^3 y + \alpha_{12} xy^3$$

has certain advantages. In particular, along any line $x = const$ or $y = const$ the displacement $w(x,y)$ will change by cubic law. All external boundaries and boundaries between elements consist precisely of such lines. Since the third degree polynomial is uniquely determined by four constants, the displacements along the boundary are uniquely determined by the values of displacements and angles of inclination at the nodal points at the ends of this boundary. And since for adjacent elements the values at the ends of the boundary are the same, then along any boundary between finite elements the function $w(x,y)$ will be continuous.

Constants $\alpha_1, ..., \alpha_{12}$ are determined from a system of twelve equations relating the values $w(x,y)$ and angles of inclination at the node points, which are obtained as a result of substituting the coordinates of these points. Arbitrary constants are determined (is founded) by inversion of the twelfth order matrix or by other linear algebra methods [6].

As possible states, deflection functions can be adopted on the basis of Hermite polynomials, sometimes used in calculations by the finite element method in displacements. For example, in the case of a rectangular plate under arbitrary lateral load, the deflection function of a rectangular finite element can be represented as a polynomial of the fourth degree. Such a deflection function $w(x,y)$ can be obtained in the form of Hermite polynomials [1]

$$w(x,y) = z_1 H_{01}(x)H_{01}(y) - z_2 H_{11}(x)H_{01}(y) + z_3 H_{01}(x)H_{11}(y) + z_4 H_{02}(x)H_{01}(y) -$$

$$- z_5 H_{12}(x)H_{01}(y) + z_6 H_{02}(x)H_{11}(y) + z_7 H_{02}(x)H_{02}(y) + z_8 H_{12}(x)H_{02}(y) +$$

$$+ z_9 H_{02}(x)H_{12}(y) + z_{10} H_{02}(x)H_{02}(y) + z_{11} H_{12}(x)H_{02}(y) + z_{12} H_{02}(x)H_{12}(y),$$

where the values of the nodal displacements $z_1, ..., z_{12}$ are equal to the followig values

$$z_{3n-2} = w(\gamma_n, \delta_n), \quad z_{3n-1} = -\frac{\partial w}{\partial x}(\nu_n, \omega_n), \quad z_{3n} = \frac{\partial w}{\partial y}(\rho_n, \tau_n), \quad n = 1, 2, 3, 4;$$

$$\gamma_1 = \gamma_{10} = \delta_1 = \delta_4 = \nu_2 = \nu_{11} = \omega_2 = \omega_5 = \rho_3 = \rho_{12} = \tau_3 = \tau_6 = 0;$$

$$\gamma_4 = \gamma_7 = \nu_5 = \nu_8 = \rho_6 = \rho_9 = a, \quad \delta_7 = \delta_{10} = \omega_8 = \omega_{11} = \tau_9 = \tau_{12} = b.$$

Conclusion. The deflection function of a rectangular finite element determines uniquely the deformed state of an element by means of its nodal displacements. As it was said above, on the basis of the explicit shape of the deflection function, all necessary matrices (deformation, stress, stiffness and load) can be obtained. Note that the deflection function of the finite element forms the basis for calculating the plate by the finite element method. On its basis it is not difficult to develop an algorithm for calculation with the implementation on a PC.

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Пластинаның тікбұрышты соңғы элементін есептеу туралы

Мақала соңғы элементтер әдісімен жіңішке пластинаның иілуін зерттеуге арналған. Пластинаның иілу есебін шешуге соңғы элементтер әдісінің қолданылуы пластинаның тікбұрышты соңғы элементін зерттеу қажеттілігіне әкеледі. Ауытқулар функциямен анықталатын, пластинаның барлық деформациясы және статикалық мәндері қалыпты бағытта пластинаның орта бетінде жылжу функциясы болып табылады. Мақалада пластиналардың ауытқулар функциясы айқын түрде қалыптастыру қарастырылды. Ауытқулар функциясы пластинаның тепе-теңдік теңдеуіндегі айнымалы мәндерді төртінші дәрежелі толық емес полиномдық және Эрмит полиномы түрінде бөлу арқылы табу әдісі берілген. Мақала негізінен механиктерге, инженерлер мен техникалық қызметкерлерге бағытталған.

Кілт сөздер: соңғы элементтер әдісі, тікбұрышты соңғы элемент, ауытқулар функциясы, бұрыштық жылжу, Эрмит полиномы.

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О расчете прямоугольного конечного элемента пластины

Статья посвящена исследованию изгиба тонкой пластины методом конечных элементов. Приложение метода конечных элементов к решению задачи об изгибе пластины приводит к необходимости исследования прямоугольного конечного элемента пластины. Все деформационные и статические величины пластины являются функциями перемещения в направлении нормали к срединной поверхности пластины, которое определяется функцией прогибов. В статье приведено построение функции прогибов пластин в явном виде. Приведены способы нахождения функции прогибов разделением переменных в уравнении равновесия пластины, через неполный полином четвертой степени и в виде полиномов Эрмита. Статья ориентирована, главным образом, на механиков, инженеров и сотрудников технических специальностей.

Ключевые слова: метод конечных элементов, прямоугольный конечный элемент, функция прогибов, угловые перемещения, полиномы Эрмита.

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ТАҢДАМАЛЫ ТАҚЫРЫПТАРҒА ШОЛУ
ОБЗОРЫ ИЗБРАННЫХ ТЕМАТИК
REVIEWS OF SELECTED TOPICS

UDC 512.54

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On the automorphism groups of relatively free groups of infinite rank: a survey

The paper is intended to be a survey on some topics within the framework of automorphisms of a relatively free groups of infinite rank. We discuss such properties as *tameness, primitivity, small index, Bergman property*, and so on.

Key words: variety of groups, relatively free group, countably infinite rank, automorphism group, tame automorphism, small index property, cofinality, Bergman property.

Introduction

Let F_∞ be a free group of infinite rank, in particular, let F_ω be a free group of countably infinite rank. In further of the paper $X_\omega = \{x_1, ..., x_i, ...\}$ be a basis of F_ω, and $X_n = \{x_1, ..., x_n\}$ be a basis of a free group F_n of any finite rank n. Thus F_n is naturally embedded into F_ω, and F_n is naturally embedded into every group F_m where $m \geq n$. More generally, let Λ be a set (finite or infinite) and F_Λ be the free group of rank $|\Lambda|$ with basis $X_\Lambda = \{x_\lambda : \lambda \in \Lambda\}$. Then for each subset Ξ of Λ free group F_Ξ is a subgroup of F_Λ, and for every $\Psi, \Xi \subseteq \Psi \subseteq \Lambda$, free group F_Ξ is a subgroup of F_Ψ.

For any variety of groups $\mathcal{C}$, let $V = \mathcal{C}(F_\Lambda)$ denote the verbal subgroup of F_Λ corresponding to $\mathcal{C}$ (see [1] for information on varieties and related concepts.). Then $G_\Lambda = F_\Lambda/V$ is the free group of rank $|\Lambda|$ in $\mathcal{C}$. In particular, G_ω is the free group of countably infinite rank in $\mathcal{C}$. Write $\bar{x}_i = x_iV$ for $i = 1, ..., i, ...$. Then $X_V = \{\bar{x}_1, ..., \bar{x}_i, ...\}$ is a basis of G_ω. For each Λ there is the standard homomorphism of F_Λ onto G_Λ. Then for each subset Ξ of Λ free group G_Ξ is a subgroup of G_Λ, and for every $\Psi, \Xi \subseteq \Psi \subseteq \Lambda$, free group G_Ξ is a subgroup of G_Ψ.

If α is an automorphism of G_Λ then $\{\alpha(\bar{x}_\lambda) : \lambda \in \Lambda\}$ is also a basis of G_Λ and every basis of G_Λ has this form.

Any automorphism ϕ of F_Λ induces an automorphism $\bar{\phi}$ of G_Λ. Thus every basis of F_Λ induces a basis of G_Λ. The converse however is not always true; in general, there are automorphisms of G_Λ which are not induced by automorphisms of F_Λ. See [2, 3] for relevant results.

An automorphism of G_Λ which is induced by an automorphism of F_Λ is called *tame.* If $\{g_\xi : \xi \in \Xi\}$ are distinct elements of G_Λ such that $\{g_\xi : \xi \in \Xi\}$ is contained in a basis of G_Λ then $\{g_\xi : \xi \in \Xi\}$ is called a *primitive system* of G_Λ.

Any primitive system $\{f_\xi : \xi \in \Xi\}$ of F_Λ induces a primitive system of G_Λ that is called *tame.* But, in general, not every primitive system of G_Λ is induced by a primitive system of F_Λ. We observe different tameness properties in the following Section 2.

An other topic of this paper is small index property. A countable first-order structure M is said to have the small index property if every subgroup of the automorphism group $\mathrm{Aut}(M)$ of index less than $2^{\aleph_0}$ contains the pointwise stabilizer $\mathrm{C}(U)$ of a finite subset U of the domain of M. In Section 3, we give results about small index property for relatively free groups of countably infinite rank.

Further in the paper, $\mathcal{A}$ denotes the variety of all abelian groups, $\mathcal{N}_k$ means the variety of all nilpotent groups of class $\leq k$, and $\mathcal{A}^2$ stands for the variety of all metabelian groups (for any varieties $\mathcal{C}$ and $\mathcal{D}$, $\mathcal{CD}$ denotes the variety of all groups with a normal subgroup in $\mathcal{C}$ and factor group in $\mathcal{D}$, thus $\mathcal{A}^2 = \mathcal{AA}$). We also denote by $A_\infty, N_{k,\infty}$ and M_ω the free groups of the countably infinite rank in the varieties $\mathcal{A}, \mathcal{N}_k$ and $\mathcal{A}^2$, respectively. For any group H and each positive integer k we denote by $\gamma_k(H)$ the kth member of the low central series in H. In particular, $\gamma_1(H) = H$ and $\gamma_2(H) = H'$, the derived subgroup of H.

Tameness

In this section, we present known results about tame automorphisms of relatively free groups of infinite rank. It is well known that every automorphism of A_∞ can be lifted to an automorphism of F_∞, thus is tame. The following results also belong to this direction.

Theorem 1 (Bryant and Macedonska [4]). Let F_∞ be a free group of infinite rank and let V be a characteristic subgroup of F_∞ such that F_∞/V is nilpotent. Then $G_\infty = F_\infty/V$ is relatively free group of infinite rank in a nilpotent variety and every automorphism of G_∞ is induced by an automorphism of F_∞, thus is tame.

If F_∞ and V are as in the statement of the theorem then V contains $\gamma_k(F_\infty)$ for some positive integer k. Since V is characteristic in F, it follows by a result of Cohen [5] that V is fully characteristic in F_∞. Thus V defines nilpotent variety $\mathcal{N}$ of groups, then $V = \mathcal{N}(F_\infty)$ is the corresponding verbal subgroup of F_∞. Hence F_∞/V is a relatively free group in $\mathcal{N}$.

To prove Theorem 1 the authors defined a property called the *finitary lifting property* (see details below) and obtained the following two results.

Proposition 1. Every nilpotent variety of groups has the finitary lifting property.

Proposition 2. If $\mathcal{C}$ is any variety of groups with the finitary lifting property and F_∞ is a free group of infinite rank then every automorphism of $F_\infty/\mathcal{C}(F_\infty)$ is induced by an automorphism of F_∞.

Let F_∞ be a free group of infinite rank and let $\{x_\lambda : \lambda \in \Lambda\}$ be a basis of F_∞. An automorphism ϕ of F_∞ will be called *finitary* if there is a finite subset U of this basis such that $\phi(x) = x$ for each free generator $x \notin U$. Let $\mathcal{C}$ be a variety of groups and write $V = \mathcal{C}(F_\infty)$. Suppose that Γ and Δ are subsets of Λ such that $\Gamma \cap \Delta = \emptyset$, Δ is finite, and $\Lambda \setminus (\Gamma \cup \Delta)$ is infinite. Let α be automorphism of F_∞/V such that $\alpha(x_\lambda V) = x_\lambda V$ for all $\lambda \in \Gamma$. We say that the triple (Γ, Δ, α) can be lifted if there exists a finitary automorphism ξ of F_∞ such that $\xi(x_\lambda) = x_\lambda$ for all λ in Γ and $\xi(x_\lambda V) = \alpha(x_\lambda V)$ for all $\lambda \in \Delta$. Such a finitary automorphism ξ is called a *lifting* of (Γ, Δ, α). We say that $\mathcal{C}$ has the *finitary lifting property* if, for every F_∞ of infinite rank, every triple (Γ, Δ, α) can be lifted.

The theorem generalises some previously known results. The case where $V = \gamma_2(F_\infty)$ is a result of Swan (see [6]). A closely related result had been obtained a few years earlier by Burns and Farouqi [7] who proved that if $A_\omega(p)$ is a free abelian of exponent p group of countably infinite rank and p is a prime number then every automorphism of $A_\omega(p)$ is induced by an automorphism of A_ω. In [8], Gawron and Macedonska proved the discussed property in the cases $V = \gamma_i(F_\omega)$ for $i = 3, 4$.

For each positive integer m, we denote by $\mathcal{A}(m)$ the variety of all abelian groups of exponent dividing m. Also we denote by $\mathcal{A}(0)$ the variety of all abelian groups $\mathcal{A}$.

Theorem 2 (Bryant and Groves [9]). Let m and n be non-negative integers. Every automorphism a free group of infinite rank in the metabelian product variety $\mathcal{A}(m)\mathcal{A}(n)$ is tame. In particular, every automorphism of M_∞ is tame.

Theorem 3 (Bryant and Gupta [10]). Let $\mathcal{C}$ be a variety such that $\mathcal{A}^2 \subseteq \mathcal{C} \subseteq \mathcal{N}_k\mathcal{A}$ for some k, and G_∞ be a free group of infinite rank in $\mathcal{C}$. Then every automorphism of G_∞ is tame.

The following result generalizes Theorems 1, 2 and 3.

Theorem 4 (Bryant and Roman'kov [11]). Let $\mathcal{C}$ be a subvariety of $\mathcal{N}_k\mathcal{A}$ for some k. Let G_∞ be a free group of infinite rank in $\mathcal{C}$. Then every automorphism of G_∞ is tame.

The main ingredient in the proof of this theorem are the following result that has its own interest.

Theorem 5 (Bryant and Roman'kov [11]). Let $\mathcal{C}$ be a subvariety of $\mathcal{N}_k\mathcal{A}$, where $k \geq 1$. Let n be a positive integer and write $l = 2^n(n+1) + 2k$. Then every primitive system of $F_n/\mathcal{C}(F_n)$ is induced by some primitive system of F_l.

Above we presented some positive results about tameness of the automorphisms of relatively free groups of infinite rank. However, there are negative results for other varieties.

Theorem 6 (Bryant and Groves [12]). Let $\mathcal{K} = \mathrm{var}(K)$ be the variety generated by a non-abelian finite simple group K, and G_∞ is the free group of the countable infinite rank in $\mathcal{K}$. Then there is an automorphism of G_∞ which is not induced by an automorphism of F_∞.

Small index property

Hodges, Hodkinson, Lascar and Shelah established in [13] that ω-categorical and ω-stable structures, and so called random graph have the small index property. In [14], Bryant and Evans use the methods of the paper [13] to show that the free group of countably infinite rank and certain relatively free groups of countably infinite rank have the small index property.

Theorem 7 has some immediate consequences through the results of [14] and [15].

Theorem 8 (Bryant and Roman'kov [11]). Let F_ω be a free group of countably infinite rank and let $\mathcal{C}$ be a subvariety of $\mathcal{N}_k\mathcal{A}$, where $k \geq 1$. Then $F_\omega/\mathcal{C}(F_\omega$ has the basis cofinality property and the small index property. The automorphism group $\mathrm{Aut}(F_\omega/\mathcal{C}(F_\omega)$ is not the union of a countable chain of proper subgroups. Also, $\mathrm{Aut}(F_\omega/\mathcal{C}(F_\omega)$ has no proper normal subgrou of index less than $2^{\aleph_0}$ and it is a perfect group.

Recall that a group is called *perfect* if it equals its derived subgroup.

Other properties

Completeness. A group G is said to be complete if G is centreless and every automorphism of G is inner. By the Burnside's criterion for a centerless group G its the automorphism group $\mathrm{Aut}(G)$ is complete if and only if the subgroup $\mathrm{Inn}(G)$ of all inner automorphisms of G is a characteristic subgroup of the group $\mathrm{Aut}(G)$ (that is, preserved under the action of all automorphisms of the group $\mathrm{Aut}(G)$).

Theorem 9 (Tolstykh [16]. The automorphism group $\mathrm{Aut}(F_\infty)$ of any free group of infinite rank is complete.

This statement was derived from the following assertions:

• The family of all inner automorphisms of F_∞ determined by powers of primitive elements of F_∞ is first-order definable in $\mathrm{Aut}(F_\infty)$, hence $\mathrm{Inn}(F_\infty)$ is a characteristic subgroup of $\mathrm{Aut}(F_\infty)$.

• The subgroup $\mathrm{Inn}(F_\infty)$ is then first-order definable in $\mathrm{Aut}(F_\infty)$.

Theorem 10 (Tolstykh [17, 18]). For any $k \geq 2$, the automorphism group $\mathrm{Aut}(N_{\infty,k})$ of any free nilpotent group $N_{\infty,k}$ of infinite rank is complete.

Note, that $\mathrm{Inn}(\mathrm{Aut}(A_\infty)) = \mathrm{Aut}(\mathrm{Aut}(A_\infty))$ ([19], [20]). Anyway $\mathrm{Aut}(A_\infty)$ is not complete because it contains a non-central the inverting authomorphism.

In [17], this statement was proved for the case $k = 2$, and in [18], for the general case.

Theorem 11 (Tolstykh [21]). Let F_∞ be an infinitely generated free group, $R \leq F'_\infty$ a fully characteristic subgroup of F_∞ such that the quotient group F_∞/R is residually torsion-free nilpotent. Then the group $\mathrm{Aut}(F_\infty/R)$ is complete.

Corollary. Let G_∞ be a free abelian-by-nilpotent (in particular metabelian or free solvable of class ≥ 3) relatively free group of infinite rank. Then the group $\mathrm{Aut}(G_\infty)$ is complete.

Generalized small index property. Let F be a relatively free algebra of infinite rank κ. We say that F has the *generalized small index property* if any subgroup of $\mathrm{Aut}(F)$ of index at most κ contains the pointwise stabilizer $\mathrm{C}(U)$ of a subset U of the domain of F of cardinality less than κ.

Theorem 12 (Tolstykh [22] Every infinitely generated free nilpotent (in particular free abelian) group N_∞ has the generalized small index property.

Bergman property. A group G is said to have the *Bergman property* (the property of *uniformity of finite width*) if given any generating X with $X = X^{-1}$ of G, we have that $G = X^l$ for some natural l, that is, every element of G is a product of at most l elements of X. The property is named after Bergman, who found in [23] that it is satisfied by all infinite symmetric groups. The first example of an infinite group with the Bergman property was apparently found by Shelah in the 1980s.

Theorem 13 (Tolstykh [24]). The automorphism group $\mathrm{Aut}(F_\omega)$ of the free group F_ω of countably infinite rank has the Bergman property.

Theorem 14 (Tolstykh [24]). For any positive integer k, the automorphism group $\mathrm{Aut}(N_{\infty,k})$ of any free nilpotent group $N_{\infty,k}$ of infinite rank has the Bergman property.

Some other discussion on the automorphism groups of free relatively free groups can be found in survey [25].

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В.А. Романьков

Шексіз рангті еркін группаларға қатысты автоморфизмдер группалары жайында: шолу

Мақаланың мақсаты шексіз рангті еркін группаларға қатысты автоморфизмдер тобын зерттеудің кейбір мәселелерін шолу болып табылады. Автоморфизмнің «нұсқаулы» болуы, қарабайырлық, кіші индекс, Бергман қасиеті және тағы басқадай қасиеттері жан-жақты талқыланады.

Кілт сөздер: группалардың көпбейнелілігі, салыстырмалы еркін топ, санамалы шексіз ранг, автоморфизмдер группасы, «нұсқаулы» автоморфизм, кіші индекс қасиеті, кофиналдылық, Бергман қасиеті.

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О группах автоморфизмов относительно свободных групп бесконечного ранга: обзор

Целью статьи является обзор некоторых вопросов исследований групп автоморфизмов относительно свободных групп бесконечного ранга. Обсуждаются такие свойства, как быть «ручным» автоморфизмом, примитивность, свойства малого индекса, Бергмана и т.п.

Ключевые слова: многообразие групп, относительно свободная группа, счетный бесконечный ранг, группа автоморфизмов, «ручной» автоморфизм, свойство малого индекса, кофинальность, свойство Бергмана.

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