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M.T. Kosmakova

Al-Farabi Kazakh National University, Almaty
(E-mail: svetik_mir69@mail.ru)

On an integral equation of the Dirichlet problem for the heat equation in the degenerating domain

This paper considers the first boundary value problem of heat conduction in a degenerating domain with a moving boundary, the boundary of the domain moves at a variable velocity. The problem is reduced to singular Volterra integral equation of the second kind. The solution of equation is represented explicitly. Eigenfunction is found.

Key words: degenerating domain, singular Volterra integral equation, moving boundary, eigenfunction.

Introduction

The mathematical description of many thermal processes leads to solving the boundary value problems for the heat equation in domains with the moving boundary, namely in the domains degenerating into a point at the initial time. Using the apparatus of thermal potentials solving these problems is reduced to the investigation of the homogeneous Volterra integral equations of the second kind. Since the original domain has been a degenerating domain at the initial time, the obtained integral equation has the singularity: norm of the integral operator is equal to one. Therefore, the corresponding non-homogeneous equation can not be solved by method of successive approximations. Namely these problems have great practical importance. In work [1] the approximate solutions to the certain applied problems in the degenerating domains were constructed.

Modelling the thermal processes in the electric arc of high-current breaking device, process electric contactdevices in the related fields of constructing the plasmatrons leads to the study of the boundary value problems for the heat equation in non-cylindrical domains [2, 3]. These questions were the subject of research work [4–6], in which the solutions to the given problems are being constructed in the domain with the uniformly moving boundary: $G = \{x,\, t|\, t > 0,\, 0 < x < a + kt, a \neq 0\}$.

In work [7] the exact solutions to the boundary value problems of non-stationary heat conduction in the degenerating domain with the uniformly moving boundary are constructed, classes of uniqueness for solutions to the given problems are established.

The special interest is the case of the degenerating domain when the boundary of the domain moves along the automodeling law. Research of the homogeneous boundary value problem and its reduction to a singular Volterra integral equation of the second kind and finding explicitly the eigenfunction to this equation in this domain determines the contents of this work.

1. Statement of the problem

We consider the first boundary value problem of heat conduction in the degenerating domain (domain with moving boundary, the boundary of the domain is moving with variable velocity): In the domain $G = \left\{(x;\ t):\quad t > 0,\quad 0 < x < \sqrt{t}\right\}$ to find a solution to the heat equation

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}, \tag{1}$$

that satisfies the boundary conditions:

$$u(x,\ t)|_{x=0} = 0, \quad u(x,\ t)|_{x=\sqrt{t}} = 0. \tag{2}$$

In [8] the second homogeneous boundary value problem is solved in an unbounded angular domain, that is in the degenerating domain when the boundary moves with constant velocity.

2. Reducing the problem to an integral equation

The solution of the problem (1)–(2) we look for as the sum of the heat potentials of the double layer [9]:

$$u(x,t) = \frac{1}{4a^3\sqrt{\pi}} \int_0^t \frac{x}{(t-\tau)^{\frac{3}{2}}} \exp\left\{-\frac{x^2}{4a^2(t-\tau)}\right\} \nu(\tau)\, d\tau +$$

$$+\frac{1}{4a^3\sqrt{\pi}} \int_0^t \frac{x-\tau}{(t-\tau)^{\frac{3}{2}}} \exp\left\{-\frac{(x-\tau)^2}{4a^2(t-\tau)}\right\} \varphi(\tau)\, d\tau. \tag{3}$$

It is known that function (3) satisfies equation (1) for any $\nu(t)$ and $\varphi(t)$ [9]. Note that any solution of heat equation (1) can be represented by the formula (3). It has been argued, for example, in [9; 476–480].

Using conditions (1) and properties of heat potentials, we obtain an integral equation:

$$\varphi(t) - \int_0^t K(t,\,\tau)\, \varphi(\tau)\, d\tau = 0, \tag{4}$$

where

$$K(t,\tau) = \frac{1}{2a\sqrt{\pi}} \left\{ \frac{\sqrt{t}+\sqrt{\tau}}{(t-\tau)^{3/2}} \exp\left(-\frac{\left(\sqrt{t}+\sqrt{\tau}\right)^2}{4a^2(t-\tau)}\right) + \frac{\sqrt{t}-\sqrt{\tau}}{(t-\tau)^{3/2}} \exp\left(-\frac{\left(\sqrt{t}-\sqrt{\tau}\right)^2}{4a^2(t-\tau)}\right) \right\}. \tag{5}$$

We rewrite equation (4) in the form

$$\varphi(t) - \int_0^t k\left(\frac{\tau}{t}\right) \frac{\varphi(\tau)}{t}\, d\tau = 0, \tag{6}$$

where

$$k\left(\frac{\tau}{t}\right) = \frac{1}{2a\sqrt{\pi}} \left\{ \frac{1+\sqrt{\tau/t}}{(1-\tau/t)^{3/2}} \exp\left(-\frac{\left(1+\sqrt{\tau/t}\right)^2}{4a^2(1-\tau/t)}\right) + \right.$$

$$\left. + \frac{1-\sqrt{\tau/t}}{(1-\tau/t)^{3/2}} \exp\left(-\frac{\left(1-\sqrt{\tau/t}\right)^2}{4a^2(1-\tau/t)}\right) \right\}. \tag{7}$$

3. Solving the integral equation

The solution of equation (6) we will look for in the form $\varphi(t) = t^\beta$, where β is arbitrary and unknown number.

Substituting $\varphi(t) = t^\beta$ into equation (6), taking into account equality (7), we get

$$t^\beta - \int_0^t k\left(\frac{\tau}{t}\right) \frac{\tau^\beta}{t}\, d\tau = 0.$$

After replacement $\tau = tx$ for the integral last equality takes the form

$$t^{\beta} - t^{\beta}\int\limits_0^1 k(x)x^{\beta}\,dx = 0.$$

We will find the roots of the following equation [10]

$$\int\limits_0^1 k(x)x^{\beta}\,dx = 1. \tag{8}$$

We calculate the integral in the right-hand side of last equality

$$\int\limits_0^1 k(x)x^{\beta}\,dx = \frac{1}{2a\sqrt{\pi}}\int\limits_0^1\left\{\frac{1+\sqrt{x}}{(1-x)^{3/2}}\exp\left(-\frac{(1+\sqrt{x})^2}{4a^2(1-x)}\right)+\right.$$

$$\left.+\frac{1-\sqrt{x}}{(1-x)^{3/2}}\exp\left(-\frac{(1-\sqrt{x})^2}{4a^2(1-x)}\right)\right\}x^{\beta}\,dx =$$

$$= \frac{1}{2a\sqrt{\pi}}\int\limits_0^1\left\{\frac{1}{(1+\sqrt{x})^{1/2}(1-\sqrt{x})^{3/2}}\exp\left(-\frac{1+\sqrt{x}}{4a^2(1-\sqrt{x})}\right)+\right.$$

$$\left.+\frac{1}{(1+\sqrt{x})^{3/2}(1-\sqrt{x})^{1/2}}\exp\left(-\frac{1-\sqrt{x}}{4a^2(1+\sqrt{x})}\right)\right\}x^{\beta}\,dx.$$

After replacement $y = \sqrt{x}$ we obtain

$$\int\limits_0^1 k(x)x^{\beta}\,dx = \frac{1}{a\sqrt{\pi}}\left(I_1(\beta) + I_2(\beta)\right), \tag{9}$$

where

$$I_1(\beta) = \int\limits_0^1 \frac{y^{2\beta+1}}{(1+y)^{1/2}(1-y)^{3/2}}\exp\left(-\frac{1+y}{4a^2(1-y)}\right)dy,$$

$$I_2(\beta) = \int\limits_0^1 \frac{y^{2\beta+1}}{(1+y)^{3/2}(1-y)^{1/2}}\exp\left(-\frac{1-y}{4a^2(1+y)}\right)dy.$$

After replacement $z = \sqrt{(1-y)/(1+y)}$ last integrals take the form:

$$I_1(\beta) = \int\limits_0^1 \frac{\left(1-z^2\right)^{2\beta+1}}{(1+z^2)^{2\beta+1}z^2}\exp\left(-\frac{1}{4a^2z^2}\right)dz,$$

$$I_2(\beta) = \int\limits_0^1 \frac{\left(1-z^2\right)^{2\beta+1}}{(1+z^2)^{2\beta+1}}\exp\left(-\frac{z^2}{4a^2}\right)dz.$$

The first integral can be rewritten as

$$I_1(\beta) = \int\limits_1^\infty \frac{\left(z^2-1\right)^{2\beta+1}}{\left(z^2+1\right)^{2\beta+1}} \exp\left(-\frac{z^2}{4a^2}\right) dz.$$

Substituting expressions for $I_1(\beta)$ and $I_2(\beta)$ into (9), we get

$$\int\limits_0^1 k(x) x^\beta \, dx = \frac{1}{a\sqrt{\pi}} \left\{ \int\limits_1^\infty \frac{\left(z^2-1\right)^{2\beta+1}}{\left(z^2+1\right)^{2\beta+1}} \exp\left(-\frac{z^2}{4a^2}\right) dz + \right.$$

$$\left. + \int\limits_0^1 \frac{\left(1-z^2\right)^{2\beta+1}}{\left(1+z^2\right)^{2\beta+1}} \exp\left(-\frac{z^2}{4a^2}\right) dz \right\}. \qquad (10)$$

For $\beta = -1/2$ from (10) we obtain

$$\int\limits_0^1 k(x) \;\; x^{-1/2} \;\; dx = \frac{1}{a\sqrt{\pi}} \left\{ \int\limits_1^\infty \exp\left(-\frac{z^2}{4a^2}\right) dz + \int\limits_0^1 \exp\left(-\frac{z^2}{4a^2}\right) dz \right\} = 1.$$

Thus, equation (8) has a real solution $\beta = -1/2$. Therefore, integral equation (6) has eigenfunction

$$\varphi(t) = t^{-1/2}. \qquad (11)$$

Shall show later that equation (6) has no other eigenfunctions.

4. Conclusion

Stated problem (1)–(2) is reduced to singular integral equation (4). Exact solution (11) to the obtained integral equation is found.

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М.Т.Космакова

Құлдырайтын облыстағы жылуөткізгіштік теңдеуі үшін Дирихле есебінің интегралдық теңдеуі туралы

Мақалада құлдырайтын облыста жылжымалы шекарасы бар жылуөткізгіштің бірінші шеттік есебі қарастырылды, сондай-ақ облыстың шекарасы ауыспалы жылдамдықпен қозғалады. Есеп Вольтерраның екінші текті ерекше интегралдық теңдеуіне келтірілген. Теңдеудің шешімі айқын түрде алынған. Өзіндік функциясы табылған.

М.Т.Космакова

Об интегральном уравнении задачи Дирихле для уравнения теплопроводности в вырождающейся области

В статье рассмотрена первая краевая задача теплопроводности в вырождающейся области с подвижной границей, причем граница области движется с переменной скоростью. Задача сведена к особому интегральному уравнению Вольтерра второго рода. Решение уравнения представлено в явном виде. Найдена собственная функция.

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