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On conditions for the weighted integrability of the sum of the series with monotonic coefficients with respect to the multiplicative systems

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In this paper, we studied the issues of integrability with the weight of the sum of series with respect to multiplicative systems, provided that the coefficients of the series are monotonic. The conditions for the weight function and the series' coefficients are found; the sum of the series belongs to the weighted Lebesgue space L_p $(1 < p < \infty)$. In addition, the case of $p = 1$ was considered. In this case, other conditions for the weighted integrability of the sum of the series under consideration are found. In the case of the generating sequence's boundedness, the proved theorems imply an analogue of the well-known Hardy-Littlewood theorem on trigonometric series with monotone coefficients.

Keywords: the multiplicative systems, the weighted integrability of the sum of series, generator sequence, monotone coefficients, Hardy-Littlewood theorem.

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Introduction

The Hardy-Littlewood theorem concerning series with monotone coefficients in the theory of trigonometric series states the following [1, 2]: for the series $\sum_{n=0}^{\infty} a_n \cos nx$, where the coefficients a_n decrease to zero as n approaches infinity, to represent the Fourier series of a function $f(x) \in L_p[0, 2\pi]$, where $1 < p < \infty$, it is both necessary and sufficient that $\sum_{n=1}^{\infty} a_n^p n^{p-2} < \infty$.

An analogue of this theorem for the Walsh system was proved by Moritz F. [3]. For multiplicative systems with bounded generating sequences p $(1 \leq \sup_n p_n < c)$ it was proved by Timan M.F., Tukhliev K. [4].

The weighted integrability of the sum of trigonometric series with generalized monotone coefficients was studied in the works of Tikhonov S.Yu., Dyachenko M.I. [5, 6] and others. Weighted integrability for the sum of series for multiplicative systems is considered in the works of Volosivets S.S., Fadeev R.N. [7, 8], Bokayev N.A., Mukanov Zh.B. [9].

In this paper, we consider weight functions with other conditions. This article is a continuation of the article [10].

1 *Notation and Preliminaries*

This paper examines a series characterized by monotonic coefficients concerning the multiplicative systems. We delve into the inquiry: what criteria regarding the weight function and series coefficients ensure that the sum of the series falls within the L_p space with weighting? Before delving into the main discussion, we define multiplicative systems.

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Definition 1. Let $\{p_k\}_{k=1}^{\infty}$ be a sequence of natural numbers $p_k \geq 2,\ k \in \mathbb{N},\ \sup_k p_k = N < \infty$. By definition, let us put

$$m_0 = 1,\ m_n = p_1 p_2 \cdots p_n,\ n \in \mathbb{N}.$$

Then every point $x \in [0,1)$ has decomposition

$$x = \sum_{k=1}^{\infty} \frac{x_k}{m_k}, \quad x_k \in \mathbb{Z} \cap [0, p_k), \tag{1}$$

where $\mathbb{Z}$ is the set of integers. Decomposition (1) is uniquely defined if $x = n/m_k$ takes a decomposition with a finite number of nonzero x_k. If $n \in \mathbb{Z}_+ := \{0, 1, 2, \dots\}$ is represented as

$$n = \sum_{j=1}^{\infty} \alpha_j m_{j-1}, \quad \alpha_j \in \mathbb{Z} \cap [0, p_j),$$

then for the numbers $x \in [0,1)$ we put by definition

$$\psi_n(x) = exp\left(2\pi i \sum_{j=1}^{\infty} \frac{\alpha_j x_j}{p_j}\right), \quad n \in \mathbb{Z}_+.$$

It is known that the system $\{\psi_n\}_{n=0}^{\infty}$, called the Price system, is an orthonormalized system complete in $L^1(0,1)$ (see [10] or [11]). If all $p_k = 2$, then $\{\psi_n\}_{n=0}^{\infty}$ coincides with Walsh system in the Paley numbering.

Let $L^p(G),\ G := [0,1),\ 1 \leq p < \infty,$ denote Lebesgue space with norm

$$\|f\|_p = \left(\int_G |f(x)|^p dx\right)^{\frac{1}{p}}, \quad \|f\|_\infty = ess \sup_{x \in G} |f(x)|.$$

Definition 2. Let $\varphi(x)$ be a non-negative measurable function on $(0, \infty)$. We say that $\varphi(x)$ satisfies condition B_1, if for all $x \geq 1$

$$\int_x^{\infty} \frac{\varphi(t)}{t^2} dt \leq C \frac{\varphi(x)}{x},$$

where C is a positive number independent of x.

For example, the function $\varphi(t) = t^{\alpha}$ $(\alpha < 1)$ satisfies condition B_1.

To prove the main results, we need the following auxiliary assertions.

By

$$D_n(x) = \sum_{k=0}^{n-1} \psi_k(x), \quad n = 1, 2, ...,$$

denote the Dirichlet kernel of the system $\{\psi_n(x)\}$.

Lemma A. (see [11] or [12]) For any $k \in \mathbb{N}$ and $x \in [0,1)$ the Dirichlet kernels have the following properties:

$$D_{m_k} = \begin{cases} m_k, & \text{если } x \in \left[0, \frac{1}{m_k}\right), \\ 0, & \text{если } x \notin \left[0, \frac{1}{m_k}\right). \end{cases} \tag{2}$$

2 Main Results

In this section, under certain conditions imposed on the weight function, a necessary and sufficient condition is given for the function to belong to a Lebesgue space with the weight of the sum of series for the multiplicative systems. The following theorem about integrability with the weight of the sum of series with monotone coefficients is valid.

Theorem 1. [10] Let $1 < p < \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$ and $\sup p_n = N < \infty$

$$f(x) = \sum_{k=0}^{\infty} a_k \psi_k(x), \quad a_k \downarrow 0 \ at \ k \to \infty$$

and let $\varphi(x)$ be a non-negative measurable function on $[1, \infty)$. Then

1^0. If the function $\varphi^p(x)$ satisfies condition B_1 and

$$\sum_{n=1}^{\infty} a_n^p \cdot n^p \int_n^{n+1} \frac{\varphi^p(x)}{x^2} dx < \infty,$$

then $\varphi\left(\frac{1}{x}\right) f(x) \in L_p(0, 1)$.

2^0. If $\varphi^{-p'}(x)$ it satisfies the condition B_1 and $\varphi\left(\frac{1}{x}\right) f(x) \in L_p(0, 1)$, then it takes place (6).

Remark. If the weight function $\varphi(x)$ has the form $\varphi(x) = x^\alpha$, then in this case $\varphi^p(x)$ and $\varphi^{-p'}(x)$ satisfy condition B_1 at $-\frac{1}{p'} < \alpha < \frac{1}{p}$ and condition (7) has the form

$$\sum_{n=1}^{\infty} a_n^p \cdot n^{p(\alpha+1)-1} < \infty.$$

Let $1 \leq p < \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$ and $\varphi(x) \geq 0$ is some locally integrable on function on $[0, 1]$. Lebesgue measurable on $[0, 1]$ function $f(x)$, belongs to space $L_{p,\varphi}$, if

$$\|f\|_{p,\varphi} = \left(\int_0^1 |f(x)\ \varphi(x)|^p dx\right)^{\frac{1}{p}} < \infty.$$

Let us put

$$A_p = \sup_{0 \leq t \leq 1} \left(\int_0^t \varphi^p(x) dx\right)^{\frac{1}{p}} \left(\int_t^1 (x\varphi(x))^{-p'} dx\right)^{\frac{1}{p'}},$$

$$B_p = \sup_{0 \leq t \leq 1} \left(\int_t^1 \left(\frac{\varphi(x)}{x}\right)^p dx\right)^{\frac{1}{p}} \left(\int_0^t \varphi^{-p'}(x) dx\right)^{\frac{1}{p'}}.$$

The following theorem is true.

Theorem 2. Let $1 \leq p < \infty, \frac{1}{p} + \frac{1}{p'} = 1$, $\sup\limits_n p_n = N < \infty$, and $\varphi(x)$ be some locally integrable function on $[0, 1]$ and

$$f(x) \equiv \sum_{k=0}^{\infty} a_k(f) \Psi_k(x),$$

where $a_k(f) \downarrow 0$ at $k \to \infty$. Then

1^0. If $A_p < \infty$ and

$$D_p = \sum_{n=1}^{\infty} a_n^p \cdot n^p \int_{\frac{1}{n+1}}^{\frac{1}{n}} \varphi^p(x) dx < \infty, \tag{3}$$

that $f(x) \in L_{p,\varphi}(0,1)$ and
$$\|f\|_{p,\varphi}^{p} \leq C_p D_p.$$

2^0. If $B_p < \infty$ и $f(x) \in L_{p,\varphi}$, then the series (3) converges and
$$\|f\|_{p,\varphi}^{p} \geq C_p D_p.$$

Combining points 1^0 and 2^0 of this theorem, we can formulate the following statement:

Theorem 3. Let $1 \leq p < \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$, $\sup\limits_n p_n = N < \infty$ and $\varphi(x)$ be a non-negative, locally integrable function such that
$$\max(A_p,\ B_p) < \infty.$$

Then, in order for the function $f(x) \equiv \sum_{k=0}^{\infty} a_k(f)\,\Psi_k(x)$, where $a_k(f) \downarrow 0$ at $k \to \infty$ to belong to the class $L_{p,\varphi}(0,\ 1)$ it is necessary and sufficient to satisfy the condition
$$D_p = \sum_{n=1}^{\infty} a_n^p \cdot n^p \int_{\frac{1}{n+1}}^{\frac{1}{n}} \varphi^p(x)\,dx < \infty.$$

In this case, there exists a relation
$$\|f\|_{L_{p,\varphi}} \approx D_p^{\frac{1}{p}}.$$

To prove the theorem, we need the following auxiliary statements.

Lemma 1. Let $a_n \downarrow 0$ at $n \to \infty$, $\sup\limits_n p_n = N < \infty$ and
$$S_n(x) = \sum_{n=1}^{n-1} a_k \psi_k(x).$$

Then for any integer $n \geq 1$ and for any number $x \in (0,1)$
$$|S_n(x)| \leq C \sum_{k=0}^{\left[\frac{1}{x}\right]} a_k. \tag{4}$$

Proof. At $n < \left[\frac{1}{x}\right]$ inequality (4) is obvious, since $|\psi_k(x)| = 1, \quad k = 0,\ 1,\ 2\ldots$

Let $n > \left[\frac{1}{x}\right]$, Then
$$|S_n(x)| \leq \sum_{k=0}^{\left[\frac{1}{x}\right]} a_k + \left| \sum_{k=\left[\frac{1}{x}\right]+1}^{n-1} a_k \psi_k(x) \right|.$$

Using inequality
$$|D_n(x)| = \left| \sum_{k=0}^{n-1} a_k \psi_k(x) \right| \leq \frac{C}{x}, \quad x \in (0,\ 1),$$

and applying the Abel transform to the last sum, due to the monotonicity of the sequence $\{a_n\}$, we get
$$|S_n(x)| \leq \sum_{k=0}^{\left[\frac{1}{x}\right]} a_k + \frac{C}{x} a_{\left[\frac{1}{x}\right]+1} \leq \sum_{k=0}^{\left[\frac{1}{x}\right]} a_k + C\left(\left[\frac{1}{x}\right] + 1\right) a_{\left[\frac{1}{x}\right]+1} \leq C_1 \sum_{k=0}^{\left[\frac{1}{x}\right]} a_k.$$

Lemma 1 is proved.

Lemma B. [13] Let $1 < p < \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$, $u = \{u_n\}_{n=1}^{\infty}$, $v = \{v_n\}_{n=1}^{\infty}$, $u_n \geq 0$, $v_n \geq 0$ and

$$\|a\|_{p,v} = \left(\sum_{n=1}^{\infty} |a_n v_n|^p\right)^{\frac{1}{p}}.$$

Then to satisfy the inequality $\|\sum_{k=1}^{n} a_k\|_{p,v} \leq C \|a\|_{p,u}$ it is necessary and sufficient that

$$A = \sup_l \left(\sum_{n=l+1}^{\infty} u_n^p\right)^{\frac{1}{p}} \left(\sum_{m=1}^{l-1} v_m^{-p'}\right)^{\frac{1}{p'}} < \infty. \tag{5}$$

Lemma C. [14] Let $1 < p < \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$, $u(x) \geq 0$, $v(x) \geq 0$, $Pf(x) = \int_0^x f(t)\,dt$. Then, to satisfy the inequality $\|Pf\|_{p,v} \leq C \|f\|_{p,u}$ it s necessary and sufficient that

$$B = \left\{\int_t^1 (u(x))^p\,dx\right\}^{\frac{1}{p}} \left\{\int_0^t [v(x)]^{-p'}\,dx\right\}^{\frac{1}{p'}} < \infty.$$

Proof of Theorem 2. 1^0. According to Lemma 1 and by monotony of sequence $a_n(f)$ we have

$$\int_0^1 |f(x)\varphi(x)|^p\,dx = \sum_{n=1}^{\infty} \int_{\frac{1}{n+1}}^{\frac{1}{n}} |f(x)\varphi(x)|^p\,dx = \sum_{n=1}^{\infty} \int_{\frac{1}{n+1}}^{\frac{1}{n}} \left(\sum_{k=0}^{n+1} a_k\right)^p \varphi^p(x)\,dx \leq \tag{6}$$

$$\leq C \sum_{n=1}^{\infty} \left[\sum_{k=0}^{n} a_k(f)\right]^p \cdot \int_{\frac{1}{n+1}}^{\frac{1}{n}} \varphi^p dx.$$

To apply Lemma B, we set

$$u_n = \left(\int_{\frac{1}{n+1}}^{\frac{1}{n}} \varphi^p(x)\,dx\right)^{\frac{1}{p}}, \quad v_n = n\left(\int_{\frac{1}{n+1}}^{\frac{1}{n}} \varphi^p(x)\,dx\right)^{\frac{1}{p}}.$$

Condition (5), in this case, has the form:

$$A = \sup_l \left[\sum_{n=l+1}^{\infty} \int_{\frac{1}{n+1}}^{\frac{1}{n}} \varphi^p(x)\,dx\right]^{1/P} \left\{\sum_{n=1}^{l-1} n^{-p'} \left[\int_{\frac{1}{n+1}}^{\frac{1}{n}} \varphi^p(x)\,dx\right]^{\frac{-p'}{p}}\right\}^{\frac{1}{p'}}. \tag{7}$$

It is clear that

$$\sum_{n=l+1}^{\infty} \int_{\frac{1}{n+1}}^{\frac{1}{n}} \varphi^p(x)\,dx \leq \int_0^{\frac{1}{l}} \varphi^p(x)\,dx.$$

Now, we will show that condition (5) of Lemma B is satisfied. To do this, we will show that $A \leq CA_P$ where A_P is from the condition of Theorem 2. Let us transform the second sum in (7).

$$\sum_{n=1}^{l-1} n^{-p'} \left(\int_{\frac{1}{n+1}}^{\frac{1}{n}} \varphi^p(x)\,dx\right)^{\frac{-p'}{p}} = \sum_{n=1}^{l-1} n^{-p'} \left\{\left(\int_{\frac{1}{n+1}}^{\frac{1}{n}} \varphi^p(x)\,dx\right)^{\frac{1}{p}} \cdot \left(\int_{\frac{1}{n+1}}^{\frac{1}{n}} \varphi^{-p'}(x)\,dx\right)^{\frac{1}{p'}}\right\}^{-p'} \times \tag{8}$$

$$\times \int_{\frac{1}{n+1}}^{\frac{1}{n}} \varphi^{-p'}(x)\,dx.$$

By Holder's inequality, we have

$$\frac{1}{n(n+1)} = \int_{\frac{1}{n+1}}^{\frac{1}{n}} dx = \int_{\frac{1}{n+1}}^{\frac{1}{n}} \varphi(x)\varphi(x)^{-1} dx \le \left(\int_{\frac{1}{n+1}}^{\frac{1}{n}} \varphi^p(x)\,dx\right)^{\frac{1}{p}} \left(\int_{\frac{1}{n+1}}^{\frac{1}{n}} \varphi^{-p'}(x)\,dx\right)^{\frac{1}{p'}}.$$

Hence,

$$\left\{\left(\int_{\frac{1}{n+1}}^{\frac{1}{n}} \varphi^p(x)\,dx\right)^{\frac{1}{p}} \left(\int \varphi^p(x)\,dx\right)^{\frac{1}{p}}\right\}^{-p'} \le n^{p'}(n+1)^{p'}.$$

Then from (8) we have

$$\sum_{n=1}^{l-1} n^{-p'} \left(\int_{\frac{1}{n+1}}^{\frac{1}{n}} \varphi^p(x)\,dx\right)^{\frac{-p'}{p}} \le \sum_{n=1}^{l-1} (n+1)^{p'} \int_{\frac{1}{n+1}}^{\frac{1}{n}} \varphi^{-p'}(x)\,dx \le \int_{\frac{1}{l}}^{1} (x\varphi(x))^{-p'}\,dx. \quad (9)$$

Therefore, by (7), (9) we get

$$A = \sup_l \left(\int_0^{\frac{1}{l}} \varphi^p(x)\,dx\right)^{\frac{1}{p}} \left(\int_{\frac{1}{l}}^{1} (x\varphi(x))^{-p'}\,dx\right)^{\frac{1}{p'}} \le$$

$$\le \sup_{t\in[0,1]} \left(\int_0^t \varphi^p(x)\,dx\right)^{\frac{1}{p}} \left(\int_t^1 (x\varphi(x))^{-p'}\,dx\right)^{\frac{1}{p'}} = A_p < \infty.$$

Therefore, according to Lemma A and conditions $A_p < \infty$ it follows that

$$\int_0^1 |f(x)\varphi(x)|^p\,dx \le c\sum_{n=1}^{\infty} a_n^p(f)\cdot n^p \int_{\frac{1}{n+1}}^{\frac{1}{n}} \varphi^p(x)\,dx < \infty,$$

that is $f(x)\varphi(x) \in L_p[0,1]$.

2^0. Due to the monotonicity of the sequence $a_n(f)$

$$\sum_{n=1}^{\infty} a_n^p(f)\,n^p \int_{1/(n+1)}^{1/n} \varphi^p(x)\,dx \le \sum_{n=1}^{\infty}\left(\sum_{k=0}^{n} a_k(f)\right)^p \int_{1/(n+1)}^{1/n} \varphi^p(x)\,dx =$$

$$= \sum_{n=0}^{\infty}\left[\sum_{j=m_n}^{m_{n+1}-1}\left[\sum_{k=0}^{j} a_k(f)\right]^p \int_{1/j+1}^{1/j} \varphi^p(x)\,dx\right] \le$$

$$\le \sum_{n=0}^{\infty}\sum_{k=0}^{m_{n+1}-1} [a_k(f)]^p \int_{\frac{1}{m_{n+1}}}^{\frac{1}{m_n}} \varphi^p(x)\,dx. \quad (10)$$

By equality (2)

$$\sum_{k=0}^{m_n-1} a_k(f) = \int_0^1 f(x)\,D_{m_n}(x)\,dx = m_n\int_0^{1/m_n} f(x)\,dx \le m_n F\left[\frac{1}{m_n}\right], \quad (11)$$

where $F(x) = \int_0^x |f(t)|\,dt$.

Now from (10) and (11) follows that

$$\sum_{n=1}^{\infty} a_n^p(f)\, n^p \int_{1/(n+1)}^{1/n} \varphi^p(x)\,dx \le \sum_{n=1}^{\infty} a_n^p(f)\, n^p \left[m_{n+1} F\left[\frac{1}{m_{n+1}}\right]\right]^p \times$$

$$\times \int_{1/m_{n+1}}^{1/m_n} \varphi^p(x)\,dx \le \sum_{n=0}^{\infty} \int_{1/m_{n+1}}^{1/m_n} \left(\frac{1}{x} F(x)\right)^p \varphi^p(x)\,dx =$$

$$= \int_0^1 \left(\int_0^x |f(t)|dt\right)^p \left(\frac{\varphi(x)}{x}\right)^p dx. \tag{12}$$

According to Lemma C from the condition $B_p < \infty$ it follows that

$$\int_0^1 \left(\int_0^x |f(t)|\,dt\right)^p \left(x^{-1}\varphi(x)\right)^p dx \le C_p \int_0^1 |f(x)\,\varphi(x)|^p\,dx.$$

Then from the inequality (12) we have

$$\sum_{n=1}^{\infty} a_n^p(f)\, n^p \int_{1/n+1}^{1/n} \varphi^p(x)\,dx \le C_p \int_0^1 |f(x)\,\varphi(x)|^p\,dx < \infty.$$

Theorem 2 is proved.

Remark. By direct calculation, it can be shown that the function $\varphi(x) = \left(\frac{x}{(\ln x)^{1+\alpha}}\right)^{\frac{1}{p}}$, $\alpha > 0$ satisfies the condition from 1^0 of Theorem 2 (i.e. $A_p < \infty$) , but does not satisfy the condition from 1^0 of Theorem 1 (i.e. $\varphi^p(x)$ does not satisfy condition B_1).

In addition, the function $\varphi(x) = \left(\frac{x}{(\ln x)^{1+\alpha}}\right)^{-\frac{1}{p'}}$, $\alpha > 0$ satisfies the condition from 2^0 of Theorem 1 (i.e. $B_p < \infty$), but does not satisfy the condition from 2^0 of Theorem 2 (i.e. $\varphi^{-p'}(x)\,(x)$ does not satisfy condition B_1).

Therefore, the conditions of Theorem 1 and Theorem 2 are, generally speaking, different.

3 *About belonging to space $L_1(0,\ 1)$ with the weight of the sum of series with monotonic coefficients*

In the previous paragraphs, we considered the conditions for functions to belong to the class $L_{p,\varphi}(0,\ 1)$ at $1 < p < \infty$.

In this section, we will consider the case p=1, i.e. questions of belonging of functions to space $L_{1,\varphi}(0,\ 1)$.

Let $\varphi(x)$ be a non-negative measurable on $(1,\ \infty)$ function. They say the function $\varphi(x)$ satisfies the condition B_2, if for everyone $x \ge 1$ the following inequality holds

$$\int_1^x \frac{\varphi(t)}{t}dt \le C\varphi(x),$$

where C is a positive number independent of x.

We need the following auxiliary statement.

Lemma D. [15] *If $R_n \downarrow 0,\ 0 \le B_n \uparrow$ at $n \to \infty$, then the series*

$$\sum_{n=1}^{\infty} R_n(B_{n+1} - B_n) \ \ \textit{and} \ \sum_{n=1}^{\infty} B_n(R_{n-1} - R_n)$$

converge or diverge at the same time.

The main goal of this section is to prove the following statement.

Theorem 4. Let $a_k \downarrow 0$ at $k \to \infty$, $\sup\limits_n p_n = N < \infty$ and

$$f(x) = \sum_{k=0}^{\infty} a_k \psi_k(x),$$

and let $\varphi(x) \geq 0$ is measurable on $[1, \infty)$ function such that

$$\varphi\left(\frac{1}{x}\right) \in L_1[0, 1], \quad \frac{1}{x}\varphi\left(\frac{1}{x}\right) \bar{\in} L(0, 1).$$

Then

1^0. If

$$\sum_{k=1}^{\infty} a_k \int_k^{\infty} \frac{\varphi(x)}{x^2} dx < \infty, \tag{13}$$

then

$$\varphi\left(\frac{1}{x}\right) f(x) \in L_1(0,1).$$

2^0. If $\varphi(x)$ satisfies the condition B_2 and $\varphi\left(\frac{1}{x}\right) f(x) \in L_1(0,1)$, that is the case (13) .

3^0. If $\varphi(x) \downarrow$ at $x \geq 1$, positive function and

$$\lim_{x\to\infty} \frac{1}{\varphi(x)} \int_1^x \frac{\varphi(t)}{t} dt = \infty, \tag{14}$$

then there is a sequence $a_k \downarrow 0$ at $k \to \infty$, such that the function

$$f(x) = \sum_{k=1}^{\infty} a_k \psi_k(x)$$

integrates on (0,1), $\varphi\left(\frac{1}{x}\right) f(x) \in L_1(0,1)$, however the theries (13) diverges.

Point 3 of this theorem shows that the condition B_2 is essential for fulfilling point 2 of this theorem.

Proof. 1^0**.** By lemma 1 for any $x \in \left[\frac{1}{m_{\nu+1}}, \frac{1}{m_\nu}\right]$ we have

$$|f(x)| \leq C \sum_{k=0}^{m_{\nu+1}} a_k.$$

Therefore

$$\int_0^1 \varphi\left(\frac{1}{x}\right) |f(x)| dx = \sum_{k=1}^{\infty} \int_{1/m_{k+1}}^{1/m_k} \varphi\left(\frac{1}{x}\right) |f(x)| dx \leq \sum_{k=1}^{\infty} \sum_{j=0}^{m_{k+1}} a_j \int_{1/m_{k+1}}^{1/m_k} \varphi\left(\frac{1}{x}\right) dx \leq$$

$$\leq C_1 + \sum_{k=1}^{\infty} a_{m_{k+1}} \int_k^{\infty} \frac{\varphi(x)}{x^2} dx \leq C_1 + \sum_{k=1}^{\infty} a_k \int_k^{\infty} \frac{\varphi(x)}{x^2} dx < \infty.$$

2^0. From the conditions of Theorem 4 follows that follows that $f(x) \in L[0, 1]$, hence, $a_n = a_n(f)$ and

$$\sum_{k=0}^{m_n - 1} a_k = \int_0^1 f(x) D_{m_n}(x) dx = m_n \int_0^{1/m_n} f(x) dx.$$

Therefore, due to the monotonicity of the sequence a_n,

$$\int_{1/m_{n+1}}^{1/m_n} f(x)\,dx = \left[(p_{n+1}-1)\sum_{k=0}^{m_n-1} a_k + \sum_{k=m_n}^{m_{n+1}-1} a_k\right] m_{n+1}^{-1} \geq 0. \tag{15}$$

Let us evaluate

$$\sum_{k=1}^{\infty} a_k \int_k^{\infty} \frac{\varphi(t)}{t^2}dt \leq \sum_{n=0}^{\infty}\sum_{k=m_n}^{m_{n+1}-1} a_k \int_k^{\infty} \frac{\varphi(t)}{t^2}dt \leq \sum_{n=0}^{\infty}\int_{m_n}^{\infty}\frac{\varphi(t)}{t^2}dt\left(\sum_{k=m_n}^{m_{n+1}-1} a_k\right) =$$

$$= \sum_{n=0}^{\infty} R_n\left(B_{n+1}-B_n\right), \tag{16}$$

where $R_n = \int_{m_n}^{\infty} \frac{\varphi(t)}{t^2}dt$, $B_n = \sum_{k=0}^{m_n-1} a_k.$

By lemma D for the convergence of the series (16) it is sufficient for the convergence of the series:

$$\sum_{n=0}^{\infty} B_n\left(R_{n-1}-R_n\right) \leq \sum_{n=1}^{\infty} \frac{1}{m_{n-1}}\int_{m_{n-1}}^{m_n}\frac{\varphi(t)}{t}dt\left(\sum_{k=0}^{m_n-1} a_k\right) \leq C\sum_{n=0}^{\infty}\frac{1}{m_{n+1}}\sum_{k=0}^{m_{n+1}-1} a_k\left[D_{n+1}-D_n\right],$$

where

$$D_n = \int_1^{m_n}\frac{\varphi(t)}{t}dt.$$

Applying Lemma D again, taking into account the conditions B_2 and (15), we have

$$\sum_{n=0}^{\infty}\frac{1}{m_{n+1}}\sum_{k=0}^{m_{n+1}-1} a_k\left[D_{n+1}-D_n\right] \leq \sum_{n=1}^{\infty}\int_1^{m_n}\frac{\varphi(t)}{t}dt\int_{1/m_{n+1}}^{1/m_n}|f(x)|\;dx \leq$$

$$\leq \sum_{n=1}^{\infty}\int_{1/m_{n+1}}^{1/m_n}|f(x)|\;\left(\int_1^{x^{-1}}\frac{\varphi(t)}{t}dt\right) \leq C\sum_{n=1}^{\infty}\int_{1/m_{n+1}}^{1/m_n}|f(x)|\;\varphi\left(\frac{1}{x}\right)\;dx < \infty.$$

3^0. Let $\varphi(x)\downarrow$ at $x \geq 1$. From the condition $\frac{1}{x}\varphi\left(\frac{1}{x}\right)\bar{\in}L$ follows that

$$\sum_{n=1}^{\infty}\int_n^{\infty}\frac{\varphi(t)}{t^2}dt = \infty.$$

Hence,

$$\sum_{n=0}^{\infty} m_n\int_{m_n}^{\infty}\frac{\varphi(t)}{t^2}dt = \infty. \tag{17}$$

Let's put $a_0 = 1$ and if $m_n \leq k \leq m_{n+1}-1,\ n = 0,\ 1,\ 2,\ldots$, then

$$a_n = \lambda_n = \left[\sum_{j=0}^{n} m_j\int_{m_j}^{\infty}\frac{\varphi(t)}{t^2}dt\right]^{-1}.$$

It means , $a_n\downarrow 0$ at $n\to\infty$. From the sequence definition $\{a_n\}$ using the Abel transform, we have

$$f(x) = \sum_{n=0}^{\infty} a_n\psi_n(x) = \sum_{n=1}^{\infty}\Delta\lambda_{n-1}D_{m_n}(x).$$

Then at $x \in \left[\frac{1}{m_{k+1}}, \ \frac{1}{m_k}\right), \ k = 0, \ 1, \ 2, \ldots$ based on the property of the Dirichlet kernel by Lemma 1 we have

$$f(x) = \sum_{n=1}^{k} \Delta\lambda_{n-1} m_n,$$

where $f(x) \geq 0, \ f(x) \downarrow 0$ in $(0, \ 1)$.

Further, taking into account the condition $\varphi(x) \downarrow 0, \ x \geq 1$ we get that

$$\int_0^1 \varphi\left(\frac{1}{x}\right) |f(x)| \, dx \leq \varphi(1) \int_0^1 f(x) \, dx \leq \varphi(1) \sum_{k=0}^{\infty} \left(\sum_{n=1}^{k} \Delta\lambda_{n-1} m_n\right) \cdot \frac{1}{m_k} \leq$$

$$\leq C\varphi(1) \sum_{n=1}^{\infty} \Delta\lambda_{n-1} = 2\varphi(1) \lambda_0 < \infty,$$

i.e. $f \in L_{1,\varphi}(0, \ 1)$.

Let us show that relation (13) does not hold, i.e.

$$\sum_{n=1}^{\infty} a_n \int_n^{\infty} \frac{\varphi(t)}{t^2} dt = \infty.$$

Based on property $a_n \downarrow 0$ where this $n \to \infty$ condition is equivalent to the divergence of the series

$$S \equiv \sum_{n=1}^{\infty} m_n a_{m_n} \int_{m_n}^{\infty} \frac{\varphi(t)}{t^2} dt = \sum_{n=1}^{\infty} \lambda_n m_n \int_{m_n}^{\infty} \frac{\varphi(t)}{t^2} dt.$$

According to the famous Kronecker theorem [1; 905] and (14), (17) we have

$$S = \sum_{n=1}^{\infty} m_n \int_{m_n}^{\infty} \frac{\varphi(t)}{t^2} dt \left[\sum_{i=0}^{n} m_i \int_{m_i}^{\infty} \frac{\varphi(t)}{t^2} dt\right]^{-1} = \infty,$$

i.e.

$$\sum_{n=1}^{\infty} \lambda_n \cdot m_n \int_{m_n}^{\infty} \frac{\varphi(t)}{t^2} dt = \infty.$$

Point 3^0 of Theorem 4 proven.

Conclusion

In this paper, we have considered series with respect to the multiplicative systems with monotonic coefficients. Conditions have been obtained for the weight function and for coefficients under which the sum of the series under consideration is L_p $(1 < p < \infty)$ integrable on the interval $[0, 1]$. In addition, the obtained conditions are compared with the previously known conditions for the weight integrability of the sum of such series.

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Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

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Коэффициенттері монотонды мультипликативтік жүйелер бойынша қатарлардың қосындысының салмақты интегралдану шарттары туралы

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Мақалада біз қатарлардың коэффициенттері монотонды болған жағдайда мультипликативті жүйелер бойынша қатарлар қосындысының салмағымен интегралдылық мәселелерін қарастырамыз. Салмақ функциясы үшін және қатардың қосындысы салмақпен L_p $(1 < p < \infty)$ Лебег кеңістігіне жататын коэффициенттері үшін шарттар табылды. Сонымен қатар, $p = 1$ жағдайы қарастырылған. Бұл жағдайда қарастырылып отырған қатардың қосындысының салмағымен интегралдылықтың басқа шарттары табылған. Генеративті тізбектің шектелуі жағдайында дәлелденген теоремалар Харди-Литтлвудтың монотонды коэффициенттері бар тригонометриялық қатарлар туралы белгілі теоремасының аналогын білдіреді.

Кілт сөздер: мультипликативті жүйелер, қатарлар қосындысының салмақты интегралдануы, жасаушы тізбек, монотонды коэффициенттер, Харди-Литтлвуд теоремасы.

Об интегрируемости с весом суммы рядов с монотонными коэффициентами по мультипликативным системам

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В статье мы изучили вопросы интегрируемости с весом суммы рядов по мультипликативным системам при условии, что коэффициенты рядов монотонны. Найдены условия для весовой функции и коэффициентов ряда, для которых сумма ряда принадлежит пространству Лебега L_p $(1 < p < \infty)$ с весом. Кроме того, рассмотрен случай $p = 1$. В этом случае найдены другие условия интегрируемости с весом суммы рассматриваемого ряда. В случае ограниченности порождающей последовательности доказанные теоремы подразумевают аналог известной теоремы Харди-Литтлвуда о тригонометрических рядах с монотонными коэффициентами.

Ключевые слова: мультипликативные системы, весовая интегрируемость суммы рядов, образующая последовательность, монотонные коэффициенты, теорема Харди-Литтлвуда.

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