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Controllability and optimal speed-in-action of linear systems with boundary conditions

The paper proposes a method for solving the problem of optimal performance for linear systems of ordinary differential equations in the presence of phase and integral restrictions, when the initial and final states of the system are elements of given convex closed sets, taking into account the control value restriction. The presented work refers to the mathematical theory of optimal processes from L.S. Pontryagin and his students and the theory of controllability of dynamic systems from R.E. Kalman. We study the problem of optimal speed for linear systems with boundary conditions from given sets close to the presence of phase and integral constraints, as well as constraints on the control value. A theory of the boundary value problem has been created and a method for solving it based on the study of solvability and the construction of a general solution to the Fredholm integral equation of the first kind has been developed. The main results are the distribution of all controls' sets, each subject of which transfers the trajectory of the system from any initial state to any final state; reducing the initial boundary point to a special initial optimal control problem; constructing a system of algorithms for the gamma-algorithm study on the derivation of problems and rational execution with restrictions on the solution of the optimal speed' problem with restrictions.

Keywords: optimal performance, integrity constraints, functional gradient, integral equation.

Introduction

Methods are proposed for constructing program and positional controls for processes described by linear ordinary differential equations in the presence of boundary conditions, as well as phase and integral constraints, taking into account constraints on controls. Two problems were solved: the problem of a control existence and the problem of constructing a set of all controls that transfers the trajectory of the system from any initial state to a given final state [1–2]. The proposed methods for constructing programs and positional controls are based on the Fredholm integral equation of the first kind. A necessary and sufficient condition for the existence of a solution to a linear integral equation is obtained. A general solution is found for a class of Fredholm integral equations of the first kind [3–5]. It is shown that the boundary value problems of linear ordinary differential equations are reduced to the original optimal control problems with a quadratic functional. Algorithms for constructing minimizing sequences and estimating their convergence are given [6]. Algorithms for solving the optimal performance problem based on solving the controllability problem are presented [7–8]. One of the complex and unsolved problems of control theory is the existence of a solution to the boundary value problem of optimal control in the presence of phase and integral constraints. To solve the problem of the existence of a solution, it is necessary to create a general theory of controllability of dynamical systems. This work is devoted to solving problems of controllability of complex dynamic systems with boundary conditions and constraints [9]. It should be noted that in these works special cases of the general problem of controllability and speed of dynamic systems without phase and integral constraints were studied [10–12]. Actual and unsolved problems of controllability and optimal performance are: obtaining necessary and sufficient conditions for the solvability of general problems of controllability and performance; development of constructive methods for constructing solutions to general problems of controllability and optimality of ordinary differential equations.

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1 Statement of the problem

Consider a controlled process described by a linear ordinary differential equation with an integral and a control of the following form:

$$\dot{x} = A(t)x + B(t)u(t) + C(t)\int_a^b K(t,\tau)v(\tau)d\tau + \mu(t), \quad t \in I = [t_0, t_1], \tag{1}$$

$$\tau \in I_2 = [a, b]$$

with boundary conditions

$$(x(t_0) = x_0) \in S_0, \ (x(t_1) = x_1) \in S_1, \ S_0 \subset R^n, \ S_1 \subset R^n \tag{2}$$

as well as restrictions on control values

$$u(t) \in U(t) = \{u(\cdot) \in L_2(I_1, R^m) \mid u(t) \in U_1(t) \subset L_2(I_1, R^m), \ a.e., \ t \in I_1\}, \tag{3}$$

$$v(\tau) \in V(\tau) = \{v(\cdot) \in L_2(I_2, R^{n_1}) \mid v(\tau) \in V_1(\tau) \subset L_2(I_2, R^{n_1}), \ a.e., \tau \in I_2\}. \tag{4}$$

Here $A(t)$, $B(t)$, $C(t)$, $t \in I_1$ are matrices of orders $n\times n$, $n\times m$, $n\times m_1$ respectively, with piecewise continuous elements; $K(t,\tau)$ is a known matrix of order $m_1 \times n_1$ with elements from L_2, $\mu(t) \in L_2(I_2, R^n)$ of a given function $S_0 \subset R^n$, $S_1 \subset R^n$ of given convex closed sets, which defines restrictions on the initial and final state of the phase variables $U_1(t) \subset L_2(I_1, R^m)$, $V_1(\tau) \subset L_2(I_2, R^{n_1})$ of given convex closed sets. In particular, the sets

$$S_0 = \{x_0 \in R^n \mid |x_0 - \overline{x_1}| \le r\}, \quad S_0 = \{x_0 \in R^n \mid c_i \le x_{0i} \le d_i, \ i = \overline{1,n}\}$$

$$S_1 = \{x_1 \in R^n \mid |x_1 - \overline{x_1}| \le R\}, \quad S_1 = \{x_1 \in R^n \mid \overline{c_i} \le x_{1i} \le \overline{d_i}, \ i = \overline{1,n}\},$$

where $\overline{x_0} \in R^n$, $\overline{x_1} \in R^n$ are fixed vectors, r, R are given numbers, $x_0 = (x_{01}, \dots x_{0n}) \in R^n$, $x_1 = (x_{11}, \dots x_{1n}) \in R^n$, c_i, d_i, $\overline{c_i}$, $\overline{d_i}$, $i = \overline{1,n}$ are fixed numbers.

There are sets

$U_1 = \{u(\cdot) \in L_2(I_1, R^m) \mid \|u - \overline{u}\| \le r, \text{ a.e., } t \in I_1\}$,

$U_1 = \{u(\cdot) \in L_2(I_1, R^m) \mid \alpha_i(t) \le u_i(t) \le \beta_i(t), \text{ a.e., } i = \overline{1,n}, \ t \in I_1\}$,

$V_1(\tau) = \{v(\cdot) \in L_2(I_1, R^{n_1}) \mid \|v - \overline{v}\| \le R, \text{ a.e., } \tau \in I_1\}$,

$V_1(\tau) = \{v(\cdot) \in L_2(I_1, R^{n_1}) \mid \overline{\alpha_i}(\tau) \le v_i(\tau) \le \overline{\beta_i}(\tau), \text{ a.e. } i = \overline{1,n}, \ \tau \in I_2\}$,

where $r > 0$, $R > 0$ are given numbers, $u(t) = (u_1(t), \dots, u_m(t))$, $v(\tau) = (v_1(\tau), \dots, v_{n_1}(\tau))$, $\alpha_i(t)$, $\beta_i(t)$, $t \in I_1$, $\alpha_i(\tau)$, $\beta_i(\tau)$, $\tau \in I_2$ are given continuous functions.

There are the possible cases: 1) when the moments are fixed; 2) t_0 is fixed, to find the smallest value t_1, $t_1 > 0$ when boundary value problem (1)–(4) has a solution. Boundary value problem (1)–(4) in the second case is called the optimal performance problem.

Definition 1. Let the moments be fixed. The solution of the differential equation with subintegral control (1) is called controllable at the time of control $u_*(t) \in U(t)$, $v_*(\tau) = V(\tau)$ which transfers the trajectory of the equation (1) from point $x_{0_*}(t) = x_*(t_0) \in S_0$ at time t_0 points to $x_{1_*}(t) = x_*(t_1) \in S_1$ time t_1.

Definition 2. A quadruple $(u_*(t), v_*(\tau), \ x_{0_*}, x_{1_*}) \in U(t) \times V(\tau) \times S_0 \times S_1$ is called correct if the function $x_*(t) = x_*(t; t_0, x_{0_*}, u_*, v_*)$, $t \in I_1$ that is a solution of differential equation (1) satisfies condition (2). The set of all admissible quadruples is denoted by Σ.

2 Necessary and sufficient conditions for controllability

To solve problems (1)–(4), we consider the controllability problem of a linear system

$$\dot{y} = A(t)y + B(t)w_1(t) + C(t)w_2(t) + \mu(t),\ t \in I_1, \tag{5}$$

$$y(t_0) = x_0 = x(t_0) \in S_0,\ y(t_1) = x_1 = x(t_1) \in S_1, \tag{6}$$

$$w_1(\cdot) \in L_2(I_2, R^m),\ w_2(\cdot) \in L_2(I_2, R^{m_1}). \tag{7}$$

Theorem 1. The integral equation

$$Kw = \int_{t_0}^{t_1} \mathrm{K}(t_0, t)w(t)dt = \beta, \quad t \in I = [t_0, t_1], \tag{8}$$

have a solution for any fixed $\beta \in R^{n_1}$ if and only if the matrix

$$C(t_0, t_1) = \int_{t_0}^{t_1} K(t_0, t)K^*(t_0, t)dt \tag{9}$$

of order $n_1 \times n_1$ is positive definite, where $(*)$ is the transposition sign.

Proof. Sufficiency. Let the matrix $C(t_0, t_1) > 0$. Let us show that integral equation (8) have a solution for any $\beta \in R^n$. Let's choose

$$w(t) = K^*(t_0,\ t)C^{-1}(t_0, t_1)\beta,\ \ t \in I = [t_0,\ t_1].$$

Then

$$Kw = \int_{t_0}^{t_1} K(t_0, t)K^*(t_0,\ t)dt\ \ C^{-1}(t_0, t_1)\beta = \beta.$$

Thus, for $C(t_0, t_1) > 0$, integral equation (8) have at least one solution

$$w(t) = K^*(t_0,\ t)C^{-1}(t_0, t_1)\beta,\ t \in I,\ \beta \in R^n.$$

The sufficiency is proved.

Necessity. Let integral equation (8) have a solution for any fixed $\beta \in R^n$. Let's prove that the matrix $C(t_0, t_1) > 0$. Since $C(t_0, t_1) \geq 0$, then to prove $C(t_0, t_1) > 0$ it is necessary to show that the matrix $C(t_0, t_1)$ is nonsingular.

Suppose, by contradiction, that the matrix $C(t_0, t_1)$ is singular. Then there is a vector $c \in R^n$, $c \neq 0$ such that $c^*C(t_0, t_1)c = 0$. Let's define the function $\bar{v}(t) = K^*(t_0, t)c$, $t \in I$, $\bar{v}(\cdot) \in L_2(I, R^m)$. Note that

$$\int_{t_0}^{t_1} \bar{v}^*(t)\bar{v}(t)dt = c^* \int_{t_0}^{t_1} K^*(t_0,\ t)K(t_0,\ t)dt \cdot c = c^*\, C(t_0, t_1)c = 0.$$

Therefore, the function $\bar{v}(t) = 0,\ \ t \in I$. Since integral equation (8) have a solution for any $\beta \in R^n$, then, in particular, there exists function (7) such that $\bar{w}(\cdot) \in L_2(I, R^m)$ and $(\beta = c)$

$$\int_{t_0}^{t_1} K(t_0, t)\bar{w}(t)dt = c.$$

Thus the identity

$$0 = \int_{t_0}^{t_1} v^*(t)\bar{w}(t)dt = c^* \int_{t_0}^{t_1} K(t_0,\ t)w(t)dt = c^*c$$

is true. This contradicts the condition that $c \neq 0$. The necessity is proved. The theorem is proved.

Theorem 2. The existence of a control $w_*(\cdot) = w_{1*}(\cdot), w_{2*}(\cdot) \in L_2(I_2, R^m)\times L_2(I_2, R^{m_1})$ transferring the trajectory of equation (5) from the starting point $y(t_0) = x_0 \in S_0$ to the point $y(t_1) = x_1 \in S_1$ it is necessary and sufficient condition for the matrix

$$W(t_0, t_1) = \int_{t_0}^{t_1} \Phi(t_0, t)B_1(t)B_1^*(t)\Phi^*(t_0, t)dt \tag{10}$$

the order $n \times n$ be positive defined, where $B_1(t) = (B(t), C(t))$. Linear control system (5)–(7) differs from (1)–(3) in that the point, $x_0 \in S_0$, $x_1 \in S_1$. Let the matrix $W(t_0, t_1)$ determined by formula (8) be positive defined. Then a control $w_*(\cdot) = (w_{1*}(\cdot), w_{2*}(\cdot)) \in L_2(I_2, R^{m+m_1})$ transfers the trajectory of equation (5) from point $y_*(t_0) = x_{0*} \in S_0$ to point $y_*(t_1) = x_{1*} \in S_1$, if and only if

$$w_*(t) \in W_1 = \{w_*(\cdot) \in L_2(I_2, R^{m+m_1})|w_*(t) = p_*(t) + \lambda_1(t,\ x_{0*},\ x_{1*}) + N_1(t)z(t_1, p_*),$$

$$x_{0*} \in S_0,\ x_{1*} \in S_1,\ \forall p_*(\cdot) = (p_{1*}(\cdot),\ p_{2*}(\cdot)) \in L_2(I_2, R^{m+m_1})\}, \tag{11}$$

where

$$\lambda_1(t,\ x_{0*},\ x_{1*}) = B_1^*(t)\Phi^*(t_0, t)W^{-1}(t_0, t_1)a, \tag{12}$$

$$a = \Phi(t_0, t_1)x_{1*} - x_{0*} - \int_{t_0}^{t_1} \Phi(t_0, t)\mu(t)dt,\ N_1(t) = -B_1^*(t)\Phi^*(t_0, t)W^{-1}(t_0, t_1)\Phi(t_0, t_1),$$

$$p_*(\cdot) \in L_2(I_2, R^{m+m_1}) \tag{13}$$

and the function $z(t) = z(t, p_*)$, $t \in I_1$ is a solution of the differential equation

$$\dot{z}(t) = A(t)z + B_1(t)p_*(t),\ z(t_0) = 0,\ t \in I_1.$$

The solution of differential equation (5) corresponding to the controller, is determined by the formula

$$y_*(t) = z(t,\ p_*) + \lambda_2(t,\ x_{0*},\ x_{1*}) + N_2(t)z(t_1,\ p_*),\ t \in I_1,$$

where

$$\lambda_2(t,\ x_{0*},\ x_{1*}) = \Phi(t, t_0)W(t, t_1)W^{-1}(t_0, t_1)x_{0*} + \Phi(t, t_0,)W(t, t_1)\, W^{-1}(t_0, t_1)\Phi(t, t_0)x_{1*}+$$

$$+\int_{t_0}^{t} \Phi(t_0, \tau)\mu(\tau)d\tau - \Phi(t, t_0)W(t_0, t)\, W^{-1}(t_0, t_1) \int_{t_0}^{t_1} \Phi(t_0, t)\mu(t)dt,\ \ t \in I_1,$$

$$x_{0*} \in S_0, x_{1*} \in S_1 N_2(t) = -\Phi(t, t_0)W(t_0, t)W^{-1}(t_0, t_1)\Phi(t_0, t_1),\ t \in I_1.$$

Proof. Indeed, from (8) for $K(t_0, t) = \Phi(t_0, t)B_1(t)$ we have (10). Then

$$C(t_0, t_1) = \int_{t_0}^{t_1} K(t_0, t)K^*(t_0, t)dt = \int_{t_0}^{t_1} \Phi(t_0, t)B_1(t)B_1^*(t)\Phi^*(t_0, t)dt = W(t_0, t_1)$$

for the existence of a solution to integral equation (10) it is necessary and sufficient that the matrix $W(t_0,t_1)>0$, the control $w_1(t),\ t\in I$ is determined by the formula $w(t)=v(t)+K^*(t_0,\ t)C^{-1}(t_0,t_1)\beta-K^*(t_0,t)C^{-1}(t_0,t_1)\int\limits_{t_0}^{t_1}K(t_0,t)v(t)dt,\ \ t\in I$. Then (see (9))

$$w_1(t)=v(t)+K^*(t_0,t)\,W^{-1}(t_0,t_1)a-K^*(t_0,t)\,W^{-1}(t_0,t_1)\int\limits_{t_0}^{t_1}K(t_0,t)v(t)dt=$$

$$=v(t)+B_1^*(t)\Phi^*(t_0,t)W^{-1}(t_0,t_1)[\Phi(t_0,t_1)\varepsilon_1-\varepsilon_0-\int\limits_{t_0}^{t_1}\Phi(t_0,t)\bar{\mu}(t)dt]-$$

$$-B_1^*(t)\Phi^*(t_0,t)W^{-1}(t_0,t_1)\int\limits_{t_0}^{t_1}\Phi(t_0,t)B_1(t)v(t)dt=$$

$$=v(t)+T_1(t)\varepsilon_0+T_2(t)\varepsilon_1+\bar{\mu}(t)+M_1(t)z(t_1,v),\ \ t\in I,\ \ \forall v,\ \ v(\cdot)\in L_2(I,R^m),$$

where matrices $T_1(t)$, $T_2(t)$, $M_1(t)$, $t\in I$, are defined by relations (10),

$$\int\limits_{t_0}^{t_1}\Phi(t_0,t)B_1(t)v(t)dt=\Phi(t_0,t_1)z(t_1,v),v(\cdot)\in L_2(I,R^m),$$

$z(t,v),\ \ t\in I$, is a solution of differential equations (11)–(13). The set U_1 is generated when an arbitrary function $v(\cdot)\in L_2(I,R^m)$ runs through all elements of the space $L_2(I,R^m)$. The theorem is proved.

3 Creating and solving controllability problems

Consider optimization problem (5)–(7), in the form of

$$J(\theta)=\int_{t_0}^{t_1}F_0(q(t),t)dt\to\inf,\,\theta\in X\subset H,$$

where $q(t)=(\theta(t),z(t_1,p))$, $p_1(t)\in L_2^{\rho}(I_1,R^{m_1})=\{p_1(\cdot)\in L_2(I_1,R^m)|\|\rho_1\|\le\rho\}$,

$$p_2(t)\in L_2^{\rho}(I_1,R^{m_1})=\{p_2(\cdot)\in L_2(I_1,R^{m_1})|\|\rho_2\|\le\rho\},$$
$$F_0(q(t),t)=|F_1(q(t),t)|^2+|F_2(q(t),t)|^2,F_1(q(t),t)=w_1-u,$$
$$F_2(q(t),t)=w_1-\int_a^b K(t,\tau)v(\tau)d\tau.$$

Note that:

1) $U(t)$, $V(t)$, S_0, S_1 are bounded convex closed sets, then X is a bounded convex closed set in a reflexive Banach space H, where $L_2(I_1,R^m)$, $L_2^{\rho}(I_1,R^{m_1})$ are bounded convex closed sets in the Hilbert space L_2.

2) the functional $J(\theta)$, $\theta\in X$ is bounded from below $J(\theta)\ge 0$, $\forall\theta\in X$. It is easy to see that the quadratic functional $J(\theta)$, $\theta\in X$ is convex since $z(t,\alpha\overline{p}+(1-\alpha)\overline{\overline{p}})=\alpha z(t,\overline{p})+(1-\alpha)z(t,\overline{\overline{p}}),\forall\overline{p},\,\overline{\overline{p}}\in L_2^{\rho}(I_1,R^{m+m_1})$, $\alpha\in[0,1]$.

3) It is known that a bounded convex closed set X in the reflexive Banach space H is weakly bicompact, and a continuous convex functional $J(\theta)$, $\theta\in X$ is weakly semicontinuous from below.

4) A weakly lower semicontinuous functional $J(\theta)$, $\theta \in X$ on a weekly bicompact set reaches the infimum on the set X, and hence the set.

$X_* = \theta_* \in X | J(\theta_*) = J_* = \inf\limits_{\theta \in X} J(\theta) = \min\limits_{\theta \in X} J(\theta)\} \neq \emptyset$ where $\emptyset$ is the empty set.

The partial derivative of the function $F_0(q,t)$ are:

$$F_{0x_0}(q,t) = 2\,T_0^*(t)F_1(q,t) + 2\,T_2^*(t)F_2(q,t),$$

$$F_{0x_1}(q,t) = 2\,T_1^*(t)F_1(q,t) + 2\,T_2^*(t)F_2(q,t),$$

$$F_{0z(t_1)} = 2\,N_{11}^*(t)F_1(q,t) + 2\,N_{12}^*(t)F_2(q,t),$$

$$F_{0p_1}(q,t) = 2\,F_1(q,t), \;\; F_{0p_2}(q,t) = 2\,F_2(q,t), \; F_{0u}(q,t) = -2\,F_1(q(t),t).$$

Theorem 3. Let the matrix $W(t_0,t_1)$ be positively defined. Then the functional under the conditions is continuously differentiable with respect to Frechet, the gradient

$$J'(\theta) = (J_u'(\theta),\, J_v'(\theta),\, J_{p_1}'(\theta),\, J_{p_2}'(\theta),\, J_{x_0}'(\theta),\, J_{x_1}'(\theta)) \in H$$

at any point $\theta \in X$ is determined by the formula

$$J'(\theta) = -F_{0u}(q(t),t),$$

$$J_v'(\theta) = 2\int_{t_0}^{t_1} K^*(t,\tau)w_2(t)dt + 2\int_{t_0}^{t_1}\int_a^b K^*(t,\tau)K(t,\tau)v(\tau)d\tau dt,$$

$$J_{p_1}'(\theta) = 2F_1(q,t) - B^*(t)\psi(t),\; J_{p2}'(\theta) = 2F_2(q,t) - C^*(t)\psi(t), \tag{14}$$

$$J_{x_0}'(\theta) = \int_{t_0}^{t_1} F_{0x_0}(q(t),t)dt,\; J_{x_1}'(\theta) = \int_{t_0}^{t_1} F_{0x1}(q(t),t)dt,$$

where $\psi(t), t \in I_1$ is the solution of the coupled system

$$\dot{\psi} = -A^*(t)\psi(t),\; \psi(t_1) = -\int_{t_0}^{t_1} F_{0z_{(t_1)}}(q(t),t)dt, \tag{15}$$

$$F_{0z(t_2)}(q(t),t) = 2\,N_{11}^*(t)F_1(q(t),t) + 2\,N_{12}^*(t)F_2(q(t),t),\; t \in I_1,$$

function $z(t) = z(t,p)$, $t \in I_1$ solution of the differential equation (13).

In addition, the gradient satisfies $J'(\theta)$, $\theta \in X$ the Lipschitz condition

$$\|J'(\theta_1) - J'(\theta_2)\| \leq K\|\theta_1 - \theta_2\|,\; \forall \theta_1, \theta_2 \in X. \tag{16}$$

Proof. Let $\theta, \theta + \Delta\theta \in X$, $\Delta\theta = (\Delta u,\, \Delta v,\, \Delta p_1,\, \Delta p_2,\, \Delta x_{0*},\, \Delta x_{1*})$. As in the proof of Theorem 3, the functional increment can be represented as

$$\Delta J = J(\theta + \Delta\theta) - J(\theta) = \int_{t_0}^{t_1} \{\Delta u^*(t)F_{0u}(q(t) + \Delta p_1^*(t)[F_{0p1}(q,t) - B^*(t)\psi(t)] +$$

$$+\Delta p_2^*(t)[F_{0p2}(q,t) - C^*(t)\psi(t)] + \Delta x_0^*(t)F_{0x0}(q,t) + \Delta x_1^*(t)F_{0x_1}(q,t)\}dt +$$

$$+ \int_a^b \Delta v^*(\tau)J_v' d\tau + R_1 + R_2 + R_3 + R_4, \tag{17}$$

where

$$R_1 = \int_{t_0}^{t_1} |\Delta w_1 - \Delta v|^2 dt,\; R_2 = \int_{t_0}^{t_1} |\int_a^b K(t,\tau)\Delta v(\tau)d\tau - \Delta w_2(t)|^2 dt,$$

$$R_3 = \int_{t_0}^{t_1} \Delta x_0^*[F_{0x0}(q+\Delta q,t)+F_{0x_0}(q,t)]dt,\ R_4 = \int_{t_0}^{t_1} \Delta x_1^*[F_{0x_1}(q+\Delta q,t)-F_{0x_1}(q,t)]dt,$$

$$|R| \le c_1\|\Delta\theta\|^2,\ \Delta q(t) = (\Delta\theta(t), z(t_1,p)).$$

From (15)–(17) it follows that the Freschi derivative of functional (16) under conditions (15)–(17) is determined by formula (14), where $\varphi(t)$, $t \in I_1$ is a solution of differential equation (15)–(17).

Let $\theta_1 = \theta + \Delta\theta$, $\theta_2 = \theta$. Then

$$J'(\theta_1) - J'(\theta_2) = F_{0u}(q+\Delta q,t) - F_{0u}(q,t),\ J'_v(\theta+\Delta\theta) - J'_v(\theta),$$

$$2F_1(q+\Delta q,t) - 2F_1(q,t) - B^*(t)\Delta\psi(t),$$

$$2F_2(q+\Delta q,t) - 2F_2(q,t) - C^*(t)\Delta\psi(t),$$

$$\int_{t_0}^{t_1}[F_{0x0}(q+\Delta q,t)-F_{0x_0}(q,t)]dt,\ \int_{t_0}^{t_1}[F_{0x_1}(q+\Delta q,t)-F_{0x_1}(q,t)]dt.$$

$$|J'(\theta_1) - J'(\theta_2)| \le L_1|\Delta q(t)| + L_2|\Delta\psi(t)| + L_3\|\Delta q\|,$$

$$\|J'(\theta_1) - J'(\theta_2)\|^2 = \int_{t_0}^{t_1}|J'(\theta_1) - J'(\theta_2)|^2 dt \le L_4\|\Delta q\|^2 + L_5\int_{t_0}^{t_1}|\Delta\psi(t)|^2 dt. \tag{18}$$

Since

$$\Delta\dot{\psi} = -A^*(t)\Delta\psi(t),\ t \in I_1,\ \Delta\psi(t_1) = -\int_{t_0}^{t_1}[F_{0z_{(t_1)}}(q+\Delta q,t) - F_{0z_{(t_1)}}(q,t)]dt,$$

that

$$\Delta\psi(t) = \Delta\psi(t_1) + \int_t^{t_1} A^*(t)\Delta\psi(t),\ t \in I_1,$$

$$|\Delta\psi(t)| \le |\Delta\psi(t_1)| + A^*_{\max}\int_t^{t_1}|\Delta\psi(\tau)|d\tau \le L_6\int_{t_0}^{t_1}|\Delta q(t)|dt + A^*_{\max}\int_t^{t_1}|\Delta\psi(\tau)|d\tau,$$

$$\|\Delta q(t)\| \le c_2\|\Delta\theta\|,\ |\Delta z(t,p)| \le c_3\|\Delta p_1\| + c_4\|\Delta p_2\|.$$

Then, applying the Gronwall lemma, we obtain

$$|\Delta\psi(t)| \le L_7 e^{A^*_{\max}(t_1-t_2)}\|\Delta\theta\|. \tag{19}$$

From (18), (19) follows estimate (16).

Based on the results of Theorem 3, we construct the sequences $\{\theta_n\} = \{u_n, v_k, p_{1n}, p_{2n}, x_{0n}, x_{1n}\} \subset X$ by algorithm

$$\begin{array}{l} u_{n+1} = P_U[u_n - \alpha_n J'_u(\theta_n)], \quad v_{n+1} = P_V[v_n - \alpha_n J'_v(v_n)], \\ p_{1n+1} = P_{L_2^p}[p_{1n} - \alpha_n J'_{p_1}(\theta_n)], \quad p_{2n+1} = P_{L_2^p}[p_{2n} - \alpha_n J'_{p_2}(\theta_n)], \\ x_{0n+1} = P_{S_0}[x_{0n} - \alpha_n J'_{x0}(\theta_n)], \quad x_{1n+1} = P_{S_1}\left[x_{1n} - \alpha_n J'_{x1}(\theta_n)\right], \\ n = 0,1,2,\ldots, \quad 0 < \xi_0 \le \alpha_n \le \frac{2}{K+2\varepsilon_1},\ \varepsilon_1 > 0, \end{array} \tag{20}$$

where $K > 0$ is the Lipschitz constant of equation (14), in particular, $\varepsilon_1 = \frac{K}{2}$ in the case of $\varepsilon_0 = \alpha_n = \frac{1}{K}$. We get that U, V, S_0, S_1 are bounded convex closed sets, $P_\Omega[\cdot]$ is the projection of a point onto the set Ω. Any point has a unique projection onto a convex closed set.

Theorem 4. Let the matrix $W(t_0, t_1) > 0$, the sequence $\{\theta_n\}$ be defined by the formula (20). Then:

1. the numeric sequence $\{J(\theta_n)\}$ is strictly decreasing;

2. $\|\theta_n - \theta_{n+1}\| \to 0$ when $n \to \infty$;

3. the sequence $\{\theta_n\} \subset X$ is minimized: $\lim\limits_{n\to\infty} J(\theta_n) = J_* = \inf\limits_{\theta\to X} J(\theta)$;

4. the set $X_* = \{\theta_* \in X | J(\theta_*) = J_* = \inf\limits_{\theta\to X} J(\theta) = \min\limits_{\theta\to X} J(\theta)\}$ is not empty, the lower bound functional $J(\theta)$, $\theta \in X$ is reached on the set X;

5. the sequence $\{\theta_n\} \subset X$ converges weakly to the set X_*, $u_n \overset{weak}{\to} u_*$, $v_n \overset{weak;}{\to} v_*$, $p_{2n} \overset{weak}{\to} p_{2*}$, $x_{0n} \overset{weak}{\to} x_{0*}$, $x_{1n} \overset{weak}{\to} x_{1*}$ at $n \to \infty$, where $(u_*,\ v_*, p_{1*}, p_{2*}, x_{0*}, x_{1*}) \in X_*$;

6. the following convergence rate estimate is valid

$$0 \le J\left(\theta_n\right) - J_* \le \frac{m_0}{n}\ n = 1, 2,\ \ldots,\ m_0 = const > 0,$$

where $J(\theta_*) = J_*$;

7. controllability problem (1)–(4) has a solution if and only if the value $J(\theta_*) = 0$. In this case, the solution of controllability problem (1)–(4) is the function

$$x_*(t) = z(t, p_*) + \lambda_*(t, x_{0*}, x_{1*}) + N_2(t)z(t_1, p_*),\ t \in I_1.$$

If $J(\theta_*) > 0$, then controllability problem (1)–(4) has no solution, $x_*(t)$, $t \in I_1$ is the best necessary solution to controllability problem (1)–(4).

Proof. From the property for the projection of a point onto a set, we have

$$< J'(\theta_n),\ \theta - \theta_{n-1} > \ge \frac{1}{\alpha_n} < \theta_n - \theta_{n-1},\ \theta - \theta_{n-1},\ \ \forall\theta,\ \theta \in X. \tag{21}$$

Since $J'(\theta) \in C^{1,1}(X), X$ is a convex set, the estimate is true

$$J(\theta_1) - J(\theta_2) \ge < J'(\theta_1),\ \theta_1 - \theta_2 > -\frac{K}{2}\|\theta_1 - \theta_2\|^2,\ \forall\theta_1,\ \theta_2 \in X. \tag{22}$$

From (16) and (17) $\theta = \theta_n$, $\theta_1 = \theta_n$, $\theta_2 = \theta_{n+1}$, we get

$$J(\theta_n) - J_1(\theta_{n+1}) \ge (\frac{1}{\alpha_n} - \frac{K}{2})\ \|\theta_n - \theta_{n+1}\|^2 \ge \xi_1\|\theta_n - \theta_{n+1}\|^2,\ \frac{1}{\alpha_n} - \frac{K + 2\varepsilon_1}{2}. \tag{23}$$

It follows from equality (23) that the numerical sequence $\{J(\theta_n)\}$ is strictly decreasing, and also because of the limited value of the functional at $\|\theta_n - \theta_{n+1}\| \to 0$ by $n \to \infty$. Thus, assertions 1) and 2) of the theorems are proved.

The functional $J(\theta), \theta \in M_0$ is weakly lower semicontinuous on a weakly bicompact set X, then the set is empty. The sequence $\{\theta_n) \subset M_0$. Then, due to the weakly bicompactness of the set M_0 it follows that $\theta_n \overset{weak}{\to} \theta_*$, $n \to \infty$, $\theta_* \in X_*$. Thus, statements 4), 5) of the theorem are proved.

For convex functional $J(\theta) \in C^{1,1}(M_0)$, the following inequality holds

$$\begin{aligned} &J(\theta_n) - J(\theta_*) \le < J'(\theta_n),\ \theta_n - \theta_* > = < J'(\theta_n),\ \theta_n - \theta_{n+1} + \theta_{n+1} - \theta_* > = \\ &= < J'(\theta_n),\ \theta_n - \theta_{n+1} > - < J'(\theta_n), \theta_* - \theta_{n-1} > . \end{aligned}$$

Hence, taking into account the inequality for $\theta \in \theta_*$, we have

$$\begin{aligned} &0 \le J'(\theta_n) - J_* \le < J'(\theta_n), \theta_n - \theta_{n+1} > -\tfrac{1}{\alpha_n} < \theta_n - \theta_{n+1}, \theta_* - \theta_{n+1} > = \\ &= < J'(\theta_n) - \tfrac{1}{\alpha_n}(\theta_* - \theta_{n+1}), \theta_n - \theta_{n+1} > \le \\ &\le \|J'(\theta_n) - \tfrac{1}{\alpha_n}(\theta_* - \theta_{n+1})\|\|\theta_n - \theta_{n+1}\| \le \\ &\le (\|J'(\theta_n)\| + \tfrac{1}{\alpha_n})\|\theta_* - \theta_{n+1}\| = \|\theta_n - \theta_{n+1}\|c_0\|\theta_n - \theta_{n+1}\|, \end{aligned} \tag{24}$$

where $\|\theta_* - \theta_{n+1}\| \leq D$, $\frac{1}{\alpha_n} \leq \frac{r}{\xi_0}$, $c_0 = \sup\|J'(\theta_n)\| + \frac{D}{\xi_0}$, D is the diameter of the set M_0. Since with $\|\theta_n - \theta_{n+1}\| \to 0$, then $n \to \infty$ that $\lim\limits_{n\to\infty} J(\theta_n) = J_* = J(\theta_*)$. This means that the $\{\theta_n\} \subset M_0$ sequence reaches a minimum.

It follows from inequalities (23), (24) that $J(\theta_n) - J(\theta_{n+1}) = a_n - a_{n+1} \geq \varepsilon_1\|\theta_n - \theta_{n+1}\|^2$, $a_n - a_{n+1} \geq c_0\|\theta_n - \theta_{n+1}\|$, $a_n = J(\theta_n) - J_* = J(\theta_n) - J(\theta_*)$. Then $a_n > 0$,$a_n - a_{n+1} \geq \frac{\varepsilon_1}{c_0^2} a_n^2$,$n = 1, 2, ...$,$m_0 \geq \frac{c_0^2 \varepsilon_1}{\varepsilon_1}$. The theorem is proved.

4 Solving a model problem

Consider a controlled process described by a differential equation with an integral equation of the form

$$\dot{x_1} = x_2,\ \dot{x_2} = u + \int_1^2 e^{(t+1)\tau} v(\tau)d\tau,\ t \in I_1 = [0,2],\ \tau \in I_2 = [1,2], \tag{25}$$

$$(x_{10}(0),\ x_{20}(0)) \in S_0 = \{-1 \leq x_{10} \leq 1,\ 1 \leq x_{20} \leq 2\},$$

$$(x_{11}(2),\ x_{21}(2)) \in S_1 = \{-1 \leq x_{11}(2) \leq 1,\ -2 \leq x_{21}(0) \leq -1\},$$

$$u(t) \in U = \{u(\cdot) \in L_2(I_2,\ R^1)|\ -1 \leq u(t) \leq 1,\ a.e.\ t \in I_1\},$$

$$v(t) \in V = \{v(\cdot) \in L_2(I_2,\ R^1)|\ \tau \leq v(\tau) \leq 2\tau,\ a.e.\,\tau \in I_2\}. \tag{26}$$

1. The necessary sufficient conditions satisfy controllability defined by the ratios:

a) Matrix $W(0,2) = \begin{pmatrix} 16/3 & -4 \\ -4 & 4 \end{pmatrix} > 0$;

b) $u_*(t) = w_{1*}(t) = p_{1*}(t)T_0(t)x_{0*} + T_1(t)x_{1*} + \mu_{11}(t) + N_{11}(t)z(t_1,\ p_*)$, $t \in I_1$,

$$T_0(t) = (\frac{3}{4}(t-1),\ \frac{3t-4}{4}),\ T_1(t) = (\frac{3t-4}{4},\ \frac{(3t-4)}{4}),\ \mu_{11}(t) \equiv 0,$$

$$N_{11}(t) = (\frac{3(t-1)}{4},\ \frac{(2-3t)}{4}),\ p_{1*}(\cdot) \in L_2(I_2,\ R^1),\ x_{0*} \in (x_{10*},\ x_{20*}) \in S_0,$$

$$(x_{11*},\ x_{21*}) \in S_1,\ u_*(t) \in U,\ v_*(t) \in V;$$

c) $\int_1^2 e^{(t+1)\tau} v_*(\tau)d\tau = w_{2*}(t) = p_{2*}(t) + T_2(t)x_{0*} + T_3(t)x_{1*} + \mu_{12}(t) + N_{12}(t)z(t_1,\ p_*)$,

$$T_2(t) = (\frac{3}{4}(t-1),\ \frac{3t-4}{4}),\ T_3(t) = (\frac{3(1-t)}{4},\ \frac{3t-2}{4}),\ N_{12}(t) = (\frac{3(t-1)}{4},\ \frac{(2-3t)}{4}),$$

$$x_{0*} \in (x_{10*},\ x_{20*}) \in S_0,\ x_{1*} = (x_{11*},\ x_{21*}) \in S_1,\ p_{2*}(\cdot) \in L_2(I_2,\ R^1), \mu_{12}(t) \equiv 0,$$

$$S_0 = S_{10},\ S_{20},\ S_{10} = \{x_{10} \in R^1|-1 \leq x_{10} \leq 1,\ S_{20} = \{x_{20} \in R^1|1 \leq x_{20} \leq 2\};$$

$$S_1 = S_{11},\ S_{21},\ S_{11} = \{x_{11}(2) \in R^1|-1 \leq x_{11}(2) \leq 1\},\ S_{21} = \{x_{21}(2) \in R^1|-2 \leq x_{21}(2) \leq -1\}.$$

2. Construction of a solution to the controllability problem. The desired controls $u_*(t) \in U$, $v_*(\tau) \in V$, $p_{1*}(t) \in L_2^{\rho}(I_1,\ R^1)$, $p_{2*}(t) \in L_2^{\rho}(I_1,\ R^1)$, $x_{0*} \in S_0$, $x_{1*} \in S_1$, can be found when solving the optimal control problem: minimize the functional

$$J(u,\ v, p_1, p_2, x_{10}, x_{20}, x_{11}, x_{21}) = \int_{t_0}^{t_1} \{|w_1(t) - u(t)|^2 + |w_2(t) - \int_a^b K(t,\tau)v(\tau)d\tau|^2\}dt \to \inf \tag{27}$$

under conditions

$$u(t) \in U,\ v(\tau) \in V,\ p_1(t) \in L_2^{\rho}(I_2, R^1),\ p_2(t) \in L_2(I_1, R^1),\ x_{10} \in S_{10},\ x_{20} \in S_{20},\ x_{11} \in S_{11},\ x_{21} \in S_{21}, \tag{28}$$

$$w_1(t) = p_1(t) + T_{10}(t)x_{10} + T_{20}(t)x_{20} + T_{11}(t)x_{11} + T_{21}(t)x_{21} + N_{11}(t)z(t_1,p),\ t \in I_1,$$
$$w_2(t) = p_2(t) + T_{20}(t)x_{10} + T_{30}(t)x_{20} + T_{31}(t)x_{11} + T_{41}(t)x_{21} + N_{12}(t)z(t_1,p),\ t \in I_1,$$
$$T_{10} = T_{10}(t) = \frac{3}{4}(t-1),\ T_{20} = T_{20}(t) = \frac{3t-4}{4}, T_{11} = T_{11}(t) = \frac{3t-4}{4},\ T_{21} = T_{21}(t) = \frac{3t-2}{4},$$
$$T_{20} = T_{20}(t) = \frac{3}{4}(t-1),\ T_{30} = T_{30}(t) = \frac{3t-4}{4}, T_{31} = T_{31}(t) = \frac{3(t-1)}{4},\ T_{41} = T_{41}(t) = \frac{3t-2}{4},$$

where a function $z(t,p)$, $t \in I_1$ is a solution of the differential equation

$$\dot{z}_1 = z_2,\ \dot{z}_2 = p_1(t) + p_2(t),\ z_1(0) = 0,\ z_2(0) = 0,\ t \in I_1.$$

Let us calculate at $\theta = (u,\ v, p_1, p_2, x_{10}, x_{20}, x_{11}, x_{21})$, $q = (\theta, z(2))$ a gradient of functional (24) under conditions (25)–(28):

a) $J'_u(\theta) = F_{0u}(q,t) = -2(w_1 - u)$, $F_0(q,t) = |F_1(q,t)|^2 + |F_2(q,t)|^2$,

$$F_1(q,t) = (w_1 - u),\ F_2(q,t) = w_2 - \int_1^2 e^{(t+1)\tau} w(\tau)d\tau,$$
$$J'_v(\theta) = -2\int_0^2 e^{(t+1)\tau} w_2(t)dt + 2\int_0^2\int_1^2 e^{(t+1)\tau} e^{(t+1)\tau} v(\tau)d\tau dt,$$
$$J'_{\rho_1}(\theta) = 2F_1(q,t) - B^*(t)\psi(t),\ J'_{\rho_2}(\theta) = 2F_2(q,t) - C^*(t)\psi(t),\ t \in I_1,$$
$$J'_{x_{10}}(\theta) = \int_0^2 [2T_{10}(t)F_1(q,t) + 2T_{20}(t)F_2(q,t)]dt,\ J'_{x_{20}}(\theta) = \int_0^2 [2T_{20}(t)F_1(q,t) + 2T_{30}(t)F_2(q,t)]dt,$$
$$J'_{x_{11}}(\theta) = \int_0^2 [2T_{11}(t)F_1(q,t) + 2T_{31}(t)F_2(q,t)]dt,\ J'_{x_{21}}(\theta) = \int_0^2 [2T_{21}(t)F_1(q,t) + 2T_{40}(t)F_2(q,t)]dt.$$

b) partial derivative

$$F_{0z_{(t_1)}}(q,t) = \begin{pmatrix} \frac{3(t-1)}{4} \\ \\ \frac{2-3t)}{4} \end{pmatrix},\ F_1(q,t) = \begin{pmatrix} \frac{3(t-1)}{4} \\ \\ \frac{2-3t)}{4} \end{pmatrix}.$$

c) coupled system

$$\dot{\psi}_1 = 0,\ \dot{\psi}_2 = -\psi_1,\ \ \psi(2) = \begin{pmatrix} \psi_1(2) \\ \psi_2(2) \end{pmatrix} = -\int_{t_0}^{t_1} F_{0z(t_1)}\left(q\left(t\right), t\right) dt.$$

d) minimizing sequences are:

$$\begin{array}{l} u_{n+1} = P_U[u_n - \alpha_n J'_u(\theta_n)],\ \ p_{n+1} = P_V[v_n - \alpha_n J'_v(\theta_n)], \\ p_{1n+1} = P_{L_2^p}[p_{1n} - \alpha_n J'_{p_1}(\theta_n)],\ \ p_{2n+1} = P_{L_2^p}[p_{2n} - \alpha_n J'_{p_2}(\theta_n)], \\ x_{10}^{n+1} = P_{S_{10}}[x_{10}^{(n)} - \alpha_n J'_{x_{10}}(\theta_n)],\ \ x_{20}^{n+1} = P_{S_{20}}[x_{20}^{(n)} - \alpha_n J'_{x_{20}}(\theta_n)], \\ x_{11}^{n+1} = P_{S_{11}}[x_{11}^{(n)} - \alpha_n J'_{x_{11}}(\theta_n)],\ \ x_{20}^{n+1} = P_{S_{21}}[x_{21}^{(n)} - \alpha_n J'_{x_{21}}(\theta_n)], \\ n = 0,1,2,\ldots,\ \ \alpha_n \le \frac{1}{K}. \end{array}$$

e) projections of a point onto sets

$$P_V[v_n - \alpha_n J'{}_v(\theta_n)] = \left\{ \begin{array}{c} \tau,\ \text{if}\ v_n - \alpha_n J'{}_v(\theta_n) < \tau; \\ v_n - \alpha_n J'{}_v(\theta_n,\ \text{if}\ \tau \le v_n - \alpha_n J'{}_v(\theta_n) \le 2\tau; \\ 2\tau,\ \text{if}\ v_n - \alpha_n J'{}_v(\theta_n) > 2\tau; \end{array} \right.$$

$$P_{L_2^p}[p_{1n} - \alpha_n J'_{p_1}(\theta_n)] = p_{1n} - \alpha_n J'_{p_1}(\theta_n),\ if\ \left\|p_{1n} - \alpha_n J'_{p_1}(\theta_n)\right\| \le \rho,$$

$$P_{L_2^p}[p_{2n} - \alpha_n J'_{p_2}(\theta_n)] = p_{2n} - \alpha_n J'_{p_2}(\theta_n),\ if\ \left\|p_{2n} - \alpha_n J'_{p_2}(\theta_n)\right\| \le \rho,$$

$\rho > 0$ is quite large;

$$P_{S_{10}}[x_{10}^{(n)} - \alpha_n J'_{x_{10}}(\theta_n)] = \begin{cases} -1, \text{ if } x_{10}^{(n)} - \alpha_n J'_{x_{10}}(\theta_n) < -1; \\ x_{10}^{(n)} - \alpha_n J'_{x_{10}}(\theta_n), \text{ if } -1 \le x_{10}^{(n)} - \alpha_n J'_{x_{10}}(\theta_n) \le 1; \\ 1, \text{ if } x_{10}^{(n)} - \alpha_n J'_{x_{10}}(\theta_n) > 1; \end{cases}$$

$$P_{S_{20}}[x_{20}^{(n)} - \alpha_n J'_{x_{20}}(\theta_n) = \begin{cases} 1, \text{ if } x_{20}^{(n)} - \alpha_n J'_{x_{20}}(\theta_n) < 1; \\ x_{20}^{(n)} - \alpha_n J'_{x_{20}}(\theta_n), \text{ if } 1 \le P_{S_{20}}[x_{20}^{(n)} - \alpha_n J'_{x_{20}}(\theta_n) \le 2; \\ 2, \text{ if } P_{S_{20}}[x_{20}^{(n)} - \alpha_n J'_{x_{20}}(\theta_n) > 2; \end{cases}$$

$$P_{S_{11}}[x_{11}^{(n)} - \alpha_n J'_{x_{11}}(\theta_n)] = \begin{cases} -1, \text{ if } x_{11}^{(n)} - \alpha_n J'_{x_{11}}(\theta_n) < -1; \\ x_{11}^{(n)} - \alpha_n J'_{x_{11}}(\theta_n), \text{ if } -1 \le x_{11}^{(n)} - \alpha_n J'_{x_{11}}(\theta_n) \le 1; \\ -2, \text{ if } x_{11}^{(n)} - \alpha_n J'_{x_{11}}(\theta_n) > -2; \end{cases}$$

$$P_{S_{21}}[x_{21}^{(n)} - \alpha_n J'_{x_{21}}(\theta_n)] = \begin{cases} -2, \text{ if } x_{21}^{(n)} - \alpha_n J'_{x_{21}}(\theta_n) < -2; \\ x_{21}^{(n)} - \alpha_n J'_{x_{21}}(\theta_n), \text{ if } -2 \le x_{21}^{(n)} - \alpha_n J'_{x_{21}}(\theta_n) \le -1; \\ -1, \text{ if } x_{21}^{(n)} - \alpha_n J'_{x_{21}}(\theta_n) > -1. \end{cases}$$

f) limit points of minimizing sequences:

$$u_n(t) \overset{weak}{\to} u_*(t),\ v_n(\tau) \overset{weak}{\to} v_*(\tau),\ p_1(t) \overset{weak}{\to} p_{1*}(t),\ p_2(t) \overset{weak}{\to} p_{2*}(t),\ t \in I_1,$$

$$x_{10}^{(n)} \to x_{10}^*,\ x_{20}^{(n)} \to x_{20}^*,\ x_{11}^{(n)} \to x_{11}^*,\ x_{21}^{(n)} \to x_{21}^*.$$

g) solvability of the controllability problem (22), (23):

1) if $J(u_*,\ v_*, p_{1*}, p_{2*}, x_{10}^*, x_{20}^*, x_{11}^*, x_{21}^*) = 0$, the solution of problem (21)–(23) is a function

$$x_*(t) = z(t, p_*) + \lambda(t, x_{10}^*, x_{20}^*, x_{11}^*, x_{21}^*) + N_2(t) z(t_1, p_*),\ t \in I_1;$$

2) if $J(u_*,\ v_*, p_{1*}, p_{2*}, x_{10}^*, x_{20}^*, x_{11}^*, x_{21}^*) > 0$, then the controllability problem (22), (23) has no solution. In this case, the function $x_*(t)$, $t \in I_1$, is a given approximation of the controllability problem.

5 *Conclusion*

The main results obtained in this work are: the choice of a set of program and positional controls for the process described by a linear ordinary differential equation, in the absence of restrictions on the values of the controls, by constructing a general solution of the Fredholm integral equation of the first kind; determination of program and positional control, as well as solving problems of optimal performance in the presence of restrictions on the control values and phase and integral restrictions; reduction of the initial-boundary value problem with restrictions to a special initial-boundary value problem of the optimal control and the construction of minimizing sequences and successive narrowing of the area of admissible controls solution of the optimal performance problem.

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Шектік шарттары бар сызықтық жүйелердің басқарылуы және оңтайлы әсері

Мақалада фазалық және интегралдық шектеулер болған кезде жай дифференциалдық теңдеулердің сызықтық жүйелері үшін оңтайлы жылдамдық әсерін шешу әдісі ұсынылған, мұнда жүйенің бастапқы және соңғы күйі басқару мәнінің шектеулілігін ескере отырып, берілген дөңес тұйық жиындардың элементтері болып табылады. Ұсынылған жұмыс Л.С. Понтрягин мен оның шәкірттерінің оңтайлы процестерінің математикалық теориясына, сонымен бірге Р.Е. Кальманның динамикалық жүйелерін басқару теориясына жатады. Фазалық және интегралдық шектеулер, сондай-ақ басқару шектеулері болған кезде берілген жиындардың шектік шарттары бар сызықтық жүйелер үшін оңтайлы жылдамдық әсері зерттелді. Шектік есептің теориясы құрылуы және оны шешу әдісі шешімділікті зерттеу, бірінші типтегі Фредгольм интегралдық теңдеуінің жалпы шешімін құру негізінде жасалды. Негізгі нәтижелер: жүйенің траекториясын кез келген бастапқы күйден кез келген қажетті соңғы күйге ауыса алатын әрбір элементті барлық басқару жиындарынан бөліп алу; алынған басқарудың қажетті және жеткілікті шарттарының бар болуы; шектеулері бар оңтайлы жылдамдық әсерінің мәселесін шешудің алгоритмі.

Кілт сөздер: оңтайлы тиімділік, толығымен шектеу, функционалды градиент, интегралдық теңдеу.

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Управляемость и оптимальное быстродействие линейных систем с граничными условиями

В статье предложен метод решения задачи оптимальной скорости для линейных систем обыкновенных дифференциальных уравнений при наличии фазовых и интегральных ограничений, когда начальное и конечное состояния системы являются элементами заданных выпуклых замкнутых множеств с учетом ограничения контрольного значения. Представленная работа относится к математической теории оптимальных процессов Л.С. Понтрягина и его учеников и теории управляемости динамических систем Р.Е. Кальмана. Исследована задача оптимальной скорости для линейных систем с граничными условиями из заданных множеств, близких к наличию фазовых и интегральных ограничений, а также ограничения по управляющему значению. Создана теория граничной задачи, и разработан метод ее решения на основе изучения разрешимости и построения общего решения интегрального уравнения Фредгольма первого рода. Основными результатами являются распределение всех наборов элементов управления, каждый субъект которых переводит траекторию системы из любого начального состояния в любое конечное состояние; сведение начальной граничной точки к специальной исходной задаче оптимального управления; построение системы алгоритмов гамма-алгоритма учения о выводе задач и рациональном выполнении с ограничениями решения задачи оптимальной скорости с ограничениями.

Ключевые слова: оптимальная производительность, ограничения целостности, функциональный градиент, интегральное уравнение.

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