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M.M. Aripov[1], D. Utebaev[2,*], M.M. Kazimbetova[2], R.Sh. Yarlashov[2]

[1]*M. Ulugbek National University of Uzbekistan, Tashkent, Uzbekistan;*
[2]*Berdakh Karakalpak State University, Nukus, Uzbekistan*
(E-mail: mirsaidaripov@mail.ru, dutebaev_56@mail.ru, q.muxabbat-1511@mail.ru, rinat030391@mail.ru)

On convergence of difference schemes of high accuracy for one pseudo-parabolic Sobolev type equation

Difference schemes of the finite difference method and the finite element method of high-order accuracy in time and space are proposed and investigated for a pseudo-parabolic Sobolev type equation. The order of accuracy in space is improved in two ways using the finite difference method and the finite element method. The order of accuracy of the scheme in time is improved by a special discretization of the time variable. The corresponding a priori estimates are determined and, on their basis, the accuracy estimates of the proposed difference schemes are obtained with sufficient smoothness of the solution to the original differential problem. Algorithms for the implementation of the constructed difference schemes are proposed.

Keywords: pseudo-parabolic equation, difference schemes, finite difference method, finite element method, generalized solutions, a priori estimates, stability, convergence, accuracy.

Introduction

Applied problems of engineering and technology lead to the solution of pseudo-parabolic Sobolev type equations. By pseudo-parabolic equations, we mean all high-order equations with a first-order time derivative of the following form

$$\frac{\partial}{\partial t}\left(A(u)+B(u)\right)=0,$$

$A(u)$ and $B(u)$ are elliptic operators, generally speaking, the nonlinear ones [1]. They refer to constitutive equations. Such problems arise in many fields of modern science. For example, problems in the physics of semiconductors, plasma physics, hydrodynamics of stratified and filterable liquids, the theory of "creep" of structural elements, etc. For example, the equation of waves in thin layers of liquid on the surface of a rotating globe (Rossby waves in oceanology) has the following form [2]

$$\frac{\partial}{\partial t}\Delta_3 u+\beta\frac{\partial}{\partial x_2}u=-f(x,t),\quad (x,t)\in Q_T, \tag{1}$$

where $\Delta_3=\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\frac{\partial^2}{\partial x_3^2}$ is the three-dimensional Laplace operator, β is constant, and the equation of pseudo-parabolic type has the following form [1]:

$$\left(\Delta_3 u-u\right)_t+\Delta_3 u+\beta u=-f(x,t),\quad (x,t)\in Q_T. \tag{2}$$

This equation describes the filtration process in a fractured porous fluid. The equation of moisture transfer in soil can be added to these equations [3]

$$u_t=Lu+f(x,t),\quad (x,t)\in Q_T, \tag{3}$$

*Corresponding author.
E-mail: dutebaev_56@mail.ru

where $Lu = \sum\limits_{\alpha=1}^{p} L_\alpha u, L_\alpha u = \frac{\partial}{\partial x_\alpha}\left(k_\alpha(x)\frac{\partial u}{\partial x_\alpha}\right) + \frac{\partial}{\partial t}\frac{\partial}{\partial x_\alpha}\left(k_\alpha(x)\frac{\partial u}{\partial x_\alpha}\right)$. Mathematical models of nonstationary processes in anuniaxial ferroelectric semiconductor lead to initial-boundary value problems for pseudo-parabolic equations of the following form [3]:

$$\frac{\partial}{\partial t}\left(\Delta_3 u - \gamma_1 u\right) + \alpha_1\left(\Delta_2 u - \gamma_1 u\right) + \beta_1 \frac{\partial^2 u}{\partial x_3^2} = -f(x,t), \ \ (x,t) \in Q_T. \tag{4}$$

Here $\Delta_2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$ is the two-dimensional Laplace operator, $\gamma_1 = 1/r_d^2$, $\beta_1 = 4\pi\alpha_2 + \alpha_1$, $\alpha_l > 0$ is constant $(l = 1,2)$, $r_d = \sqrt{T^2/(4\pi e^2 n_0)}$ is the Debye screening effect (Debye radius), e is the absolute value of the electron charge, n_0 is the unperturbed particle density, $Q_T = \{(x,t) : \ x \in \Omega, \ t \in (0,T]\}$, $\Omega = \{x = (x_1, \, x_2, x_3) : 0 < x_k < l_k, \ k = 1,2,3\}$.

The above equations are supplemented with initial and various boundary conditions, for example, local ones - classical boundary conditions and nonlocal ones, where, instead of classical boundary conditions, a certain relationship is specified between the values of the sought-for function on the boundary of the domain and inside it. General questions of unique solvability and analytic properties of such problems were studied in [1–6].

Recently, more attention has been paid to numerical methods for solving the above equations. In particular, in [1, 2], problems of type (1)–(4) were reduced by some transformation to two equations (one contains differentials in time, the other contains differentials in space) and then these equations were solved by the finite difference method using quasi-uniform grids. Difference schemes built on quasi-uniform grids have the second-order of accuracy in time and space variables, with sufficient smoothness of the solution to the original differential problem. Similar problems were studied in [7–10], where high-order Sobolev type equations with a second-order time derivative were considered. High-order accurate schemes of the finite element method were constructed and investigated with minimal requirements for the smoothness of the solution to the original differential problem. Difference schemes for an equation with nonlocal boundary conditions were studied in [11–15], where difference schemes of the first and second orders of accuracy were investigated.

The knowledge of the laws and features of non-stationary processes plays a primary role in the development and improvement of technological processes, technical installations and devices in a number of industries; this determines the relevance of research in the above areas. This implies the need to construct and search for numerical methods of high accuracy (more than the second accuracy) for various non-stationary initial-boundary value problems, including pseudo-parabolic equations. However, numerical methods have their limitations in terms of stability, accuracy, and economy. Therefore, the problem of determining the optimal method is an urgent issue.

In this article, we consider the construction and study of high-accuracy difference schemes of boundary value problems for equation (4). Here, the initial-boundary value problem for this equation is first approximated in spatial variables by the finite difference method and the finite element method; then, for the resulting system of ordinary differential equations, the second-order finite difference method and the fourth-order finite element method (constructed and investigated in [7]) were used.

1 *Statement of the problem*

Consider equation (4) with the following initial and boundary conditions

$$u(x,0) = u_0(x), x \in \overline{\Omega} = \Omega + \Gamma, \tag{5}$$

$$u(x,t) = \mu(t), x \in \Gamma = \partial\overline{\Omega}, t \in (0,T]. \tag{6}$$

As already mentioned, instead of boundary condition (6), one can consider any classical boundary conditions. In addition, nonlocal boundary conditions can be considered. At that, the matrices of difference schemes may turn out to be asymmetric but with the methods of linear algebra, they can be symmetrized, for example, by the bordering method [16].

Let us formulate a generalized statement of problem (4)–(6). Function $u(x,t) \in W_2^1(\Omega)$, is called a generalized solution of the problem, for each $t \in [0,T]$, it has derivative $\frac{\partial u}{\partial t} \in L_2[0,T]$ and satisfies the following relations almost everywhere on $[0,T]$:

$$a_3\left(\frac{du(t)}{dt},\vartheta\right) + a_2(u(t),\vartheta) + a_1(u(t),\vartheta) = (f(t),\vartheta), \;\; u(0) = u_0, \forall\vartheta(x) \in H, \tag{7}$$

where

$$a_3(u,\vartheta) = -\iint\limits_{\Omega}\left(\sum_{k=1}^{3} u_{x_k}\vartheta_{x_k} + \gamma_1 u\vartheta\right)dx, \;\; a_2(u,\vartheta) = -\alpha_1\iint\limits_{\Omega}\left(\sum_{k=1}^{2} u_{x_k}\vartheta_{x_k} + \gamma_1 u\vartheta\right)dx,$$

$$a_1(u,\vartheta) = -\beta_1\iint\limits_{\Omega} u_{x_3}\vartheta_{x_3}dx,$$

$u = u(t)$ is the function of abstract argument $t \in [0,T]$ with values in H. Here $\overset{\circ}{W}{}_2^1(\Omega)$ is the Sobolev space vanishing at the boundaries, where scalar product and norm are defined as follows:

$$(u(x),\vartheta(x)) = \iint\limits_{\Omega}\left(u\vartheta + \sum_{m=1}^{3}\frac{\partial u}{\partial x_m}\cdot\frac{\partial\vartheta}{\partial x_m}\right)dx,$$

$$\|u(x_1,x_2,x_3)\|^2_{W_2^1(\Omega)} = \iint\limits_{\Omega}\left(u^2 + \sum_{m=1}^{3}\left(\frac{\partial u}{\partial x_m}\right)^2\right)dx.$$

It's obvious that

$$c_3\|u\|_1^2 \le a_3(u,u) \le C_3\|u\|_1^2, \;\; c_2\|u\|_1^2 \le a_2(u,u) \le C_2\|u\|_1^2, \;\; 0 \le a_1(u,u) \le C_1\|u\|_1^2,$$

where c_2, c_3, C_1, C_2, C_3 are the positive constants. Constant c_1 depends on β_1, c_2 depends on α_1, γ_1, and c_3 depends on γ_1.

The existence and uniqueness of the solution to this problem were studied in [2].

2 *Discretization in space*

Let us construct the subspace $H_h \subset H$ that approximates H. Consider the following two cases.

The first case corresponds to the approximation of equation (4) in spatial variables by the method of finite differences. Let us introduce a grid uniform in each direction $\overline{\omega}_h = \overline{\omega}_{h_1} \times \overline{\omega}_{h_2} \times \overline{\omega}_{h_3}$, in $\overline{\Omega}$ where $\overline{\omega}_{h_m} = \left\{x_m = i_m h_m, i_m = \overline{0,N}_m, \; h_m = l_m/N_m\right\}$, $m = 1,2,3$. Here $\overline{\omega}_h = \omega_h + \gamma_h$. We define the subspace $H_h = \overset{\circ}{W}{}_2^1(\omega_h)$, the space of grid functions $v(x_1,x_2,x_3)$ with norm $\|v\|_1^2 = \sqrt{\sum\limits_{i_1}^{N_1}\sum\limits_{i_2}^{N_2}\sum\limits_{i_3}^{N_3} h_1h_2h_3\left[(v_{\overline{x}_1})^2 + (v_{\overline{x}_2})^2 + (v_{\overline{x}_3})^2\right]} \le M$, where the constant M does not depend on h_1,h_2,h_3. Here $v = v(i_1h_1, i_2h_2, i_3h_3)$.

$$v_{\overline{x}_1} = \left[v(i_1h_1,i_2h_2,i_3h_3) - v((i_1-1)h_1,i_2h_2,i_3h_3)\right]/h_1,$$

$$v_{\overline{x}_2} = \left[v(i_1h_1,i_2h_2,i_3h_3) - v(i_1h_1,(i_2-1)h_2,i_3h_3)\right]/h_2,$$

$$v_{\overline{x}_3} = \left[v(i_1h_1, i_2h_2, i_3h_3) - v(i_1h_1, i_2h_2, (i_3-1)h_3) \right] / h_3,$$

$\overset{\circ}{W}{}_2^1(\omega_h)$ is the space of grid functions that vanish at the boundaries.

Approximating the expressions for $a_m(u, \vartheta)$ on the grid by the corresponding quadrature formulas $a_m^h(u_h, v_h) = \sum\limits_{i_1=1}^{N_1} \sum\limits_{i_2=1}^{N_2} \sum\limits_{i_3=1}^{N_3} h_1h_2h_3 u_{h\overline{x}_m} v_{h\overline{x}_m}$, we proceed from (7) to the definition of an approximate grid solution:

$$a_3^h\left(\frac{du_h(t)}{dt}, \vartheta\right) + a_2^h(u_h(t), \vartheta) + a_1^h(u_h(t), \vartheta) = (f_h(t), \vartheta), \quad \forall \vartheta(x) \in H_h, \tag{8}$$

$$u_h(0) = u_{0,h}. \tag{9}$$

Relations (8), (9) correspond to the following Cauchy problem for the function $u_h(t)$:

$$D\frac{du_h(t)}{dt} + Au_h(t) = f_h(t), \quad u_h(0) = u_{0,h}, \tag{10}$$

where

$$D = (\Lambda_1 + \Lambda_2 + \Lambda_3) + \gamma_1 E, \ A = \alpha_1(\Lambda_1 + \Lambda_2 + \gamma_1 E) + \beta_1 \Lambda_3, \tag{11}$$

$\Lambda_m y = -y_{x_m\overline{x}_m}$, $m = 1, 2, 3$, y is the value of the function at a fixed node, $x = (i_1h_1, i_2h_2, i_3h_3)$,

$$y_{x_1\overline{x}_1} = (y((i_1+1)h_1, i_2h_2, i_3h_3) - 2y(i_1h_1, i_2h_2, i_3h_3) + y((i_1-1)h_1, i_2h_2, i_3h_3))/\, h_1^2,$$

$$y_{x_2\overline{x}_2} = (y(i_1h_1, (i_2+1)h_2, i_3h_3) - 2y(i_1h_1, i_2h_2, i_3h_3) + y(i_1h_1, (i_2-1)h_2, i_3h_3))/\, h_2^2,$$

$$y_{x_3\overline{x}_3} = (y(i_1h_1, i_2h_2, (i_3+1)h_3) - 2y((i_1h_1, i_2h_2, i_3h_3)) + y(i_1h_1, i_2h_2, (i_3-1)h_3))/\, h_3^2.$$

Here $u_{h,0} = P_h u_0(x)$ is the interpolant of the initial condition, P_h is the projection operator $P_h : H \to H_h$ and $f_h(t) = P_h f(x, t)$.

Difference operators D and A approximate differential operators $\Delta_3 u - \gamma_1 u$ and $\alpha_1 (\Delta_2 u - \gamma_1 u) + \beta_1\, \partial^2 u / \partial x_3^2$ with second-order approximation errors.

The second case corresponds to the approximation of equation (4) in spatial variables by the finite element method. Let $H_h \subset H$ be the set of elements of the form $\vartheta_h = \sum\limits_{m=1}^{M} a_m \Phi_m(x)$. Here $\{\Phi_m = \Phi_m(x)\}_{m=1}^M$ is the basis of piecewise polynomial functions that are a polynomial of p degree on each finite element [17]. Let us give an example of a basis based on third degree polynomials. To do this, we introduce a partition of the domain Ω into $N_1 \times N_2 \times N_3$ parallelepipeds

$$\Omega_{ijk} = \{(i-h)h_1 \le x_1 \le ih_1, \ (j-1)h_2 \le x_2 \le jh_2, \ (k-1)h_3 \le x_3 \le kh_3 \},$$

$$i = \overline{1, N_1}, \ j = \overline{1, N_2}, \ k = \overline{1, N_3}, \ h_m = l_m/N_m \,, \ m = 1, 2, 3.$$

Let us choose the following system of basis functions:

$$\varphi_{ijk}(x_1, x_2, x_3) = \varphi_i(x_1)\varphi_j(x_2)\varphi_k(x_3), \ \ i = \overline{1, N_1 - 1}, \ \ j = \overline{1, N_2 - 1}, \ \ k = \overline{1, N_3 - 1},$$

where $\varphi_l(x)$ is the basis function built on the basis of the B_3 - spline [7]. In this case $p = 3$. Then the approximate solution can be represented as a bicubic spline:

$$\vartheta_h(x_1, x_2, x_3, t) = \sum_{k=1}^{N} a_k(t)\varphi_k(x_1, x_2, x_3), \tag{12}$$

where $\varphi_k(x_1, x_2, x_3) = \varphi_i(x_1)\varphi_j(x_2)\varphi_k(x_3)$, $i = \overline{1, N_1 - 1}$, $j = \overline{1, N_2 - 1}$, $k = \overline{1, N_3 - 1}$, and $N = (N_1 - 1)(N_2 - 1)(N_3 - 1)$.

The stiffness matrices corresponding to operators D, A are calculated as follows:

$$D = \{a_3(\varphi_l, \varphi_m)\}_{l,m=1}^{M}, \; A = \{a_2(\varphi_l, \varphi_m)\}_{l,m=1}^{M} + \{a_1(\varphi_l, \varphi_m)\}_{l,m=1}^{M}.$$

When choosing a polynomial of a degree no less than three, at each finite element in spatial variables, we have the third order of accuracy in spatial steps.

In both cases

$$D^* = D > 0, \; A^* = A > 0.$$

In what follows, for simplicity of notation, in (10), $u \in H_h$ is used instead of u_h, i.e., problem (10) is written in the following form

$$D\dot{u} + Au = f, \; u(0) = u_0, \tag{13}$$

where $\dot{u} = du/dt$.

3 Time discretization

Here we also consider two cases of approximation. Let discrete function y approximates a continuous function u.

The first case. Let us introduce grid $\omega_\tau = \{t_n = n\tau, \; n = 1, 2, ..., \; \tau > 0\}$ in time t. Then we approximate problem (13) by the following difference scheme

$$Dy_t + Ay^{(\sigma)} = \varphi, \; y^0 = u_{0,h}, \; y^n \in H_h, \tag{14}$$

where $y_t = (\hat{y} - y)/\tau$, $y = y^n = y(t_n)$, $\hat{y} = y^{n+1} = y(t_n + \tau)$, $y^{(\sigma)} = \sigma\,\hat{y} + (1 - \sigma)y$. Here D and A are defined according to (11), and σ is some arbitrary real parameter $\varphi = \overline{f} = f\,(x, t_n + \tau/2\,)$.

It is known from the theory of difference schemes [18] that the approximation error for scheme (14) is:

$$\psi = O\left(\tau^2 + |h|^2\right) \text{ for } \sigma = 0.5, \; \psi = O\left(\tau + |h|^2\right) \text{ for } \sigma \neq 0.5, \; |h|^2 = h_1^2 + h_2^2 + h_3^2.$$

The second case consists in discretizing problem (13) by the finite element method connecting the values $\dot{y}^{n+1}$, $\dot{y}^n$, y^{n+1}, y^n that approximate $\frac{du_h}{dt}(t_n + \tau)$, $\frac{du_h}{dt}(t_n)$, $u_h(t_n + \tau)$, $u_h(t_n)$, respectively. Such a scheme was constructed in [7] and it has the form:

$$Dy_t - \gamma A\dot{y}_t + Ay^{(0.5)} = \varphi_1, \; \gamma D\dot{y}_t + \alpha Ay_t - \beta A\dot{y}^{(0.5)} = \varphi_2, \tag{15}$$

where $\varphi_1 = \frac{1}{\tau}\int\limits_{t_n}^{t_{+n1}} f(t)dt$, $\varphi_2 = \frac{1}{\gamma\tau}\int\limits_{t_n}^{t_{n+1}} f(t)(s_1\vartheta_2^{(1)} + s_2\vartheta_2^{(2)})dt$, $s_1 = 15\gamma - 35\alpha/3$, $s_1 = 140\gamma - 350\alpha/3$, $\vartheta_1^{(1)} = 1/2$, $\vartheta_2^{(3)} = \tau\xi(1 - \xi)\left(\xi - 1/2\right)$, $\xi = (t - t_n)/\tau$.

The initial conditions for (15) are specified as follows: in addition to the natural condition $y^0 = u_0$, it is necessary to specify $\dot{y}^0$. For this, from the system of equations (13), at $t = 0$, we determine $\dot{u}_0 = D^{-1}(f^0 - \mathrm{A}u_0)$ and set $\dot{y}^0 = \dot{u}_0$, therefore, the initial conditions for (15) have the form:

$$y^0 = u_0, \; \dot{y}^0 = D^{-1}(f^0 - \alpha u_0). \tag{16}$$

From the calculated values of $\dot{y}^{n+1}$, $\dot{y}^n$, y^{n+1}, y^n, it is possible to restore the approximation to $u_h(t)$ for any $t \in [t_n, t_{n+1}]$, $n = 0, 1, ...$ by the following formula:

$$y(t) = y^n\varphi_{00}^n(t) + \dot{y}^n\varphi_{10}^n(t) + y^{n+1}\varphi_{01}^n(t) + \dot{y}^{n+1}\varphi_{11}^n(t).$$

Here $\varphi_{00}^n(t) = 2\xi^3 - 3\xi^2 + 1$, $\varphi_{01}^n(t) = 3\xi^2 - 2\xi^3$, $\varphi_{10}^n(t) = \tau(\xi^3 - 2\xi^2 + \xi)$, $\varphi_{11}^n(t) = \tau(\xi^3 - \xi^2)$, $\xi = \frac{t-t_n}{\tau}$.

Combining the approximation in space and time, we consider four methods for solving problem (4)–(6):

- *Scheme* 1^0 – difference approximation of the second order of accuracy in space (11) and time (14);
- *Scheme* 2^0 – approximation of the FEM with bicubic elements in space (12) and time (14);
- *Scheme* 3^0 – difference approximation of the second order of accuracy in space (11) and the FEM scheme in time (15), (16);
- *Scheme* 4^0 – approximation of the FEM with bicubic elements in space (12) and the FEM scheme in time (15), (16).

4 Stability and accuracy

Let us analyze the stability and accuracy of the selected schemes. It is known [18], that schemes (14), (15) are stable under the following conditions

$$D > 0, \quad A = A^* > 0, \quad D \geq \tau A/2. \tag{17}$$

Let us check the fulfillment of conditions (17). It is seen from (11) that $D = D^* > 0$, $A = A^* > 0$. The last condition (17) takes the form

$$\Lambda_1 + \Lambda_2 + \Lambda_3 + \gamma_1 E - \frac{\tau}{2}\ [\alpha_1(\Lambda_1 + \Lambda_2 + \gamma_1 E) + \beta_1 \Lambda_3] \geq 0.$$

To satisfy it, it is enough that

$$\tau \leq 2 \max\left(\frac{1}{\alpha_1}, \frac{1}{\beta_1}\right). \tag{18}$$

This condition is interesting because the time step is not related to the space step and is determined by the parameters α_1, β_1 of the problem. Thus, the following theorem holds.

Theorem 1. Under condition (18), the solution of scheme 1^0 converges to a sufficiently smooth solution of problem (4)-(6) and the following accuracy estimate holds

$$\|y(t) - u(t)\|_A + \|y_t(t) - u_t(t)\|_D \leq M(\tau^{m_1} + |h|^{m_2}), \tag{19}$$

where $\|\vartheta\|_D = \sqrt{(D\vartheta, \vartheta)} = \|\vartheta\|_{W_2^1(\omega_h)}$, $\|\vartheta\|_A = \sqrt{(A\vartheta, \vartheta)} = \|\vartheta_{x_1}\|_{L_2(\omega_h)}$ are the norms in the space of grid functions H_h, $y_t = (y^{n+1} - y^n)/\tau$, $m_1 = 1,\ m_2 = 2$ for $\sigma = 0.5$ and $m_1 = 2,\ m_2 = 2$ for $\sigma \neq 0.5$.

Let us formulate a result on the stability and accuracy of scheme 2^0.

Theorem 2. Under condition (18), the solution of scheme 2^0 converges to a sufficiently smooth solution of problem (4)-(6) and the following accuracy estimate (19) holds $m_1 = 1,\ m_2 = 3$ for $\sigma = 0.5$ and $m_1 = 2,\ m_2 = 3$ for $\sigma \neq 0.5$.

Now let us investigate the accuracy of scheme (15), (16). Let $z^n = y^n - u^n$, $\dot{z}^n = \dot{y}^n - \dot{u}^n$, where $u^n = u(t_n)$. Then scheme (15), (16) satisfies the relations

$$Dz_t - \gamma A\dot{z}_t + Az^{(0.5)} = \psi_1,\ \gamma D\dot{z}_t + \alpha Az_t - \beta A\dot{z}^{(0.5)} = \psi_2,\ z^0 =\ 0,\ \dot{z}^0 =\ 0,$$

the approximation error is

$$\psi_1 = \frac{\tau^4}{3840} D\overset{V}{u} + \frac{\tau^4}{720} A\overset{IV}{u} + O(\tau^5),$$

$$\psi_2 = (\alpha + \beta - \gamma)\ddot{\overline{A}} + \frac{\tau^2}{24}\left[(\alpha + 3\beta - \gamma)\dddot{\overline{A}} - (3\gamma - 2\alpha)\dddot{\overline{f}}\right] + O\left(\tau^4\right).$$

Hence, if the following conditions are met

$$\alpha+\beta=\gamma,\ \ \alpha,\ \ \beta,\ \gamma=\left(\tau^2\right), \tag{20}$$

then $\psi_1=\psi_2=O\left(\tau^4\right)$.

To prove the convergence of the two-layer vector scheme (15), (16), we reduce it to a three-layer scheme separately for y and its derivative $\dot{y}$. When operators D and A are permutable, i.e., $DA = AD$, the following estimate is obtained [7]

$$\left\|u_h(t)-u(t)\right\|_A+\left\|u_{ht}(t)-u_t(t)\right\|_D\leq M\tau^4.$$

Let us $w=D^{1/2}y,\ \dot{w}=D^{1/2}\dot{y}$ instead of $y,\dot{y}$. Note that $\left(D^{1/2}\right)^*=D^{1/2}>0$ and the inverse operator $D^{-1/2}=\left(D^{-1/2}\right)^*>0$ exists.

After obvious transformations from (15) we obtain

$$\begin{gathered}\widetilde{D}w_t-\gamma\widetilde{A}\dot{w}_t+\widetilde{A}w^{(0.5)}=\widetilde{\varphi}_1,\ \ \gamma\widetilde{D}\dot{w}_t+\alpha\widetilde{A}w_t-\beta\widetilde{A}\dot{w}^{(0.5)}=\widetilde{\varphi}_2,\\ w^0=D^{1/2}u_0,\ \ \dot{w}^0=D^{1/2}(f^0-Au_0),\end{gathered} \tag{21}$$

where $\widetilde{\varphi}_1=D^{-1/2}\varphi_1,\ \widetilde{\varphi}_2=D^{-1/2}\varphi_2,\ \widetilde{D}=E,\ \ \widetilde{A}=D^{-1/2}AD^{-1/2}$. It is clear that $\widetilde{D}=\widetilde{D}^*>0,\ \ \widetilde{A}=\widetilde{A}^*>0$ and $\widetilde{D}\ \widetilde{A}=\widetilde{A}\widetilde{D}$. Consequently, there is no need for the permutability of operators D and A. Then, eliminating from (21) first $\dot{w}$, and then $\hat{\dot{w}}$ and adding them, taking into account (16), we obtain the following three-layer difference scheme

$$B_1w^{n+1}+B_2w^n+B_3w^{n-1}=\tau F_n,\ \ n=1,2,...,\ \text{where } w^0,w^1 \text{ are given}, \tag{22}$$

$$B_1=\gamma\widetilde{D}^2+\frac{\tau}{2}(\gamma-\beta)\widetilde{A}\widetilde{D}-\left(\frac{\tau^2}{4}\beta-\alpha\gamma\right)\widetilde{A}^2,$$

$$B_2=2\gamma\widetilde{D}^2+\left(\frac{\tau^2}{2}\beta+2\alpha\gamma\right)\widetilde{A}^2,$$

$$B_3=\gamma\widetilde{D}^2-\frac{\tau}{2}(\gamma-\beta)\widetilde{A}\widetilde{D}-\left(\frac{\tau^2}{4}\beta-\alpha\gamma\right)\widetilde{A}^2,$$

$$F_n=\left(\gamma\widetilde{D}-\frac{\tau}{2}\beta\widetilde{A}\right)\widetilde{\varphi}_1^n+\gamma\widetilde{A}\widetilde{\varphi}_2^n-\left(\gamma\widetilde{D}+\frac{\tau}{2}\beta\widetilde{A}\right)\widetilde{\varphi}_1^{n-1}-\gamma\widetilde{A}\widetilde{\varphi}_2^{n-1}.$$

Equation (22) can be rewritten in the canonical form:

$$\overline{B}w_{\overset{\circ}{t}}+\tau^2\overline{R}w_{\bar{t}t}+\overline{A}w=\overline{F},\ \text{where } y^0,y^1 \text{ are given}, \tag{23}$$

and operators in (23) have the following form:

$$\begin{aligned}&\overline{B}=\tau(B_1-B_3)=\tau(\gamma-\beta)\widetilde{A}\widetilde{D}=\tau\alpha\widetilde{A}\widetilde{D},\\ &\overline{R}=\frac{1}{2}\left(B_1+B_3\right)=\gamma\widetilde{D}^2-\left(\frac{\tau^2}{4}\beta-\alpha\gamma\right)\widetilde{A}^2,\\ &\overline{A}=B_1+B_2+B_3=4\gamma(\widetilde{D}^2+\alpha\widetilde{A}^2),\\ &\overline{F}=\gamma\widetilde{D}\widetilde{\varphi}_{1,\bar{t}}^n-\beta\widetilde{A}\frac{\widetilde{\varphi}_1^n+\widetilde{\varphi}_1^{n-1}}{2}+\gamma\widetilde{A}\widetilde{\varphi}_{2,\bar{t}}^n.\end{aligned} \tag{24}$$

Hence it is clear that, $\overline{B}^*=\overline{B}>0$, $\overline{A}^*=\overline{A}>0$, $\overline{R}^*=\overline{R}$.

Now, based on the results of the theory of difference schemes [18], we check the fulfillment of the stability condition for the three-layer difference scheme (23)

$$\overline{R} \geq \frac{1}{4}\overline{A}. \tag{25}$$

A straightforward computation ensures that (25) holds if the following conditions are met

$$\tau \leq 2/\sqrt{\beta}. \tag{26}$$

Condition (26) always holds if (20) is satisfied. Then, based on the results of the theory of difference schemes [18, 19], we establish the validity of the following theorem.

Theorem 3. Under condition (26), scheme (23) is stable in $H_{\overline{A}}$ by the initial data and by the right-hand side, and its solution satisfies the following estimate

$$\|w^n\|^2_{\overline{A}} \leq \left\|w^0\right\|^2_{\overline{A}} + \frac{1}{2}\sum_{k=0}^{n} \tau \left\|\overline{F}_k\right\|^2_{\overline{B}^{-1}}. \tag{27}$$

From inequality (27), returning to the variable y and taking into account the definition of operators $\overline{A}$, $\overline{B}^{-1}$ and $\overline{F}$ in (24), we obtain the estimate

$$\begin{gathered} \|y^n\|_{\widetilde{A}^2} \leq \left\|y^0\right\|_{\widetilde{A}^2} + M\max_k \left(\frac{\gamma}{\sqrt{\alpha\beta}}\left\|\widetilde{\varphi}^k_{1,\bar{t}}\right\|_{\widetilde{A}^{-1}\widetilde{D}} \right.\\ \left. -\frac{\beta}{\alpha}\left\|\frac{\widetilde{\varphi}^k_1 + \widetilde{\varphi}^{k-1}_1}{2}\right\|_{\widetilde{A}\widetilde{D}^{-1}} + \frac{\gamma}{\sqrt{\alpha\beta}}\left\|\widetilde{\varphi}^k_{2,\bar{t}}\right\|_{\widetilde{A}\widetilde{D}^{-1}}\right), \end{gathered} \tag{28}$$

where M is a constant independent of τ and h.

Let us apply the obtained estimate to assess the error of scheme (23). The $z = y - u$ error satisfies the equation $\overline{B}z_{\overset{\circ}{t}} + \tau^2\overline{R}z_{\bar{t}t} + \overline{A}z = \psi$, where $\psi = \overline{F} - (\overline{B}u_{\overset{\circ}{t}} + \tau^2\overline{R}u_{\bar{t}t} + \overline{A}u)$. Hence, the following estimate

$$\begin{gathered} \|z_n\|_{\widetilde{A}^2} \leq \|z_0\|_{\widetilde{A}^2} + M\max_k \left(\frac{\gamma}{\sqrt{\alpha\beta}}\left\|\psi^k_{1,\bar{t}}\right\|_{\widetilde{A}^{-1}\widetilde{D}} \right.\\ \left. +\frac{\beta}{\alpha}\left\|\frac{\psi^k_1 + \psi^{k-1}_1}{2}\right\|_{\widetilde{A}\widetilde{D}^{-1}} + \frac{\gamma}{\sqrt{\alpha\beta}}\left\|\psi^k_{2,\bar{t}}\right\|_{\widetilde{A}\widetilde{D}^{-1}}\right) \end{gathered}$$

is valid for z.

Here $\psi_1,\ \psi_2$ are the errors in the approximation of the vector scheme (15).

Eliminating z and $\hat{z}$, from relation (21), we can arrive at an equation of the form (23) for $\dot{z} = \dot{y} - \dot{u}$. Then we obtain $\|z^n\|_{\widetilde{A}^2} = \|u^n - y^n\|_{\widetilde{A}^2} = O(\tau^4)$ and $\|\dot{z}^n\|_{\widetilde{A}^2} = \|\dot{u}^n - \dot{y}^n\|_{\widetilde{A}^2} = O(\tau^4)$ at the point of time $t_n,\ n = 1, 2...$. Therefore, based on estimate (28), under the conditions of (20), we obtain the convergence of scheme (15) to the solution of the original problem $u(t_n) \in C^6[0, T]$ with the fourth order, i.e.,

$$\|y(t_n) - u(t_n)\|_{\widetilde{A}^2} + \|\dot{y}(t_n) - \dot{u}(t_n)\|_{\widetilde{A}^2} \leq M\tau^4.$$

Therefore, for the error $\|y(t) - u(t)\|$, $\forall t \in [t_n, t_{n+1}],\ n = 0, 1, ...$ the following result holds.

Theorem 4. Let the stability conditions (26) be satisfied. Then, if $u(x, t) \in C^6[0, T]$, then scheme (15), (16) converges to the solution of problem (13) and the following accuracy estimates are valid for its solution:

$$\|y(t) - u(t)\|_{\widetilde{A}^2} \leq M\tau^4,\ \ \|\dot{y}(t) - \dot{u}(t)\|_{\widetilde{A}^2} \leq M\tau^4,\ \ \forall t \in [0, T].$$

The second estimate of Theorem 4 is obtained using the results of Theorem 3 for the derivative $\dot{z}$.

To estimate the accuracy of schemes 3^0 and 4^0, it is necessary to obtain an estimate of the error $z = u_h - u$. Using the technique of such an estimate in the theory of difference schemes [18] of the theory of the finite element method [17], we formulate the following results.

Theorem 5. Under condition (26), the solution to scheme 3^0 converges to a sufficiently smooth solution of problem (4)-(6) and the following accuracy estimate holds

$$\|y(t) - u(t)\|_1 \le M(\tau^4 + h^2).$$

Theorem 6. Under condition (26), the solution of scheme 4^0 converges to a sufficiently smooth solution of problem (4)-(6) and the following accuracy estimate holds

$$\|y(t) - u(t)\|_1 + \|\dot{y}(t) - \dot{u}(t)\|_1 \le M(\tau^4 + h^3).$$

5 *Schemes with skew-symmetric operator*

Let us investigate the stability by the initial data and the right-hand side of scheme (15), (16) with operators $D^* = D > 0$, $A^* = -A$, and write it in the canonical form

$$\widetilde{B}Y_t + \widetilde{A}Y = 0; \quad Y = (y, \dot{y}), \tag{29}$$

where

$$\widetilde{B} = \begin{pmatrix} D + \frac{\tau}{2}A & -\gamma A \\ \alpha A & \gamma D - \frac{\tau}{2}\beta A \end{pmatrix}, \quad \widetilde{A} = \begin{pmatrix} A & 0 \\ 0 & -\beta A \end{pmatrix}.$$

To prove stability by the initial data of scheme (29), we use the results of [20]. To do this, we take $\alpha = \tau^2/12$ and represent the operator $\widetilde{B}$ in the form $\widetilde{B} = \widetilde{D} + \widetilde{A}C$, where

$$\widetilde{D} = \begin{pmatrix} D & -\gamma A \\ \alpha A & \gamma D \end{pmatrix}, \quad C = \begin{pmatrix} \frac{\tau}{2} & 0 \\ 0 & \frac{\tau}{2} \end{pmatrix}.$$

Then, for the stability of scheme (29), on the basis of the results of the theory of difference schemes [20], it remains to check the fulfillment of condition $C^*\widetilde{D} + \widetilde{D}C \ge \widetilde{D}$. This condition is met if $\alpha > 0$, $\gamma > 0$. Thus, taking into account (18), we arrive at the following statement.

Theorem 7. If the conditions $\alpha > 0$, $\beta > 0$, $\gamma > 0$, are satisfied, then scheme (29) is stable by the initial data and the right-hand side in $H_{\widetilde{D}}$ and the following estimate

$$\left\|Y^{n+1}\right\|_{\widetilde{D}} \le \left\|Y^0\right\|_{\widetilde{D}} + \sum_{k=0}^{k} \tau \left\|\Phi_k\right\|_{\widetilde{D}}$$

is true.

Based on this estimate, likewise in the previous sections, we obtain the accuracy of scheme (15), (16) with the skew-symmetric operator A, i.e. the results of Theorems 1, 2, 5 and 6 are also valid for scheme (29).

6 *Algorithm for the implementation of the scheme* (21)

Consider one of the possible algorithms for implementing the scheme (21). We rewrite it as

$$m_{11}\hat{w} + m_{12}\overset{\wedge}{\dot{w}} = \phi_1, \; m_{21}\hat{w} + m_{22}\overset{\wedge}{\dot{w}} = \phi_2, \tag{30}$$

where

$$m_{11}=\widetilde{D}+\frac{\tau}{2}\widetilde{A},\quad m_{12}=-\gamma\widetilde{A},\quad m_{21}=\alpha\widetilde{A},\quad m_{22}=\gamma\widetilde{D}-\frac{\tau}{2}\beta\widetilde{A},$$

$$\phi_1=\tau\widetilde{\varphi}_1+\left(\widetilde{D}-\frac{\tau}{2}\widetilde{A}\right)w-\gamma\widetilde{A}\dot{w},\quad \phi_2=\tau\widetilde{\varphi}_2+\alpha\widetilde{A}w+\left(\gamma\widetilde{D}+\frac{\tau}{2}\beta\widetilde{A}\right)\dot{w}.$$

To calculate the integrals $\widetilde{\varphi}_1$ and $\widetilde{\varphi}_2$, we can use the Simpson quadrature formula.

Taking into account the permutable operators $\widetilde{A}$ and $\widetilde{D}$, we exclude $\overset{\wedge}{\dot{w}}$ from equation (30):

$$C\hat{w}=F. \tag{31}$$

Here $C=\gamma\widetilde{D}^2+\frac{\tau}{2}(\gamma-\beta)\widetilde{A}\widetilde{D}-\left(\frac{\tau^2}{4}\beta-\alpha\gamma\right)\widetilde{A}^2,\quad F=m_{22}\phi_1-m_{12}\phi_2.$

Equation (31) can be solved either directly by inverting the operator C, or by factoring it

$$C=\gamma\, C_1C_2=\gamma\left[\widetilde{D}^2-(x_1+x_2)\tau\widetilde{A}\widetilde{D}+x_1x_2\tau^2\widetilde{A}^2\right],\ C_k=\left(\widetilde{D}-x_k\tau\widetilde{A}\right),\ k=1,2.$$

Then, equation (31) is solved using the following algorithm:

$$\gamma_1C_1\overline{w}=F,\quad C_2\hat{w}=\overline{w}. \tag{32}$$

After determining $\hat{w}$ from (32), the solution of $\overset{\wedge}{\dot{w}}$ is calculated, for example, from the equation $\left(\gamma\widetilde{D}-\frac{\tau}{2}\beta\widetilde{A}\right)\overset{\wedge}{\dot{w}}=\phi_2-\alpha\widetilde{A}\hat{w}$.

The scheme (14) is implemented as follows:

$$(D+\sigma\tau A)y^{n+1}=\left[D-(1-\sigma)\tau A\right]y^n+\tau\varphi,\quad n=0,1,2,...,$$

$$y^0=u_{h0}.$$

Remark. It is possible to prove the stability of scheme (15), (16) with variable operators $A=A_n$, $D=D_n$, for example, in norm A_n. It is required that the operatorA_n be Lipschitz-continuous in t.

Conclusions

The methods of a high degree of accuracy for solving the first boundary value problem for a pseudo-parabolic equation of a special form are developed and investigated in this article. These methods are based on finite-difference and finite-element approximations in space and time. The stability and convergence of the constructed methods are proved, and the accuracy estimates are obtained. An algorithm for the implementation of the finite element method was developed. Other pseudo-parabolic equations given in the introduction, as well as other types of similar equations, are investigated likewise. We can study problems with other local and nonlocal boundary conditions.

The system of ordinary differential equations obtained by spatial approximation may turn out to be rigid. A separate study will be devoted to this issue and numerical modeling, where, based on the algorithm for implementing the method developed here, it will be tested on exact solutions in the form of a Fourier series and the constructed methods will be compared with other methods. In addition, on the basis of a computational experiment, the convergence rates of the method along the spatial and temporal directions will be checked, as well as visualizations, which confirm these theoretical results.

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М. М. Арипов[1], Д. Утебаев[2], М. М. Казымбетова[2], Р. Ш. Ярлашов[2]

[1]*М.ұлықбек атындағы Өзбекстан ұлттық университеті, Ташкент, Өзбекстан;*
[2]*Бердах атындағы Қарақалпақ мемлекеттік университеті, Нүкіс, Өзбекстан*

Соболев типті псевдопараболалық теңдеуі үшін жоғары дәлдіктегі айырмашылық схемаларының жинақтылығы туралы

Соболев типті псевдопараболалық теңдеуі үшін уақыт пен кеңістік бойынша жоғары дәлдіктегі ақырлы айырымдық әдісі мен ақырлы элементтер әдісінің әртүрлі айырымдық схемалары ұсынылған және зерттелген. Кеңістіктегі дәлдік тәртібін арттыру екі жолмен, ақырлы айырымдық схемасы және ақырлы элементтер схемасы мен жүзеге асырылды. Уақыт бойынша тізбектің дәлдігінің жоғары тәртібіне уақыт айнымалысын арнайы іріктеу арқылы қол жеткізілген. Тиісті априорлық бағалаулар және олардың негізінде бастапқы дифференциалдық есепті шешудің жеткілікті тегістігімен ұсынылған айырымдық схемаларының дәлдігінің бағалары алынды. Құрылған айырымдық схемаларын іске асыру алгоритмдері жүзеге асырылды.

Кілт сөздер: псевдопараболалық теңдеу, айырымдық схемалар, ақырлы айырымдар әдісі, ақырлы элементтер әдісі, априорлық бағалаулар, тұрақтылық, жинақтылық, дәлдік.

М. М. Арипов[1], Д. Утебаев[2], М. М. Казимбетова[2], Р. Ш. Ярлашов[2]

[1]*Национальный университет Узбекистана имени М. Улугбека, Ташкент, Узбекистан;*
[2]*Каракалпакский государственный университет имени Бердаха, Нукус, Узбекистан*

О сходимости разностных схем повышенной точности для одного псевдопараболического уравнения соболевского типа

Предложены и исследованы различные разностные схемы метода конечных разностей и метода конечных элементов высокого порядка точности по времени и по пространству для псевдопараболического уравнения соболевского типа. Повышение порядка точности по пространству осуществлено двумя способами: методом конечных разностей и методом конечных элементов. Высокий порядок точности схемы по времени достигнут за счет специальной дискретизации временной переменной. Получены соответствующие априорные оценки, и на их основе оценки точности предложенных разностных схем при достаточной гладкости решения исходной дифференциальной задачи. Реализованы алгоритмы выполнения построенных разностных схем.

Ключевые слова: псевдопараболическое уравнение, разностные схемы, метод конечных разностей, метод конечных элементов, обобщенные решения, априорные оценки, устойчивость, сходимость, точность.

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