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The Bessel equation on the quantum calculus

A large number of the most diverse problems related to almost all the most important branches of mathematical physics and designed to answer topical technical questions are associated with the use of Bessel functions. This paper introduces a h-difference equation analogue of the Bessel differential equation and investigates the properties of its solution, which is express using the Frobenius method by assuming a generalized power series. The authors find discrete analogue formulas for Bessel function and the h-Neumann function and these are solutions presented by a series with the h-fractional function $t_h^{(\alpha)}$. Lastly they obtain the linear dependencies between h-functions Bessel on T_a.

Keywords: Bessel function, modified Bessel function, Bessel difference equation, h-calculus, the h-derivative and h-fractional function.

1 Introduction and Preliminary

Now a days, the theory of transformation operators is a fully formed independent branch of mathematics, located at the junction of differential, integral, and integro-differential equations, functional analysis, function theory, complex analysis, the theory of special functions and fractional integro-differentiation, the theory of inverse problems and scattering problems, the theory of optimal control and dynamic systems. The special area of application of transformation operators has become the theory of differential equations with singularities in the coefficients, especially with Bessel operators.

The Bessel functions are widely used in solving problems in acoustics, radiophysics, hydrodynamics, problems of atomic and nuclear physics. There are numerous applications of Bessel functions to the theory of heat conduction and the theory of elasticity (problems of vibrations of plates, problems of the theory of shells, problems of determining the stress concentration near cracks) [1–5].

The theory of fractional h-calculus is a rapidly developing field of great interest from both a theoretical and an applied point of view. Especially we refer to [6–12] and the references in it. As for applications in various fields of mathematics, we refer to [13–20] and references in them. Let $h > 0$ and $T_a = \{a, a+h, a+2h...\}$, $\forall a \in \mathbb{R}$.

Definition 1. (see [9]) Let $f : T_a \to \mathbb{R}$. Then the h-derivative of the function $f = f(t)$ has the form and is defined as

$$D_h f\left(t\right)\left(x\right) = \frac{f(\delta_h(t)) - f(t)}{h}, t \in T_a, \tag{1}$$

where $\delta_h(t) = t + h$.

We assume $f \cdot g : T_a \to \mathbb{R}$. Then the product rule for h-derivation reads (see [9])

$$D_h\left(f\left(x\right)g(x)\right) = f(x+h)D_h g(x) + g(x)D_h f(x) \tag{2}$$

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and the h-integral (or the h-difference sum) is given by

$$\int_a^x f(t)d_h t = \sum_{k=a/h}^{x/h-1} f(kh)h, \quad x \in T_a. \tag{3}$$

Definition 2. (see [9]) Let $t, \alpha \in \mathbb{R}$. Then the h-fractional function $t_h^{(\alpha)}$ is defined as

$$t_h^{(\alpha)} = h^\alpha \frac{\Gamma\left(\frac{t}{h}+1\right)}{\Gamma\left(\frac{t}{h}+1-\alpha\right)},$$

where Γ is the gamma function of Euler, $\frac{t}{h} \geq 0$ and we use the convention that division at the pole gives zero. Notice that

$$\lim_{h\to 0} t_h^{(\alpha)} = t^\alpha.$$

Hence, from (1) we find that

$$t^{\alpha-1} = \frac{1}{\alpha} D_h \left[t_h^{(\alpha)}\right].$$

Let $t \in T_0$. Then, for $\alpha, \beta \in \mathbb{R}$,

$$t_h^{(\alpha+\beta)} = t_h^{(\alpha)}(t-\alpha h)_h^{(\beta)}, \tag{4}$$

Definition 3. (Foundamental theorem h-calculus) If $F(x)$ is an h-antiderivative of $f(x)$ is continus at $x = 0$, we get

$$\int_a^b f(x)d_h x = F(b) - F(a),$$

for $a, b \in T_a$.

2 The Bessel equation. Bessel functions.

2.1 The Bessel differential equation. We consider the h-difference equation in the following form:

$$t_h^{(2)} D_h^2 y(t-2h) + t_h^{(1)} D_h y(t-h) + t_h^{(2)} y(t-2h) - v^2 y(t) = 0 \tag{5}$$

which is called the h-Bessel equation of the indicator in v, where v is a real number. This equation has a special point $t = 0$ (the coefficient at the highest derivative in (5) vanishes at $t = 0$).

Theorem 2.1. Let $v \leq 0$. Then there is a particular solution to equation (5), given by a uniformly convergent series

$$J_{v,h}(t) = \sum_{k=0}^{\infty} \frac{(-1)^k t_h^{v+2k}}{k!\Gamma(v+k+1)2^{v+2k}} \tag{6}$$

which is the solution to the Bessel equation and is called the Bessel function of the first kind v-th order.

Proof. Following the classical methods (see, for example, [6], p. 379), we will look for a solution to this equation in the form of a series. Therefore, there is a solution to equation (5) in the form of a generalized power series

$$y(t) = t^{\alpha} \sum_{k=0}^{\infty} a_k (t-\alpha h)_h^{(k)}, a_0 \neq 0, \tag{7}$$

where α is the characteristic indicator to be determined. By (4) we can rewrite the expression (6) in the form

$$y(t) = \sum_{k=0}^{\infty} a_k t_h^{(\alpha+k)}$$

and using Definition 2 and (1) we find the h-derivatives:

$$\begin{aligned} D_h^2 y(t-2h) &= D_h^2 \sum_{k=0}^{\infty} a_k (t-2h)_h^{(\alpha+k)} \\ &= (\alpha+k)(\alpha+k-1) \sum_{k=0}^{\infty} a_k (t-2h)_h^{(\alpha+k-2)} \end{aligned}$$

and

$$\begin{aligned} D_h y(t-h) &= D_h \sum_{k=0}^{\infty} a_k (t-h)_h^{(\alpha+k)} \\ &= (\alpha+k) \sum_{k=0}^{\infty} a_k (t-h)_h^{(\alpha+k-1)}. \end{aligned}$$

Therefore, substituting (7) and its first and second h-derivatives into the equation (5), we get that

$$t_h^{(2)}(\alpha+k)(\alpha+k-1) \sum_{k=0}^{\infty} a_k (t-2h)_h^{(\alpha+k-2)} + t_h^{(1)}(\alpha+k) \sum_{k=0}^{\infty} a_k (t-h)_h^{(\alpha+k-1)} +$$
$$t_h^{(2)} \sum_{k=0}^{\infty} a_k (t-2h)_h^{(\alpha+k)} - v^2 \sum_{k=0}^{\infty} a_k t_h^{(\alpha+k)} = 0$$

so we can rewrite the equation:

$$(\alpha+k)(\alpha+k-1) \sum_{k=0}^{\infty} a_k t_h^{(2)} (t-2h)_h^{(\alpha+k-2)} + (\alpha+k) \sum_{k=0}^{\infty} a_k t_h^{(1)} (t-h)_h^{(\alpha+k-1)} +$$
$$\sum_{k=0}^{\infty} a_k t_h^{(2)} (t-2h)_h^{(\alpha+k)} - \sum_{k=0}^{\infty} a_k v^2 t_h^{(\alpha+k)} = 0$$

where $t_h^{(2)}(t-2h)_h^{(\alpha+k-2)} = t_h^{\alpha+k}$ and $t_h^{(1)}(t-h)_h^{(\alpha+k-1)} = t_h^{\alpha+k}$.

From here we get a general formula for all these series.

$$(\alpha+k)(\alpha+k-1) \sum_{k=0}^{\infty} a_k t_h^{(\alpha+k)} + (\alpha+k) \sum_{k=0}^{\infty} a_k t_h^{(\alpha+k)} + \sum_{k=0}^{\infty} a_k t_h^{(\alpha+k+2)} - \sum_{k=0}^{\infty} a_k v^2 t_h^{(\alpha+k)} = 0$$

and
$$\sum_{k=0}^{\infty} a_k \left((\alpha+k)(\alpha+k-1)+(\alpha+k)-v^2\right) t_h^{(\alpha+k)} + \sum_{k=0}^{\infty} a_k t_h^{(\alpha+k+2)} = 0$$
and, finally
$$\sum_{k=0}^{\infty} a_k \left((\alpha+k)^2-v^2\right) t_h^{(\alpha+k)} + \sum_{k=0}^{\infty} a_k t_h^{(\alpha+k+2)} = 0.$$

Next, for $\alpha, \alpha+1, ..., \alpha+k, ...,$, we are equating to zero the coefficients at the same powers of x, we lead at the following recurrent relations for the coefficients:
$$\begin{cases} t^{\alpha} \quad |(\alpha^2-v^2)a_0 = 0, \\ t^{\alpha+1}| \left((\alpha+1)^2-v^2\right) a_1 = 0, \\ t^{\alpha+2}| \left((\alpha+2)^2-v^2\right) a_2 + a_0 = 0, \\ \vdots \quad |\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ t^{\alpha+k}| \left((\alpha+k)^2-v^2\right) a_k + a_{k-2} = 0, \ \ \forall k \geq 2. \end{cases} \tag{8}$$

Since $a_0 \neq 0$, it follows from the first equation (8) that $\alpha^2 - v^2 = 0$, or $\alpha = \pm v$. Now from the second equation (8) we will have $a_1 = 0$.

Let us consider the case $\alpha = v > 0$ first. Let us rewrite the $k-th$ $(k > 1)$ equation of system (8) in the following form
$$a_k = \frac{-a_{k-2}}{k(2v+k)}.$$

Considering that $a_1 = 0$, we get from here $a_3 = 0$ and $a_{2k+1} = 0$ in general. On the other hand, each even coefficient can be expressed in terms of the previous one by the formula
$$a_{2k} = -\frac{a_{2k-2}}{2^2 k(v+k)}.$$

Consistent application of this formula allows us to find an expression a_{2k} through a_0 :
$$\begin{aligned} a_2 &= -\frac{a_0}{2^2 \cdot 1 \cdot (v+1)} \\ &\Rightarrow \ a_4 = -\frac{a_2}{2^2 \cdot 2 \cdot (v+2)} = \frac{a_0}{2^4 \cdot 1 \cdot 2 \cdot (v+1)(v+2)} \\ &\Rightarrow \ a_6 = -\frac{a_4}{2^2 \cdot 3 \cdot (v+3)} = -\frac{a_0}{2^6 \cdot 3! \cdot (v+1)(v+2)(v+3)} \\ &\cdots \quad \cdots \quad \cdots \\ &\Rightarrow \ a_{2k} = \frac{(-1)^k a_0}{2^{2k} k! \prod\limits_{r=1}^{k} (v+r)}. \end{aligned}$$

The coefficient a_0 has so far been left arbitrary. If $v \neq -n$, where $n > 0$ is an integer, then assuming
$$a_0 = \frac{1}{2^v \Gamma(v+1)}$$
we find
$$\begin{aligned} a_{2k} &= \frac{(-1)^k}{2^{2k} k!(v+1)(v+2)(v+3)...(v+k)} \cdot \frac{1}{2^v \Gamma(v+1)} \\ &= \frac{(-1)^k}{2^{2k+v} k! \Gamma(v+k+1)}. \end{aligned}$$

Substituting this expression for the coefficients in (7), we get

$$y_1(t) = \sum_{k=0}^{\infty} \frac{(-1)^k t_h^{v+2k}}{k!\Gamma(v+k+1)2^{v+2k}}.$$

The proof is complete.

Corollary 2.3. The equation (5) does not change when v is replaced by $-v$, then the function:

$$J_{-v,h}(t) = \sum_{k=0}^{\infty} \frac{(-1)^k t_h^{-v+2k}}{k!\Gamma(-v+k+1)2^{-v+2k}} \tag{9}$$

is also a solution to the equation (5).

Theorem 2.4. If $v \neq n$. Then the general solution to equation (5) has the form:

$$y(t) = C_1 J_{v,h}(t) + C_2 J_{-v,h}(t). \tag{10}$$

Proof. Now we prove that $y(t)$ in the following form is also a solution to equation (8):

$$\begin{aligned} y(t) &= C_1 J_{v,h}(t) + C_2 J_{-v,h}(t) \\ &= C_1 \sum_{k=0}^{\infty} \frac{(-1)^k t_h^{(v+2k)}}{k!\Gamma(v+k+1)2^{v+2k}} \\ &+ C_2 \sum_{k=0}^{\infty} \frac{(-1)^k t_h^{(-v+2k)}}{k!\Gamma(-v+k+1)2^{-v+2k}}. \end{aligned}$$

Using (1) to find the h-derivatives from the formula (10):

- $$t_h^{(2)} D_h^2 \left(C_1 J_{v,h}(t-2h) + C_2 J_{-v,h}(t-2h)\right) = C_1 \sum_{k=0}^{\infty} \frac{(-1)^k (v+2k)(v+2k-1) t_h^{(v+2k)}}{k!\Gamma(v+k+1)2^{v+2k}} + C_2 \sum_{k=0}^{\infty} \frac{(-1)^k (-v+2k)(-v+2k-1) t_h^{(-v+2k)}}{k!\Gamma(-v+k+1)2^{-v+2k}},$$
- $$t_h D_h \left(C_1 J_{v,h}(t-h) + C_2 J_{-v,h}(t-h)\right) = C_1 \sum_{k=0}^{\infty} \frac{(-1)^k (v+2k) t_h^{(v+2k)}}{k!\Gamma(v+k+1)2^{v+2k}} + C_2 \sum_{k=0}^{\infty} \frac{(-1)^k (-v+2k) t_h^{(-v+2k)}}{k!\Gamma(-v+k+1)2^{-v+2k}},$$
- $$t_h^{(2)} \left(C_1 J_{v,h}(t-2h) + C_2 J_{-v,h}(t-2h)\right) = C_1 \sum_{k=0}^{\infty} \frac{(-1)^k t_h^{(v+2k+2)}}{k!\Gamma(v+k+1)2^{v+2k}} + C_2 \sum_{k=0}^{\infty} \frac{(-1)^k t_h^{(-v+2k+2)}}{k!\Gamma(-v+k+1)2^{-v+2k}},$$
- $$-v^2 \left(C_1 J_{v,h}(t) + C_2 J_{-v,h}(t)\right) = -v^2 C_1 \sum_{k=0}^{\infty} \frac{(-1)^k t_h^{(v+2k)}}{k!\Gamma(v+k+1)2^{v+2k}} + C_2 \sum_{k=0}^{\infty} \frac{(-1)^k t_h^{(-v+2k)}}{k!\Gamma(-v+k+1)2^{-v+2k}}.$$

Now we substitute in equation (10):

$$C_1\left(\sum_{k=0}^{\infty}\frac{(-1)^k(v+2k)(v+2k-1)t_h^{(v+2k)}}{k!\Gamma(v+k+1)2^{v+2k}}+\sum_{k=0}^{\infty}\frac{(-1)^k(v+2k)t_h^{(v+2k)}}{k!\Gamma(v+k+1)2^{v+2k}}\right)+$$

$$+C_1\left(\sum_{k=0}^{\infty}\frac{(-1)^k t_h^{(v+2k+2)}}{k!\Gamma(v+k+1)2^{v+2k}}-v^2\sum_{k=0}^{\infty}\frac{(-1)^k t_h^{(v+2k)}}{k!\Gamma(v+k+1)2^{v+2k}}\right)+$$

$$+C_2\left(\sum_{k=0}^{\infty}\frac{(-1)^k(-v+2k)(-v+2k-1)t_h^{(-v+2k)}}{k!\Gamma(-v+k+1)2^{-v+2k}}+\sum_{k=0}^{\infty}\frac{(-1)^k(-v+2k)t_h^{(-v+2k)}}{k!\Gamma(-v+k+1)2^{-v+2k}}\right)+$$

$$+C_2\left(\sum_{k=0}^{\infty}\frac{(-1)^k t_h^{(-v+2k+2)}}{k!\Gamma(-v+k+1)2^{-v+2k}}-v^2\sum_{k=0}^{\infty}\frac{(-1)^k t_h^{(-v+2k)}}{k!\Gamma(-v+k+1)2^{-v+2k}}\right)=0.$$

If $C_1=-C_2$ then $y(t)=C_1J_{v,h}(t)+C_2J_{-v,h}(t)$ is a solution to the equation (5). The proof is complete.

Example 2.5. Find a general solution to the following equation:

$$t_h^{(2)}D_h^2y(t-2h)+t_h^{(1)}D_hy(t-h)+t_h^{(2)}y(t-2h)-2y(t)=0. \tag{11}$$

Proof. We consider two cases $v=1/2$ and $v=-1/2$. 1) According to the definition (see (6)) of the Bessel function $J_{\frac{1}{2},h}(t)$ we have:

$$J_{\frac{1}{2},h}(t)=\sum_{k=0}^{\infty}\frac{(-1)^k t_h^{\left(\frac{1}{2}+2k\right)}}{k!\Gamma\left(\frac{1}{2}+k+1\right)2^{\frac{1}{2}+2k}}.$$

Since

$$\begin{aligned}\Gamma\left(\frac{1}{2}\right)&=\int_0^{\infty}e^{-t}t^{-\frac{1}{2}}dt=\int_0^{\infty}e^{-t}d(2\sqrt{t})=2\int_0^{\infty}e^{-\xi^2}d(2\sqrt{\xi})=\sqrt{\pi},\\ \Gamma(t+1)&=t\Gamma(t),\quad t>0,\end{aligned}$$

then

$$\begin{aligned}k!\Gamma\left(k+\frac{3}{2}\right)&=\Gamma(k+1)\,\Gamma\left(k+1+\frac{1}{2}\right)\\ &=k\left(k+\frac{1}{2}\right)\cdot\frac{2^{2k-1}}{\sqrt{\pi}}\cdot\frac{\sqrt{\pi}}{2^{2k-1}}\cdot\Gamma(k)\,\Gamma\left(k+\frac{1}{2}\right)\\ &=\frac{\sqrt{\pi}}{2^{2k-1}}k\left(k+\frac{1}{2}\right)\Gamma(2k).\end{aligned}$$

Considering also that $\Gamma(k+1)=k!$ for $k\in\mathbb{N}$, we get

$$\begin{aligned}J_{\frac{1}{2},h}(t)&=\sum_{k=0}^{\infty}\frac{(-1)^k t_h^{\left(\frac{1}{2}+2k\right)}}{k!\Gamma\left(\frac{3}{2}+k\right)2^{\frac{1}{2}+2k}}\\ &=\sum_{k=0}^{\infty}\frac{(-1)^k t_h^{\left(-\frac{1}{2}\right)}\left(t+\frac{1}{2}h\right)^{(2k+1)}}{k!\Gamma\left(\frac{3}{2}+k\right)2^{\frac{1}{2}+2k}}\\ &=t_h^{\left(-\frac{1}{2}\right)}\sum_{k=0}^{\infty}\frac{(-1)^k\left(t+\frac{1}{2}h\right)^{(2k+1)}}{\frac{\sqrt{\pi}}{2^{2k-1}}k\left(k+\frac{1}{2}\right)\Gamma(2k)\,2^{\frac{1}{2}+2k}}\\ &=\frac{\sqrt{2}t_h^{\left(-\frac{1}{2}\right)}}{\sqrt{\pi}}\sum_{k=0}^{\infty}\frac{(-1)^k\left(t+\frac{1}{2}h\right)_h^{(2k+1)}}{(2k+1)!}.\end{aligned}$$

The row on the right hand side of the last equality represents the decomposition of the function $sin_h t$. Therefore, the following equality is true

$$J_{\frac{1}{2},h}(t) = \frac{\sqrt{2} t_h^{(-\frac{1}{2})}}{\sqrt{\pi}} sin_h \left(t + \frac{1}{2}h\right). \tag{12}$$

Now, using (1), (6), and (12) we find the h-derivatives of $sin_h t$:

$$\begin{aligned} D_h sin_h \left(t + \frac{1}{2}h\right) &= D_h \sum_{k=0}^{\infty} \frac{(-1)^k \left(t + \frac{1}{2}h\right)_h^{(2k+1)}}{(2k+1)!} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k (2k+1) \left(t + \frac{1}{2}h\right)_h^{(2k)}}{(2k)!(2k+1)} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k \left(t + \frac{1}{2}h\right)_h^{(2k)}}{(2k)!} \\ &= cos_h \left(t + \frac{1}{2}h\right). \end{aligned}$$

2) Let us now consider the case when $v = -\frac{1}{2}$. By using (9) we have that

$$J_{-\frac{1}{2},h}(t) = \sum_{k=0}^{\infty} \frac{(-1)^k t_h^{(-\frac{1}{2}+2k)}}{k!\Gamma\left(-\frac{1}{2}+k+1\right) 2^{-\frac{1}{2}+2k}}.$$

Taking into account that

$$\begin{aligned} k!\Gamma\left(k - \frac{1}{2} + 1\right) &= k\Gamma(k)\Gamma\left(k + \frac{1}{2}\right) \\ &= k\left(k + \frac{1}{2}\right) \cdot \frac{2^{2k-1}}{\sqrt{\pi}} \cdot \frac{\sqrt{\pi}}{2^{2k-1}} \cdot \Gamma(k)\Gamma\left(k + \frac{1}{2}\right) \\ &= \frac{\sqrt{\pi}}{2^{2k}} 2k\Gamma(2k), \end{aligned}$$

we get that

$$\begin{aligned} J_{-\frac{1}{2},h}(t) &= \sum_{k=0}^{\infty} \frac{(-1)^k t_h^{(-\frac{1}{2}+2k)}}{k!\Gamma\left(k+\frac{1}{2}\right) 2^{-\frac{1}{2}+2k}} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k t_h^{(-\frac{1}{2})} \left(t + \frac{1}{2}h\right)_h^{(2k)}}{k!\Gamma\left(k+\frac{1}{2}\right) 2^{-\frac{1}{2}+2k}} \\ &= t_h^{(-\frac{1}{2})} \sum_{k=0}^{\infty} \frac{(-1)^k \left(t + \frac{1}{2}h\right)_h^{(2k)}}{k\Gamma(k)\Gamma\left(k+\frac{1}{2}\right) 2^{-\frac{1}{2}+2k}} \\ &= t_h^{(-\frac{1}{2})} \sum_{k=0}^{\infty} \frac{(-1)^k \left(t + \frac{1}{2}h\right)_h^{(2k)}}{\frac{\sqrt{\pi}}{2^{2k}} 2k\Gamma(2k) 2^{-\frac{1}{2}+2k}} \\ &= \frac{\sqrt{2} t_h^{(-\frac{1}{2})}}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k \left(t + \frac{1}{2}h\right)_h^{(2k)}}{2k!}. \end{aligned}$$

The row on the right side of the last equality is a function $\cos_h t$. Therefore,

$$J_{-\frac{1}{2},h}(t) = \frac{\sqrt{2}t_h^{-\frac{1}{2}}}{\sqrt{\pi}} \cos_h\left(t + \frac{1}{2}h\right). \tag{13}$$

Now, using (1) we find the h-derivatives of:

$$\begin{aligned} D_h \cos_h\left(t + \frac{1}{2}k\right) &= D_h \sum_{k=0}^{\infty} \frac{(-1)^k \left(t + \frac{1}{2}h\right)_h^{(2k)}}{2k!} \\ &= \sum_{k=1}^{\infty} \frac{(-1)^k 2k \left(t + \frac{1}{2}h\right)_h^{(2k-1)}}{2k!} \\ &= \sum_{k=1}^{\infty} \frac{(-1)^k \left(t + \frac{1}{2}h\right)_h^{(2k-1)}}{(2k-1)!} \\ &= -\sum_{n=0}^{\infty} \frac{(-1)^n \left(t + \frac{1}{2}h\right)_h^{(2n+1)}}{(2n+1)!} \\ &= -sin_h(t + \frac{1}{2}h). \end{aligned}$$

According to (12) and (13), we get a general solution to the equation (11):

$$y(t) = C_1 \frac{\sqrt{2}t_h^{(-\frac{1}{2})}}{\sqrt{\pi}} sin_h\left(t + \frac{1}{2}h\right) + C_2 \frac{\sqrt{2}t_h^{-\frac{1}{2}}}{\sqrt{\pi}} \cos_h\left(t + \frac{1}{2}h\right).$$

The proof is complete.

Theorem 2.6. We define the h-Neumann function for non-integers ν (complex constant) by the formula:

$$N_{\nu,h}(t) = \frac{\cos_h(\nu\pi) J_{\nu,h}(t) - J_{-\nu,h}(t)}{\sin_h(\nu\pi)} \tag{14}$$

and it is a solution to equation (5).

Proof. Now, by using (1) we obtain the h-derivatives of the function (14):

$$D_h N_{\nu,h}(t) = \frac{\cos_h(\nu\pi)}{\sin_h(\nu\pi)} D_h J_{\nu,h}(t) - \frac{1}{\sin_h(\nu\pi)} D_h J_{-\nu,h}(t)$$

$$D_h^2 N_{\nu,h}(t) = \frac{\cos_h(\nu\pi)}{\sin_h(\nu\pi)} D_h^2 J_{\nu,h}(t) - \frac{1}{\sin_h(\nu\pi)} D_h^2 J_{-\nu,h}(t).$$

Substitute equation (5) into

$$t_h^{(2)}\left(\frac{\cos_h(\nu\pi)}{\sin_h(\nu\pi)} D_h^2 J_{\nu,h}(t-2h) - \frac{1}{\sin_h(\nu\pi)} D_h^2 J_{-\nu,h}(t-2h)\right) +$$

$$+t_h\left(\frac{\cos_h(\nu\pi)}{\sin_h(\nu\pi)}D_hJ_{\nu,h}(t-h)-\frac{1}{\sin_h(\nu\pi)}D_hJ_{-\nu,h}(t-h)\right)+$$
$$+t_h^{(2)}\left(\frac{\cos_h(\nu\pi)}{\sin_h(\nu\pi)}J_{\nu,h}(t-2h)-\frac{1}{\sin_h(\nu\pi)}J_{-\nu,h}(t-2h)\right)-$$
$$-\nu^2\left(\frac{\cos_h(\nu\pi)}{\sin_h(\nu\pi)}J_{\nu,h}(t)-\frac{1}{\sin_h(\nu\pi)}J_{-\nu,h}(t)\right)=0.$$

Consequently:

$$\frac{\cos_h(\nu\pi)}{\sin_h(\nu\pi)}(t_h^{(2)}D_h^2J_{\nu,h}(t-2h)+t_hD_hJ_{\nu,h}(t-h)+t_h^{(2)}J_{\nu,h}(t-2h)-\nu^2J_{\nu,h}(t))-$$

$$-\frac{1}{\sin_h(\nu\pi)}(t_h^{(2)}D_h^2J_{-\nu,h}(t-2h)+t_hD_hJ_{-\nu,h}(t-h)+t_h^{(2)}J_{-\nu,h}(t-2h)-\nu^2J_{-\nu,h}(t))=0.$$

We know that functions (5) and (9) of the first kind in the form $J_{\nu,h}(t)$ and $J_{-\nu,h}(t)$ which is the solution to the Bessel equation. Thus, we can say that the h-Neumann function (15) is the solution to equation (9).

Let $\nu>0$ and

$$L^2_{\nu,h}\,[a,b]:=\{f:\left[\int_a^b|f(x)|^2\,|x|^{2(\nu+1/2)}\,d_hx\right]^{1/2}\},$$

for $\forall\in a,b\in T_a$.

The h-Bessel operator: In this article, we consider a discrete analogue of the Bessel operator, where the h-Bessel operator has the following form:

$$(B_hy)(t):=t_h^{(-2\nu-1)}D_h\left[D_hy(t)\frac{1}{t_h^{(-2\nu-1)}}\right].$$

In addition, B_h is a linear operator, that is

$$B_h(\alpha y+\beta f)=\alpha B_h(y)+\beta B_h(f),\forall y,f\in L^2_{\nu,h}(a,b).$$

Theorem 2.7. (Orthogonality of eigenfunctions). Let (λ_1,y) and (λ_2,f) two pairs of eigenvalues and eigenfunctions, and $\lambda_1\neq\lambda_2$. Then, for both regular and periodic problems, the corresponding eigenfunctions $y(t)$ and $f(t)$ are orthogonal with weight r (therefore $\langle y(t),f(t)\rangle=0$).

Proof. The first two statements follow from the definition 3 and (1)–(3) for $\forall y,\ f\in L_{\nu,2}(a,b)$, we get that

$$\begin{aligned}\frac{(f(t+h)B_hy(t)-y(t+h)B_hf(t))}{t_h^{(-2\nu-1)}} &= D_h\left[D_hy(t)\frac{1}{t_h^{(-2\nu-1)}}\right]f(t+h)\\ &- D_h\left[D_hf(t)\frac{1}{t_h^{(-2\nu-1)}}\right]y(t+h)\\ &= D_h\left[f(t)D_hy(t)\frac{1}{t_h^{(-2\nu-1)}}\right.\\ &- \left.y(t)D_hf(t)\frac{1}{t_h^{(-2\nu-1)}}\right] \end{aligned}\qquad(15)$$

and

$$\int_0^h \frac{(f(t+h)B_h y(t) - y(t+h)B_h f(t))}{t_h^{(-2\nu-1)}} d_h t = \left[f(t) D_h y(t) \frac{1}{t_h^{(-2\nu-1)}} - y(t) D_h f(t) \frac{1}{t_h^{(-2\nu-1)}} \right] \Big|_0^h . \tag{16}$$

And using (5), we see that

$$B_h y(t) = t_h^{(-2\nu-1)} D_h \left[D_h y(t) \frac{1}{t_h^{(-2\nu-1)}} \right] = -\lambda_1^2 y(t+h) \tag{17}$$

and

$$B_h f(t) = t_h^{(-2\nu-1)} D_h \left[D_h f(t) \frac{1}{t_h^{(-2\nu-1)}} \right] = -\lambda_2^2 f(t+h) \tag{18}$$

Now multiply the first of the obtained equations (17) and (18) by $f(t)$, the second by $y(t)$, and find the difference. The resulting equation is reduced to the following form

$$\begin{aligned} f(t+h)B_h y(t) - y(t+h)B_h f(t) &= t_h^{(-2\nu-1)} D_h \left[\frac{D_h y(t)}{t_h^{(-2\nu-1)}} \right] f(t+h) \\ &- t_h^{(-2\nu-1)} D_h \left[D_h f(t) \frac{1}{t_h^{(-2\nu-1)}} \right] y(t+h) \\ &= (\lambda_2^2 - \lambda_1^2) y(t+h) f(t+h). \end{aligned}$$

We can rewrite

$$\begin{aligned} \frac{(f(t+h)B_h y(t) - y(t+h)B_h f(t))}{t_h^{(-2\nu-1)}} &= D_h \left[D_h y(t) \frac{1}{t_h^{(-2\nu-1)}} \right] f(t+h) \\ &- D_h \left[D_h f(t) \frac{1}{t_h^{(-2\nu-1)}} \right] y(t+h) \\ &= (\lambda_2^2 - \lambda_1^2) y(t+h) f(t+h) \frac{1}{t_h^{(-2\nu-1)}}. \end{aligned} \tag{19}$$

From (2), (16), and (19), we may compute

$$\begin{aligned} (\lambda_2^2 - \lambda_1^2) \int_0^h \frac{y(t+h)f(t+h)}{t_h^{(-2\nu-1)}} d_h t &= \int_0^h D_h \left[f(t) D_h y(t) \frac{1}{t_h^{(-2\nu-1)}} - y(t) D_h f(t) \frac{1}{t_h^{(-2\nu-1)}} \right] d_h t \\ &= \int_0^h \frac{(f(t+h)B_h y(t) - y(t+h)B_h f(t))}{t_h^{(-2\nu-1)}} d_h t \\ &= \int_0^h D_h \left[f(t) D_h y(t) \frac{1}{t_h^{(-2\nu-1)}} - y(t) D_h f(t) \frac{1}{t_h^{(-2\nu-1)}} \right] d_h t \\ &= h \frac{1}{h_h^{(-2\nu-1)}} - h \frac{1}{h_h^{(-2\nu-1)}} - 0 + 0. \end{aligned}$$

Here:

$$\begin{bmatrix} t = 0 \Rightarrow j_{(\nu,h)}(0) = 1; D_h j_{(\nu,h)}(0) = 0 \\ t = h \Rightarrow j_{(\nu,h)}(h) = 1; D_h j_{(\nu,h)}(h) = h. \end{bmatrix}$$

Therefore,

$$\int_0^h \frac{(f(t+h)B_h y(t) - y(t+h)B_h f(t))}{t_h^{(-2\nu-1)}} d_h t = (\lambda_2^2 - \lambda_1^2) \int_0^h \frac{y(t+h)f(t+h)}{t_h^{(-2\nu-1)}} d_h t = 0$$

and

$$\lambda_2 \neq \lambda_1 \Rightarrow \langle y(t+h), f(t+h) \rangle = 0.$$

It proves the claim.

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Кванттық есептеудегі Бессель теңдеуі

Бессель функцияларын қолдану математикалық физиканың барлық дерлік маңызды салаларына қатысты және өзекті техникалық сұрақтарға жауап беруге арналған әртүрлі есептердің үлкен санымен байланысты. Жұмыста Бессель дифференциалдық теңдеуінің аналогы болып табылатын h-айырымдық теңдеуі енгізілген және жалпыланған дәрежелер қатарын алып, Фробениус әдісі арқылы өрнектейтін оның шешімінің қасиеттері зерттелген. Бессель функциясы мен h-Нейман функциясы үшін дискретті аналогтық формулалар табылды, олардың шешімдері h-бөлшек функциясы $t_h^{(\alpha)}$ бар қатармен берілген. Сонымен қатар, T_a бойынша h-Бессель функциялары арасындағы сызықтық тәуелділіктер алынған.

Кілт сөздер: Бессель функциясы, модификацияланған Бессель функциясы, Бессель айырымдық теңдеуі, h-есептеу, h-туынды және h-бөлшек функциясы.

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Уравнение Бесселя в квантовом исчислении

С использованием функций Бесселя связано большое количество самых разнообразных задач, относящихся практически ко всем важнейшим разделам математической физики и призванных ответить на актуальные технические вопросы. В статье мы вводим h-разностное уравнение, аналог дифференциального уравнения Бесселя, и исследуем свойства его решения, которые мы выражаем с помощью метода Фробениуса, предполагая обобщенный степенной ряд. Найдены дискретные формулы-аналоги для функции Бесселя и h-функции Неймана, решения которых представлены рядом с h-дробной функцией $t_h^{(\alpha)}$. Кроме того, мы получили линейные зависимости между h-функциями Бесселя на T_a.

Ключевые слова: функция Бесселя, модифицированная функция Бесселя, разностное уравнение Бесселя, h-исчисление, h-производная и h-дробная функции.