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To the solution of the Solonnikov-Fasano problem with boundary moving on arbitrary law $x = \gamma(t)$.

In this paper we study the solvability of the boundary value problem for the heat equation in a domain that degenerates into a point at the initial moment of time. In this case, the boundary changing with time moves according to an arbitrary law $x = \gamma(t)$. Using the generalized heat potentials, the problem under study is reduced to a pseudo-Volterra integral equation such that the norm of the integral operator is equal to one and it is shown that the corresponding homogeneous integral equation has a nonzero solution.

Key words: heat equation, moving boundary, degenerating domain, pseudo-Volterra integral equation.

Introduction

In many practically important engineering problems, the process of forming a temperature field in the structure under study is accompanied by the removal of a part of the substance from the surface, which leads to a change in its boundaries over time. The need to take into account the of the boundaries mobility of the studied domain significantly complicates the solution of the corresponding problems [1-5].

For example, in mathematical modeling of thermophysical processes in an electric arc of high-current disconnecting devices, the heat equation is used, which takes into account the effect of heat sources in the arc and the effect of contracting the axial section of the arc in the cathode region into the contact spot [6]. Moreover, the diameter of the contact spot is much less than the section diameter of the developed column of the arc, which makes it possible to consider it as a mathematical point. At the initial moment of time, the contacts are in a closed state and solution domain of the problem is absent; then, the solution domain changes over time according to the conditions for opening the contacts.

From a mathematical point of view, the singularity of the problem under consideration lies, firstly, in the presence of a moving boundary, and secondly, in the degeneration of the solution domain at the initial moment [7, 8]. Problems in domains with moving boundaries are also relevant in modeling physical processes in a gas discharge plasma, during melting of electrical contacts, the effect of an electric arc on contacts, in studying the problems of thermal shock in domains with a moving boundary, in solving a number of problems in hydromechanics [9-13].

Applying the method of generalized heat potentials, a number of similar problems can be reduced to the solution of sinqular Volterra type integral equations of the second kind. It is essential here that if in the boundary value problem the variable domain does not degenerate into a point at the initial moment of time, then the integral equation equivalent to it is solved by the method of successive approximations. If the domain degenerates into a point at the initial moment of time, then the integral equation of the boundary value problem has a singularity, which is that the integral from the kernel tends to unity as the upper limit of integration tends to the lower one, and this means that the method of successive approximations is not applicable to it.

Statement of the problem

Let's study the solvability issues of the following boundary value problem:

$$\frac{\partial u}{\partial t} - a^2 \frac{\partial^2 u}{\partial x^2} = f\left(x, t\right), \quad \{0 < x < \gamma(t),\ t > 0\} \tag{1}$$

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$$\left.\frac{\partial u}{\partial x}\right|_{x=0} = u_0(t), \quad \frac{d\tilde{u}(t)}{dt} + \left.\frac{\partial u}{\partial x}\right|_{x=\gamma(t)} = u_1(t), \tag{2}$$

where $\tilde{u}(t) = u(\gamma(t), t)$, $\gamma(0) = 0$ for $\gamma(t) = [t(1+\alpha_0(t))]^{\omega}$, $\omega > \frac{1}{2}$.

Function $\gamma(t) : (0, \infty) \to (0, \infty)$ satisfies the following conditions:

1. asymptotics of the function $\gamma(t)$ as $t \to 0$ and as $t \to \infty$ has the form t^{ω}, where $\omega > \frac{1}{2}$;

2. starting from some moment of time t_1^* until moment of time t_2^* the function $\gamma(t)$ is arbitrary, strictly monotone and one-to-one, i.e. there is a reverse transformation $\gamma^{-1}(t)$.

We introduce the classes of solutions and data of the problem as follows:

$$(x + [\gamma(t)]^{\frac{3}{2\omega}-1})^{-1} u(x,t) \in L_{\infty}(G), \text{ i.e. } \quad u(x,t) \in L_{\infty}(G; (x + [\gamma(t)]^{3/2\omega-1})^{-1}),$$

$$\begin{array}{l} f(x,t) \in W_{\infty}^{1,0}\left(G; [\gamma(t)]^{3/2\omega-1} \exp\left\{[\gamma(t)]^{\frac{2\omega-1}{\omega}}/(4a^2)\right\}\right); \\ u_0(t) \in L_{\infty}\left(R_+; [\gamma(t)]^{-(3/2\omega-1)}\right); \quad u_1(t) \in L_{\infty}\left(R_+; [\gamma(t)]^{3/2\omega-1}\right). \end{array}$$

This kind of boundary value problem (1) arises, for example, in studies of the Stefan problem [14].

Transformation of problem (1) and reduction of it to the integral equation

Introducing a new unknown function $v(x,t) = \frac{\partial u}{\partial x}$, we transform problem (1)–(2) to the next problem:

$$\frac{\partial v}{\partial t} - a^2 \frac{\partial^2 v}{\partial x^2} = \tilde{f}(x,t), \quad \{0 < x < t, \ t > 0\} \tag{3}$$

$$v(x,t)|_{x=0} = v_0(t), \quad \left.\left(\frac{\partial v}{\partial x} + \frac{1+\gamma'(t)}{a^2} v\right)\right|_{x=\gamma(t)} = v_1(t), \tag{4}$$

where $\tilde{f}(x,t) \equiv \frac{\partial f(x,t)}{\partial x}$, $v_0(t) \equiv u_0(t)$, $v_1(t) \equiv \frac{u_1(t)}{a^2} + \left.\frac{f(x,t)}{a^2}\right|_{x=\gamma(t)}$.

Remark 1. Each solution to boundary value problem (3)–(4) defines a unique solution (up to a constant) of boundary value problem (1)–(2).

We will find the solution of problem (3)–(4) as the sum of heat potentials [15]:

$$\begin{array}{l} v(x,t) = \dfrac{1}{2a\sqrt{\pi}} \displaystyle\int_0^t \int_0^{\infty} \frac{1}{(t-\tau)^{1/2}} \exp\left\{-\frac{(x-\xi)^2}{4a^2(t-\tau)}\right\} \tilde{f}(\xi,\tau) d\xi d\tau + \\ + \dfrac{1}{4a^3\sqrt{\pi}} \displaystyle\int_0^t \frac{x}{(t-\tau)^{3/2}} \exp\left\{-\frac{x^2}{4a^2(t-\tau)}\right\} \nu(\tau) d\tau + \\ + \dfrac{1}{2a\sqrt{\pi}} \displaystyle\int_0^t \frac{1}{(t-\tau)^{1/2}} \exp\left\{-\frac{(x-\gamma(\tau))^2}{4a^2(t-\tau)}\right\} \varphi(\tau) d\tau. \end{array} \tag{5}$$

The function defined by equality (5) satisfies the equation (3) for any functions $\nu(t)$ and $\varphi(t)$, which are still unknown and are to be determined further.

Satisfying solution (5) with boundary conditions (4), we obtain the following integral equation:

$$\varphi(t) + \int_0^t K_{\gamma}(t,\tau)\varphi(\tau) d\tau = F(t), \tag{6}$$

kernel $K_{\gamma}(t,\tau)$ of which can be represented as a sum:

$$K_{\gamma}(t,\tau) = \sum_{i=1}^{4} K_{\gamma}^{(i)}(t,\tau),$$

where:

$$K_\gamma^{(1)} = \frac{1}{2a\sqrt{\pi}} \frac{\gamma(t)+\gamma(\tau)}{(t-\tau)^{\frac{3}{2}}} \exp\left\{-\frac{(\gamma(t)+\gamma(\tau))^2}{4a^2(t-\tau)}\right\};$$

$$K_\gamma^{(2)} = -\frac{1}{2a\sqrt{\pi}} \frac{\gamma(t)-\gamma(\tau)}{(t-\tau)^{\frac{3}{2}}} \exp\left\{-\frac{(\gamma(t)-\gamma(\tau))^2}{4a^2(t-\tau)}\right\};$$

$$K_\gamma^{(3)} = -\frac{1}{a\sqrt{\pi}} \frac{1+\gamma(t)'}{(t-\tau)^{\frac{1}{2}}} \exp\left\{-\frac{(\gamma(t)+\gamma(\tau))^2}{4a^2(t-\tau)}\right\};$$

$$K_\gamma^{(3)} = \frac{1}{a\sqrt{\pi}} \frac{1+\gamma(t)'}{(t-\tau)^{\frac{1}{2}}} \exp\left\{-\frac{(\gamma(t)-\gamma(\tau))^2}{4a^2(t-\tau)}\right\}.$$

The free term of equation (6) has the following form:

$$F(t) = -\frac{a}{\sqrt{\pi}} \int_0^t \left[\frac{1}{(t-\tau)^{3/2}} - \frac{\gamma^2(t)}{2a^2(t-\tau)^{5/2}}\right] \exp\left\{-\frac{\gamma^2(t)}{4a^2(t-\tau)}\right\} v_0(\tau)d\tau +$$

$$-\frac{1+\gamma'(\mathrm{t}))}{a\sqrt{\pi}} \int_0^t \frac{\gamma(t)}{(t-\tau)^{3/2}} \exp\left\{-\frac{\gamma^2(t)}{4a^2(t-\tau)}\right\} v_0(\tau)d\tau + 2a^2 \cdot v_1(t) -$$

$$-\frac{1}{2a\sqrt{\pi}} \int_0^t \int_0^\infty \left[\frac{\gamma(t)+\xi}{(t-\tau)^{3/2}} \exp\left\{-\frac{(\gamma(t)+\xi)^2}{4a^2(t-\tau)}\right\} - \frac{\gamma(t)-\xi}{(t-\tau)^{3/2}} \exp\left\{-\frac{(\gamma(t)-\xi)^2}{4a^2(t-\tau)}\right\}\right] \tilde{f}(\xi,\tau)d\xi d\tau -$$

$$-\frac{1}{a\sqrt{\pi}} \int_0^t \int_0^\infty \frac{1+\gamma'(t)}{(t-\tau)^{1/2}} \cdot \exp\left\{-\frac{(\gamma(t)-\xi)^2}{4a^2(t-\tau)}\right\} \cdot \tilde{f}(\xi,\tau)d\xi\, d\tau.$$

We will find the solution of integral equation (6) in the class of functions:

$$[\gamma(t)]^{\frac{3}{2\omega}-1}\varphi(t) \in L_\infty(0,\infty), \ \text{т.е.} \quad \varphi(t) \in L_\infty\left(0,\infty;\ [\gamma(t)]^{\frac{3}{2\omega}-1}\right).$$

For convenience, we represent equation (6) as follows:

$$\varphi_1(t) + \int_0^t \left[\frac{\gamma(t)}{\gamma(\tau)}\right]^{\frac{3}{2\omega}-1} K_\gamma(t,\tau)\varphi_1(\tau)d\tau = F_1(t), \tag{7}$$

where

$$\varphi_1(t) = t^{\frac{3}{2\omega}-1} \cdot \varphi(t), \quad F_1(t) = t^{\frac{3}{2\omega}-1} \cdot F(t).$$

Remark 2. ([16], p.183) If the (particular) solution of the integral equation

$$y(x) + \int_a^x K(x,t)\, y(t)dt = f(x)$$

is given by formula

$$y(x) = f(x) + \int_a^x R(x,t)\, f(t)dt,$$

then the (particular) solution of the integral equation (with a modified kernel)

$$y(x) + \int_a^x K(x,t)\, \frac{g(x)}{g(t)} y(t)dt = f(x)$$

is given by the formula

$$y(x) = f(x) + \int_a^x R(x,t)\, \frac{g(x)}{g(t)} f(t)dt.$$

The same is true for the solutions of the corresponding homogeneous equations.

Such kind of the Volterra integral equations were considered in the papers [17, 18].

Note that a feature of integral equation (7) is the following property of the kernel $K_\gamma(t,\tau)$:

$$\lim_{t\to 0+} \int_0^t K_\gamma(t,\tau)d\tau = 1.$$

In order to solve integral equation (7), consider the corresponding characteristic integral equation.

Characteristic integral equation. Estimates for kernels of integral operators

For integral equation (7) we will construct a characteristic equation

$$\varphi(t)+\int_0^t\left[\frac{\gamma(t)}{\gamma(\tau)}\right]^{\frac{3}{2\omega}-1}K_h(t,\tau)\varphi(\tau)d\tau=g(t), \tag{8}$$

where

$$K_h(t,\tau)=\sum_{i=1}^{4}K_h^{(i)}(t,\tau),$$

$$K_h^{(1)}(t,\tau)=\frac{1}{2a\sqrt{\pi}}\cdot\frac{(2\omega-1)^{\frac{3}{2}}\left([\gamma(\tau)]^{\frac{2\omega-1}{\omega}}\cdot[\gamma(t)]^{\frac{2\omega-2}{\omega}}+[\gamma(t)]^{\frac{4\omega-3}{\omega}}\right)\left\{[\gamma(\tau)]^{\frac{1}{\omega}}\right\}'}{\left([\gamma(t)]^{\frac{2\omega-1}{\omega}}-[\gamma(\tau)]^{\frac{2\omega-1}{\omega}}\right)^{\frac{3}{2}}}\cdot$$
$$\cdot\exp\left\{-\frac{(2\omega-1)\left([\gamma(t)]^{\frac{2\omega-1}{\omega}}+[\gamma(\tau)]^{\frac{2\omega-1}{\omega}}\right)^2}{4a^2\left([\gamma(t)]^{\frac{2\omega-1}{\omega}}-[\gamma(\tau)]^{\frac{2\omega-1}{\omega}}\right)}\right\};$$

$$K_h^{(2)}(t,\tau)=-\frac{1}{2a\sqrt{\pi}}\cdot\frac{(2\omega-1)^{\frac{3}{2}}\cdot[\gamma(t)]^{\frac{2\omega-2}{\omega}}\cdot\left\{[\gamma(\tau)]^{\frac{1}{\omega}}\right\}'}{\left(t^{2\omega-1}-\tau^{2\omega-1}\right)^{\frac{1}{2}}}\cdot\exp\left\{-\frac{(2\omega-1)\left([\gamma(t)]^{\frac{2\omega-1}{\omega}}-[\gamma(\tau)]^{\frac{2\omega-1}{\omega}}\right)^2}{4a^2\left([\gamma(t)]^{\frac{2\omega-1}{\omega}}-[\gamma(\tau)]^{\frac{2\omega-1}{\omega}}\right)}\right\};$$

$$K_h^{(3)}(t,\tau)=-\frac{2}{2a\sqrt{\pi}}\cdot\frac{(2\omega-1)^{\frac{3}{2}}\cdot[\gamma(t)]^{\frac{2\omega-2}{\omega}}\cdot\left\{[\gamma(\tau)]^{\frac{1}{\omega}}\right\}'}{\left(t^{2\omega-1}-\tau^{2\omega-1}\right)^{\frac{1}{2}}}\cdot\exp\left\{-\frac{(2\omega-1)\left([\gamma(t)]^{\frac{2\omega-1}{\omega}}+[\gamma(\tau)]^{\frac{2\omega-1}{\omega}}\right)^2}{4a^2\left([\gamma(t)]^{\frac{2\omega-1}{\omega}}-[\gamma(\tau)]^{\frac{2\omega-1}{\omega}}\right)}\right\};$$

$$K_h^{(4)}(t,\tau)=\frac{2}{a\sqrt{\pi}}\cdot\frac{(2\omega-1)^{\frac{3}{2}}\cdot[\gamma(t)]^{\frac{2\omega-2}{\omega}}\cdot\left\{[\gamma(\tau)]^{\frac{1}{\omega}}\right\}'}{\left(t^{2\omega-1}-\tau^{2\omega-1}\right)^{\frac{1}{2}}}\cdot\exp\left\{-\frac{(2\omega-1)\left([\gamma(t)]^{\frac{2\omega-1}{\omega}}-[\gamma(\tau)]^{\frac{2\omega-1}{\omega}}\right)^2}{4a^2\left([\gamma(t)]^{\frac{2\omega-1}{\omega}}-[\gamma(\tau)]^{\frac{2\omega-1}{\omega}}\right)}\right\};$$

Next, we show that it is indeed characteristic equation for the equation (7). Firstly, we note that the kernel $K_h(t,\tau)$ has the property:

$$\lim_{t\to0}\int_0^t K_\gamma^{(1)}(t,\tau)d\tau=1.$$

Equation (8), using the following the change of variables:

$$\gamma(t)=\left(\tfrac{1}{2\omega-1}\cdot t_1\right)^{\frac{\omega}{2\omega-1}},\quad \gamma(t)=\left(\tfrac{1}{2\omega-1}\cdot \tau_1\right)^{\frac{\omega}{2\omega-1}},$$
$$\varphi\left[\left(\tfrac{1}{2\omega-1}\cdot t_1\right)^{\frac{\omega}{2\omega-1}}\right]=\varphi_1(t_1),\quad g\left[\left(\tfrac{1}{2\omega-1}\cdot t_1\right)^{\frac{\omega}{2\omega-1}}\right]=g_1(t_1),$$

reduces to the following integral equation [8]:

$$\varphi_1(t_1)+\int_0^{t_1}\sqrt{\frac{t_1}{\tau_1}}K_1(t_1,\tau_1)\varphi_1(\tau_1)d\tau_1=g_1(t_1). \tag{9}$$

The kernel $K_1(t_1,\tau_1)$ has the form:

$$K_1(t_1,\tau_1)=\sum_{i=1}^{4}K_1^{(i)}(t_1,\tau_1),$$

where

$$K_1^{(1)}(t_1,\tau_1) = \frac{1}{2a\sqrt{\pi}} \cdot \frac{t_1+\tau_1}{(t_1-\tau_1)^{\frac{3}{2}}} \cdot \exp\left\{-\frac{(t_1+\tau_1)^2}{4a^2(t_1-\tau_1)}\right\};$$
$$K_1^{(2)}(t_1,\tau_1) = -\frac{1}{2a\sqrt{\pi}} \cdot \frac{t_1-\tau_1}{(t_1-\tau_1)^{\frac{3}{2}}} \cdot \exp\left\{-\frac{(t_1-\tau_1)^2}{4a^2(t_1-\tau_1)}\right\};$$
$$K_1^{(3)}(t_1,\tau_1) = -\frac{2}{a\sqrt{\pi}} \cdot \frac{1}{(t_1-\tau_1)^{\frac{1}{2}}} \cdot \exp\left\{-\frac{(t_1+\tau_1)^2}{4a^2(t_1-\tau_1)}\right\};$$
$$K_1^{(4)}(t_1,\tau_1) = \frac{2}{a\sqrt{\pi}} \cdot \frac{1}{(t_1-\tau_1)^{\frac{1}{2}}} \cdot \exp\left\{-\frac{(t_1-\tau_1)^2}{4a^2(t_1-\tau_1)}\right\}.$$

Solution of integral equation (9) has the form [8]:

$$\varphi(t_1) = g(t_1) + \int_0^t \sqrt{\frac{t_1}{\tau_1}} \cdot R(t_1,\tau_1) \cdot g(t_1)\, d\tau_1 + C \cdot \varphi_{hom}(t_1), \tag{10}$$

which also belongs to the class $L_\infty\left(\mathrm{R}_+;\sqrt{t_1}\exp\left\{\frac{t_1}{4a^2}\right\}\right)$.

Moreover, the following Lemma holds for the resolvent [8].

Lemma 1. The resolvent $R(\mathrm{t}_1,\tau_1)$ admits an estimate

$$|R(\mathrm{t}_1,\tau_1)| \leq C\frac{\tau_1}{(t_1-\tau_1)^{3/2}} \exp\left\{-\frac{t_1\tau_1}{a^2(t_1-\tau_1)}\right\}, \quad 0<\tau_1<t_1<+\infty.$$

.

Solution of characteristic integral equation (8)

Returning to the old variables, in equality (10), we obtain the solution of characteristic equation (8):

$$\varphi(t) = g(t) + \int_0^t \left(\frac{t}{\tau}\right)^{\frac{3}{2\omega}-1} R_h(t,\tau)\, g(t)\, d\tau + C \cdot \varphi_{\mathrm{hom}}\left((2\omega-1)(2\omega-1)[\gamma(\tau)]^{\frac{2\omega-1}{\omega}}\right),$$

and the resolvent $R_h(t,\tau)$ satisfies the estimate

$$R_h(t,\tau) \leq C_1(\omega) \cdot \frac{[\gamma(t)]^{\frac{5\omega-3}{\omega}}\left\{[\gamma(\tau)]^{\frac{1}{\omega}}\right\}'}{\left([\gamma(t)]^{\frac{2\omega-1}{\omega}} - [\gamma(\tau)]^{\frac{2\omega-1}{\omega}}\right)^{\frac{3}{2}}} \cdot \exp\left\{-\frac{(2\omega-1)[\gamma(t)]^{\frac{2\omega-1}{\omega}} \cdot [\gamma(\tau)]^{\frac{2\omega-1}{\omega}}}{a^2\left([\gamma(t)]^{\frac{2\omega-1}{\omega}} - [\gamma(\tau)]^{\frac{2\omega-1}{\omega}}\right)}\right\}. \tag{11}$$

Theorem 1. For any right side $g(t) \in L_\infty\left(R;\ [\gamma(t)]^{\frac{3}{2\omega}-1}\right)$ integral equation (8) has a general solution $\varphi(t) \in L_\infty\left(R;\ [\gamma(t)]^{\frac{3}{2\omega}-1}\right)$:

$$\varphi(t) = g(t) + \int_0^t \left(\frac{\gamma(t)}{\gamma(\tau)}\right)^{\frac{3}{2\omega}-1} R_h(t,\tau)\, g(t)\, d\tau + C \cdot \varphi_{\mathrm{hom}}\left((2\omega-1)\, t^{2\omega-1}\right),$$

where $\varphi_{hom}(t)$ is the solution of the homogeneous equation, and for the resolvent $R_h(t,\tau)$ we have estimate (11).

Solution of integral equation (7). (Regularization method for solving the characteristic equation)

Using Remark 2, we consider equation (6), which we represent as:

$$\varphi(t) + \int_0^t K_h(t,\tau)\,\varphi(\tau)\, d\tau = \int_0^t [K_h(t,\tau) - K_\gamma(t,\tau)]\,\varphi(\tau)\, d\tau + F(t). \tag{12}$$

Assuming the right-hand side of equation (12) is temporarily known, we write its solution as:

$$\varphi(t)=\int\limits_0^t\left[K_h(t,\tau)-K_\gamma(t,\tau)\right]\varphi(\tau)\,d\tau+\int\limits_0^t R_h(t,\tau)\left\{\int\limits_0^\tau\left[K_h(\tau,\tau_1)-K_\gamma(\tau,\tau_1)\right]\varphi(\tau_1)\,d\tau_1\right\}d\tau+$$

$$+F(t)+\int\limits_0^t R_h(t,\tau)\cdot F(\tau)d\tau+C_0\cdot\varphi_{\text{hom}}\left((2\omega-1)\cdot t^{2\omega-1}\right).$$

In the iterated integral, we change the order of integration and change the roles of the variables τ and τ_1, then we obtain

$$\varphi(t)+\int_0^t\bar{\tilde{K}}(t,\tau)\cdot\varphi(\tau)\,d\tau=C_0\cdot\varphi_{\text{hom}}\left((2\omega-1)\cdot t^{2\omega-1}\right)+\hat{F}(t) \tag{13}$$

The kernel $\bar{\tilde{K}}(t,\tau)$ has the form:

$$\bar{\tilde{K}}(t,\tau)=\tilde{K}(t,\tau)+\bar{K}(t,\tau)$$

where

$$\tilde{K}(t,\tau)=K_h(t,\tau)-K_\gamma(t,\tau),\quad \bar{K}(t,\tau)=\int_{\tau_1}^t R(t,\tau_1)\left[K_h(\tau_1,\tau)-K_\gamma(\tau_1,\tau)\right]d\tau_1.$$

Let's introduce the following notations:

$$K_h^{(i)}(t,\tau)=P_h^{(i)}\exp\left\{-Q_h^{(i)}\right\};\quad K_\gamma^{(i)}(t,\tau)=P_\gamma^{(i)}\exp\left\{-Q_\gamma^{(i)}\right\},\quad i=1,2,3,4,$$

where

$$P_h^{(1)}(t,\tau)=\frac{1}{2a\sqrt{\pi}}\cdot\frac{(2\omega-1)^{\frac{3}{2}}\left([\gamma(\tau)]^{\frac{2\omega-1}{\omega}}\cdot[\gamma(t)]^{\frac{2\omega-2}{\omega}}+[\gamma(t)]^{\frac{4\omega-3}{\omega}}\right)\left\{[\gamma(\tau)]^{\frac{1}{\omega}}\right\}'}{\left([\gamma(t)]^{\frac{2\omega-1}{\omega}}-[\gamma(\tau)]^{\frac{2\omega-1}{\omega}}\right)^{\frac{3}{2}}},$$

$$Q_h^{(1)}(t,\tau)=\frac{(2\omega-1)\left([\gamma(t)]^{\frac{2\omega-1}{\omega}}+[\gamma(\tau)]^{\frac{2\omega-1}{\omega}}\right)^2}{4a^2\left([\gamma(t)]^{\frac{2\omega-1}{\omega}}-[\gamma(\tau)]^{\frac{2\omega-1}{\omega}}\right)}.$$

$$P_\gamma^{(1)}(t,\tau)=\frac{1}{2a\sqrt{\pi}}\cdot\frac{\gamma(t)+\gamma(\tau)}{(t-\tau)^{\frac{3}{2}}},\quad Q_\gamma^{(1)}(t,\tau)=\frac{(\gamma(t)+\gamma(\tau))^2}{4a^2(t-\tau)}.$$

Now we prove the following theorem.

Theorem 2. If function $\gamma(t)=[t(1+\alpha_0(t))]^\omega$, where $\alpha_0(t)=t^\beta\sigma(t),\quad \beta>0$, and function $\sigma(t)$ is twice continuously differentiable for $0<t<\infty$, and $|\sigma(t)|\le C$, $\sigma(t)\neq 0$, then we have an estimate:

$$|K_\gamma(t,\tau)-K_h(t,\tau)|\le C(\omega)\frac{t^{\omega-1}}{\sqrt{t-\tau}}\times\left[\exp\left(-Q_\gamma(t,\tau)/2\right)+\exp\left(-Q_h(t,\tau)/2\right)\right] \tag{14}$$

and the limit relation:

$$\lim_{t\to+0}\int\limits_0^t\left[K_\gamma(t,\tau)-K_h(t,\tau)\right]d\tau=0 \tag{15}$$

holds.

Note that estimate (14) is obvious for the terms

$$\left|K_h^{(i)}(t,\tau)-K_\gamma^{(i)}(t,\tau)\right| \text{ for } i=2,3,4.$$

Now we prove the estimate (14) for $i=1$.

Lemma 2. If $\alpha_0(t)$ is monotonically increasing function, then the inequalities:

$$2\omega-1\leq\frac{[t(1+\alpha_0(t))]^{2\omega-1}-[\tau(1+\alpha_0(\tau))]^{2\omega-1}}{[t(1+\alpha_0(t))]^{2\omega-2}(1+\alpha_0(t_1)+t_1\alpha_0'(t_1))(t-\tau)}\leq 1 \quad \text{for} \quad \frac{1}{2}<\omega<1,$$

$$1\leq\frac{[t(1+\alpha_0(t))]^{2\omega-1}-[\tau(1+\alpha_0(\tau))]^{2\omega-1}}{[t(1+\alpha_0(t))]^{2\omega-2}(1+\alpha_0(t_1)+t_1\alpha_0'(t_1))(t-\tau)}\leq 2\omega-1 \quad \text{for} \quad \omega\geq 1,$$

hold, where $t_1=\tau+\theta_1(t-\tau), \quad 0<\theta_1<1$.

Proof of the Lemma 2. It's obvious that [20; 456]:

$$2\omega-1\leq\frac{1-x^{2\omega-1}}{1-x}\leq 1, \text{ if } \frac{1}{2}<\omega<1, \quad 0\leq x\leq 1;$$

$$1\leq\frac{1-x^{2\omega-1}}{1-x}\leq 2\omega-1, \quad \text{if} \quad \omega\geq 1, \quad 0\leq x\leq 1.$$

Let $\omega\geq 1$, then for $0\leq\tau\leq t$ we get

$$[t(1+\alpha_0(t))]^{2\omega-2}(t-\tau+t\alpha_0(t)-\tau\alpha_0(\tau))\leq[t(1+\alpha_0(t))]^{2\omega-1}-[\tau(1+\alpha_0(\tau))]^{2\omega-1}\leq$$

$$\leq(2\omega-1)[t(1+\alpha_0(t))]^{2\omega-2}(t-\tau+t\alpha_0(t)-\tau\alpha_0(\tau)).$$

Using Lagrange's interpolation formula, we have

$$t\alpha_0(t)-\tau\alpha_0(\tau)=(\alpha_0(t_1)+t_1\alpha_0'(t_1))(t-\tau),$$

where $t_1=\tau+\theta_1(t-\tau)$, $0<\theta_1<1$.

The proof is analogously for the case $\frac{1}{2}<\omega<1$.

The following lemmas are proved in a similar way.

Lemma 3. If the function $\gamma(t)=[t(1+\alpha_0(t))]^{\omega}$, where $\alpha_0(t)=t^{\beta}\sigma(t), \beta>0$ and the function $\alpha_0(t)$ increases monotonically for $0<t<\infty$, and $|\sigma(t)|\leq C$, then estimate:

$$\left|P_h^{(1)}(t,\tau)-P_\gamma^{(1)}(t,\tau)\right|\leq C_2(\omega)\frac{t^{\omega+\beta}}{(t-\tau)^{\frac{3}{2}}}$$

holds.

Lemma 4. Under the conditions of Lemma 3, estimate:

$$\left|Q_h^1(t,\tau)-Q_\gamma^1(t,\tau)\right|\leq M_1\frac{t^{2\omega+\beta}}{t-\tau}+M_2t^{2\omega-1}$$

holds.

Proof of the Theorem 2. First, we establish the following inequality:

$$P_2(t,\tau)=\cdot\frac{[t(1+\alpha_0(t))]^{\omega}}{2\sqrt{\pi}(t-\tau)^{3/2}}\leq M_3(\omega)\cdot\frac{t^{\omega}}{(t-\tau)^{3/2}}.$$

For those values of parameter ω, $0<\tau<t<\infty$, for which $\left|Q_h^1(t,\tau)-Q_\gamma^1(t,\tau)\right|>0$, the required estimate follows from the following inequalities:

$$\left|K_h^1(t,\tau)-K_\gamma^1(t,\tau)\right|\leq\left|\left(P_h^1(t,\tau)-P_\gamma^1(t,\tau)\right)\exp\left\{-Q_\gamma(t,\tau)\right\}\right|+$$

$$+\left|P_\gamma^1(t,\tau)\exp\left\{-Q_h^1(t,\tau)\right\}\left(1-\exp\left\{-Q_\gamma^1(t,\tau)+Q_h^1(t,\tau)\right\}\right)\right|\leq$$

$$\leq\left|P_h^1(t,\tau)-P_\gamma^1(t,\tau)\right|\exp\left\{-Q_\gamma^1(t,\tau)\right\}+\left|P_\gamma^1(t,\tau)\left(Q_\gamma^1(t,\tau)-Q_h^1(t,\tau)\right)\exp\left\{-Q_h^1(t,\tau)\right\}\right|.$$

Hence, taking into account the Lemmas 2 – 4, we get:

$$|K_h-K_\gamma|\leq\left\{\bar{M}\frac{t^{\omega+\beta}}{(t-\tau)^{\frac{3}{2}}}+M_3\frac{t^{\omega}}{(t-\tau)^{\frac{3}{2}}}\left(M_1\frac{t^{2\omega+\beta}}{t-\tau}+M_2t^{2\omega-1}\right)\right\}e^{-Q_h}\leq$$

$$\leq \cdot \frac{t^{\omega-1}}{(t-\tau)^{\frac{1}{2}}}\left(\bar{M}\frac{t^{\beta+1}}{t-\tau}+\bar{M}_1\frac{t^{2\omega+\beta+1}}{(t-\tau)^2}+\bar{M}_2\frac{t^{2\omega}}{t-\tau}\right)\cdot e^{-Q_h} \leq$$

$$\leq \cdot \frac{t^{\omega-1}}{\sqrt{t-\tau}}\left(\frac{t}{t-\tau}\cdot e^{\frac{-Q_h}{2}}\cdot \bar{M}t^{\beta}+\frac{t^2}{(t-\tau)^2}e^{\frac{-Q_h}{2}}\cdot \bar{M}_1 t^{2\omega+\beta-1}+\right.$$

$$\left.+\frac{t}{t-\tau}e^{\frac{-Q_h}{2}}\cdot \bar{M}_2 t^{2\omega-1}\right)e^{\frac{-Q_h}{2}} \leq C(\omega)\left[\frac{\alpha(t)}{\alpha(\tau)}\right]^{\frac{\frac{3}{2}-\omega}{\omega}}\cdot \frac{t^{\omega-1}}{\sqrt{t-\tau}}e^{\frac{-Q_h}{2}}.$$

If the values of the parameter ω and $0<\tau<t<\infty$ are such that the difference $Q_\gamma(t,\tau)-Q_h(t,\tau)<0$, then it is enough in the same inequalities to interchange the functions $Q_\gamma(t,\tau)$ and $Q_h(t,\tau)$, $P_\gamma(t,\tau)$ and $P_h(t,\tau)$ accordingly.

The validity of inequality (14) shows that the difference $K_\gamma(t,\tau)-K_h(t,\tau)$ has a weak singularity and the limit relation (15) holds.

$$\lim_{t\to+0}\int_0^t \frac{t^{\omega-1}}{\sqrt{t-\tau}}\left[\exp\left(-Q_h(t,\tau)/2\right)+\exp\left(-Q_\gamma(t,\tau)/2\right)\right]d\tau=0.$$

Consequently, equation (8) is indeed characteristic equation for equation (7). Theorem 2 is proved.

In order to obtain estimates for $\bar{K}(t,\tau)$, and at the same time for $\bar{\bar{K}}(t,\tau)$ estimate for the resolvent, we represent in the form:

$$\left|R\left\{\frac{1}{2\omega-1}\left([\alpha(\eta)]^{\frac{1-2\omega}{\omega}}-[\alpha(t)]^{\frac{1-2\omega}{\omega}}\right)\right\}\right| \leq \tilde{M}_2\frac{t^{3/2}\eta^{\frac{3}{2}(2\omega-1)}}{(t-\eta)^{3/2}}\exp\left(-C_2(\omega)\frac{t\eta^{2\omega-1}}{t-\eta}\right). \qquad (16)$$

Here $C_j(\omega)$, $\tilde{M}_j(\omega)$, $j=1,2$ are constants depending only on ω.

Using estimates (14) and (16), we obtain the following theorem:

Theorem 3. If function $\gamma(t)=[t(1+\alpha_0(t))]^{\omega}$, where $\alpha_0(t)=t^{\beta}\sigma(t)$, $\beta>0$, and the function $\sigma(t)$ is twice continuously differentiable for $0<t<\infty$, and $|\sigma(t)|\leq C$, $\sigma(t)\neq 0$, then the kernel $\tilde{K}(t,\tau)$ has a weak singularity, i.e. we have an estimate:

$$\left|\bar{\bar{K}}(t,\tau)\right| \leq \frac{t^{1/2+\varepsilon}}{\tau^{3/2-\omega+\varepsilon}\,(t-\tau)^{1/2}},\, 0<\varepsilon<\omega-\frac{1}{2},\, 0<\tau<t<\infty, \qquad (17)$$

which means that integral equation (7) for any $f(t)$, $[\gamma(t)]^{\frac{3}{2\omega-1}}\cdot f(t)\in L_\infty\ (R_+)$ has a unique nonzero solution:

$$\varphi(t)\in L_\infty\left(R_\infty;[\gamma(t)]^{3/2\omega-1}\right).$$

Proof. Since $\bar{\bar{K}}(t,\tau)=\tilde{K}(t,\tau)+\bar{K}(t,\tau)$, then estimate (17) follows from (14), estimates for resolvent (16) and below relations. Using the following double inequality [19; 55]:

$$C_1 t^{\rho-1}(t-\tau)\leq t^{\rho}-\tau^{\rho}\leq C_2\tau^{\rho-1}(t-\tau), \text{где } C_1=\min\{1,\rho\},C_2=\max\{1,\rho\},$$

first we get ($\rho=2\omega-1$):

$$\bar{K}(t,\tau)\leq M_2(\omega)\int_\tau^t \eta^{-\gamma-1}\left(\frac{\eta}{\tau}\right)^{1-\gamma/2}\frac{\eta^{\omega-1}}{\sqrt{\eta-\tau}}\frac{t^{3/2}\eta^{3\omega 3/2}}{(t-\eta)^{3/2}}\exp\left(-C_2(\omega)\frac{t\eta^{2\omega-1}}{t-\eta}\right)d\eta=I_2(t,\tau).$$

We represent the function I_2 as a sum of two terms:

$$I_2(t,\tau)=I_{21}(t,\tau)+I_{22}(t,\tau),$$

for each of which we will have:

$$\leq \frac{1}{\sqrt{t-\tau}}\left[C_1(\omega)+C_2(\omega)\,(\tau/t)^{\varepsilon}\cdot\ln|\tau/t|\cdot(t/\tau)^{\varepsilon}\right]=\frac{1}{\sqrt{t-\tau}}\left[C_1(\omega)+C_3(\omega)\,(t/\tau)^{\varepsilon}\right].$$

For the second term

$$I_{22}(t,\tau)=\int\limits_{\frac{t+\tau}{2}}^{t}\frac{t\eta^{\omega-1}}{\sqrt{\eta-\tau}(t-\eta)^{\frac{3}{2}}}\exp\left(-C_2(\omega)\frac{t\eta^{2\omega-1}}{t-\eta}\right)d\eta\leq$$

$$\leq\frac{C(\omega)}{\sqrt{t-\tau}}\int\limits_{\frac{t+\tau}{2}}^{t}\frac{t\eta^{\omega-1}}{(t-\eta)^{\frac{3}{2}}}\exp\left(-C_3(\omega)\frac{t(t+\tau)^{2\omega-1}}{t-\eta}\right)d\eta\leq$$

$$\leq\frac{C(\omega)}{\sqrt{t-\tau}}\int\limits_{\frac{t+\tau}{2}}^{t}\frac{t\eta^{\omega-1}}{(t-\eta)^{\frac{3}{2}}}\exp\left(-C_3(\omega)\frac{t^{2\omega}}{t-\eta}\left\{1+\frac{\tau}{t}\right\}^{2\omega-1}\right)d\eta\leq$$

$$\leq\frac{C(\omega)}{\sqrt{t-\tau}}\int\limits_{0}^{t}\frac{t^{\omega}}{(t-\eta)^{\frac{3}{2}}}\exp\left(-C_4(\omega)\frac{t^{2\omega}}{t-\eta}\right)d\eta=\frac{C_5(\omega)}{\sqrt{t-\tau}}\int\limits_{t^{\frac{2\omega-1}{2}}}^{\infty}\exp\left\{-z^2\right\}dz\leq\frac{C_5(\omega)}{\sqrt{t-\tau}}.$$

In these inequalities, the constants $C(\omega)$, $C_j(\omega)$, $j=1,2,3,4,5$ are different and depend only on ω. The obtained inequalities imply the required estimate (17). This completes the proof of the Theorem.

Remark 3. From relation (13) it follows that homogeneous equation

$$\varphi(t)-\int_0^t K_\omega(t,\tau)\mu(\tau)d\tau=0, t\in R_+,$$

is equivalent to the nonhomogeneous equation:

$$\varphi\left(t\right)+\int_0^t \bar{\bar{K}}\left(t,\tau\right)\cdot\varphi\left(\tau\right)d\tau=C_0\cdot\varphi_{\text{hom}}\left((2\omega-1)\cdot t^{2\omega-1}\right).$$

Study of the boundary value problem

A solution of the original boundary value problem (1)–(2) have the form:

$$u(x,t)=\int_0^x v(\xi,t)d\xi, \tag{18}$$

where $v(x,t)=v_{\text{hom}}(x,t)+v_{part}(x,t)$, and

$$v_{\text{hom}}(x,t)=\frac{1}{2a\sqrt{\pi}}\int_0^t\frac{1}{(t-\tau)^{1/2}}\left[-\exp\left\{-\frac{(x+\gamma(\tau))^2}{4a^2(t-\tau)}\right\}+\exp\left\{-\frac{(x-\gamma(\tau))^2}{4a^2(t-\tau)}\right\}\right]\cdot\varphi_0\left(\tau\right)d\tau \tag{19}$$

$$v_{part}(x,t)=\frac{1}{2a\sqrt{\pi}}\int_0^t\frac{1}{(t-\tau)^{1/2}}\left[-\exp\left\{-\frac{(x+\gamma(\tau))^2}{4a^2(t-\tau)}\right\}+\exp\left\{-\frac{(x-\gamma(\tau))^2}{4a^2(t-\tau)}\right\}\right]\cdot\varphi_{part}\left(\tau\right)d\tau+$$
$$+\frac{1}{2a\sqrt{\pi}}\int_0^t\int_0^\infty\frac{1}{(t-\tau)^{1/2}}\left[-\exp\left\{-\frac{(x+\xi)^2}{4a^2(t-\tau)}\right\}+\exp\left\{-\frac{(x-\xi)^2}{4a^2(t-\tau)}\right\}\right]\cdot\tilde{f}(\xi,\tau)d\xi\,d\tau+$$
$$+\frac{1}{2a\sqrt{\pi}}\int_0^t\frac{x}{(t-\tau)^{3/2}}\exp\left\{-\frac{x^2}{4a^2(t-\tau)}\right\}v_0(\tau)d\tau, \tag{20}$$

where the functions $(\gamma(t))^{\frac{3/2-\omega}{\omega}}\cdot\varphi(t)$ and $(\gamma(t))^{\frac{3/2-\omega}{\omega}}\cdot\tilde{f}(x,t)$ are bounded and continuous functions on R_+ and Q, respectively.

From (18)–(20) we obtain the following estimates:

$$u_{hom}(x,t)\leq a\sqrt{\pi}\left\{-2\,\mathrm{erf}\left(\frac{x}{2a\sqrt{2\omega-1}\cdot\left[\gamma(t)\right]^{\frac{2\omega-1}{2\omega}}}\right)+\right.$$

$$+2\exp\left\{\frac{(2\omega-1)\,[\gamma(t)]^{\frac{2\omega-1}{\omega}}}{4a^2}\right\}\mathrm{erfc}\left(\frac{\sqrt{2\omega-1}\cdot[\gamma(t)]^{\frac{2\omega-1}{2\omega}}}{2a}\right)-$$

$$-\exp\left\{\frac{2x+(2\omega-1)\,[\gamma(t)]^{\frac{2\omega-1}{\omega}}}{4a^2}\right\}\mathrm{erfc}\left(\frac{x+(2\omega-1)\,[\gamma(t)]^{\frac{2\omega-1}{\omega}}}{2a\sqrt{2\omega-1}\cdot[\gamma(t)]^{\frac{2\omega-1}{2\omega}}}\right)-$$

$$-\exp\left\{-\frac{2x-(2\omega-1)\,[\gamma(t)]^{\frac{2\omega-1}{\omega}}}{4a^2}\right\}\mathrm{erfc}\left(-\frac{x-(2\omega-1)\,[\gamma(t)]^{\frac{2\omega-1}{\omega}}}{2a\sqrt{2\omega-1}\,[\gamma(t)]^{\frac{2\omega-1}{2\omega}}}\right)\Bigg\}.$$

For $u_{part}(x,t)$, we have that

$$u_{part}(x,t)\le a\sqrt{\pi}\left\{-2\,\mathrm{erf}\left(\frac{x}{2a\sqrt{2\omega-1}\cdot[\gamma(t)]^{\frac{2\omega-1}{2\omega}}}\right)+\right.$$

$$+2\exp\left\{\frac{(2\omega-1)\,[\gamma(t)]^{\frac{2\omega-1}{\omega}}}{4a^2}\right\}\mathrm{erfc}\left(\frac{\sqrt{2\omega-1}\cdot[\gamma(t)]^{\frac{2\omega-1}{2\omega}}}{2a}\right)-$$

$$-\exp\left\{\frac{2x+(2\omega-1)\,[\gamma(t)]^{\frac{2\omega-1}{\omega}}}{4a^2}\right\}\mathrm{erfc}\left(\frac{x+(2\omega-1)\,[\gamma(t)]^{\frac{2\omega-1}{\omega}}}{2a\sqrt{2\omega-1}\cdot[\gamma(t)]^{\frac{2\omega-1}{2\omega}}}\right)-$$

$$-\exp\left\{-\frac{2x-(2\omega-1)\,[\gamma(t)]^{\frac{2\omega-1}{\omega}}}{4a^2}\right\}\mathrm{erfc}\left(-\frac{x-(2\omega-1)\,[\gamma(t)]^{\frac{2\omega-1}{\omega}}}{2a\sqrt{2\omega-1}\,[\gamma(t)]^{\frac{2\omega-1}{2\omega}}}\right)\Bigg\}+\frac{1}{\sqrt{\pi}}\,\mathrm{erf}\left(\frac{\sqrt{2\omega-1}\,[\gamma(t)]^{\frac{2\omega-1}{2\omega}}}{2a}\right).$$

Therefore estimates of these integrals give the statement of the Theorem.

Theorem 5. For any right side $f(t)\in L_\infty\Big(\mathrm{R}_+;\ [\gamma(t)]^{\frac{3/2\,-\,\omega}{\omega}}\ \exp\ \big\{\gamma(t)/(4a^2)\big\}\Big)$ and for given functions $f(x,t)\in W^{1,0}_\infty\Big(G;[\gamma(t)]^{3/2\omega-1}\exp\Big\{[\gamma(t)]^{\frac{2\omega-1}{\omega}}/(4a^2)\Big\}\Big)$, $u_0(t)\in L_\infty(R_+;[\gamma(t)]^{\frac{\omega-3/2}{\omega}})$; $u_1(t)\in L_\infty(\mathrm{R}_+;$ $[\gamma(t)]^{\frac{3/2-\omega}{\omega}})$ boundary value problem (1)–(2) has a general solution $u(x,t)\in L_\infty(G;\big(x+[\gamma(t)]^{3/2\omega-1})^{-1}\big)$, which is determined from formula (18)–(20).

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Шекарасы $x = \gamma(t)$ заңдылығымен қозғалатын Солонников-Фазан есебінің шешімі туралы

Жұмыста бастапқы мезетте жойылатын облыстағы жылуөткізгіштік теңдеу үшін шекаралық есептің шешімі зерттелген. Мұнда, шекарасы уақытқа байланысты $x = \gamma(t)$ заңдылығымен өзгереді. Қарастырылып отырған есеп жалпыланған жылу потенциалдарының көмегімен псевдо-вольтерралық интегралдық теңдеуге келтіріледі. Ал интегралдық оператордың нормасының бірге тең болуы оның ерекшелігі болып табылады. Сонымен қатар, сәйкес біртекті интегралдық теңдеудің нөлдік емес шешімінің болатыны көрсетілген.

Кілт сөздер: жылуөткізгіштік теңдеу, жылжымалы шекара, жойылатын облыс, псевдо-вольтерралық интегралдық теңдеу.

М.Т. Дженалиев, М.И. Рамазанов, А.О. Танин

К решению задачи Солонникова-Фазано при движении границы по произвольному закону $x = \gamma(t)$

В работе исследованы вопросы разрешимости граничной задачи для уравнения теплопроводности в области, которая вырождается в точку в начальный момент времени. При этом изменяющаяся со временем граница движется по произвольному закону $x = \gamma(t)$. С помощью обобщенных тепловых потенциалов исследуемая задача редуцируется к псевдо-вольтерровому интегральному уравнению, особенность которого заключается в том, что норма интегрального оператора равна единице. Показано, что соответствующее однородное интегральное уравнение имеет ненулевое решение.

Ключевые слова: уравнение теплопроводности, подвижная граница, вырождающаяся область, псевдовольтерровое интегральное уравнение.

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