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The problem of trigonometric Fourier series multipliers of classes in $\lambda_{p,q}$ spaces

In this article, we consider weighted spaces of numerical sequences $\lambda_{p,q}$, which are defined as sets of sequences $a = \{a_k\}_{k=1}^{\infty}$, for which the norm

$$\|a\|_{\lambda_{p,q}} := \left(\sum_{k=1}^{\infty} |a_k|^q k^{\frac{q}{p}-1}\right)^{\frac{1}{q}} < \infty$$

is finite. In the case of non-increasing sequences, the norm of the space $\lambda_{p,q}$ coincides with the norm of the classical Lorentz space $l_{p,q}$. Necessary and sufficient conditions are obtained for embeddings of the space $\lambda_{p,q}$ into the space λ_{p_1,q_1}. The interpolation properties of these spaces with respect to the real interpolation method are studied. It is shown that the scale of spaces $\lambda_{p,q}$ is closed in the relative real interpolation method, as well as in relative to the complex interpolation method. A description of the dual space to the weighted space $\lambda_{p,q}$ is obtained. Specifically, it is shown that the space is reflective, where p', q' are conjugate to the parameters p and q. The paper also studies the properties of the convolution operator in these spaces. The main result of this work is an O'Neil type inequality. The resulting inequality generalizes the classical Young-O'Neil inequality. The research methods are based on the interpolation theorems proved in this paper for the spaces $\lambda_{p,q}$.

Keywords: trigonometric Fourier coefficients, O'Neil inequality, convolution operator, $M_{p_0,q_0}^{p_1,q_1}$ class.

Introduction

Let $1 \le p \le \infty$, $L_p \equiv L_p(\mathbb{R})$ and let the convolution operator be given by

$$(Af)(x) = (K * f)(x) = \int_{\mathbb{R}} K(x-y)f(y)dy.$$

The Young convolution inequality

$$\|A\|_{L_p \to L_q} \le \|K\|_{L_r}, \qquad 1 + \frac{1}{q} = \frac{1}{p} + \frac{1}{r}, \qquad 1 \le p \le q \le \infty,$$

has a very important role both in Harmonic Analysis and PDE (see, e.g., [1, Ch. 4, § 2, 4], [2]). $K(x) = |x|^{-\gamma}$, $\gamma > 0$. Young's estimates were generalized by O'Neil [3] who showed that for $1 < p < q < \infty$, $0 < t, s_1, s_2, \le \infty$, $1/r = 1 - 1/p + 1/q$ and $1/t = 1/s_1 + 1/s_2$

$$\|A\|_{L_{p,s_1} \to L_{q,s_2}} \le C \|K\|_{L_{r,t}},$$

and in particular

$$\|A\|_{L_p \to L_q} \le C \|K\|_{L_{r,\infty}}, \tag{1.1}$$

where $L_{p,s}$ is Lorentz space.

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Note that inequality (2.2) unlike (2.1) gives the Hardy-Littlewood fractional integration theorem, which corresponds to the model case in which $K(x) = |x|^{-1/r}$.

When $1 \le p \le q \le \infty$, were considered in [4], [5]. The estimate (1.1) was improved in [6], [7].

There are several generalizations of both Young and O'Neil's inequalities for various function spaces (weighted L_p, classical and weighted Lorentz spaces, weighted Besov and Hardy spaces, Wiener amalgam spaces, Orlicz spaces; see, e.g., [8], [9], [10], [11], [12], [13], [14], [15], [16] and references there in). We also remark that the sharp Young convolution inequality was obtained in [17] and [18].

Note that the norm estimates for convolution operators in various spaces are closely related to the problem of multipliers of Fourier transforms and Fourier series [19], [20], [21], [22], [23].

Let $0 < p < \infty,\ 0 < q \le \infty$. It is said that the sequence $a = \{a_k\}_{k=-\infty}^{\infty}$ belongs to the class $\lambda_{p,q}$, if

$$\|a\|_{\lambda_{p,q}} = \left(\sum_{k=-\infty}^{\infty} |a_k|^q \bar{k}^{\frac{q}{p}-1} \right)^{\frac{1}{q}} < \infty,$$

where $\bar{k} = \max(|k|, 1)$, and

$$\left\|a\right\|_{\lambda_{p,\infty}} = \sup_k |a_k| \bar{k}^{\frac{1}{p}} < \infty,$$

if $q = \infty$.

Our aim is to study the convolution inequalities in weighted spaces $\lambda_{p,q}$.

Throughout this paper, $F \lesssim G$ means that $F \le CG$; by C we denote positive constants that may be different on different occasions. Moreover, $F \asymp G$ means that $F \lesssim G \lesssim F$.

2. Properties of the spaces $\lambda_{p,q}$

Lemma 2.1 For embedding

$$\lambda_{p_0,q_0} \hookrightarrow \lambda_{p_1,q_1} \tag{2.1}$$

to hold it is necessary and sufficient that
for $q_0 \le q_1$, $\frac{1}{q_0} - \frac{1}{q_1} \le \frac{1}{p_0} - \frac{1}{p_1}$,
for $q_1 < q_0$, $\frac{1}{p_0} - \frac{1}{p_1} > 0$.

Proof. We consider the case $q_0 \le q_1$. Let $\frac{1}{q_0} - \frac{1}{q_1} \le \frac{1}{p_0} - \frac{1}{p_1}$. Then, using the inequality $(a+b)^\alpha \le$ $\le a^\alpha + b^\alpha$ for $\alpha < 1$, we get:

$$\left\|a\right\|_{\lambda_{p_1,q_1}} = \left(\sum_{k=-\infty}^{\infty} |a_k|^{q_1} \bar{k}^{(\frac{q_1}{p_1}-1)} \right)^{\frac{1}{q_1}} = \left(\sum_{k=-\infty}^{\infty} \left(|a_k|^{q_1} \bar{k}^{(\frac{1}{p_1}-\frac{1}{q_1})} \right)^{q_1} \right)^{\frac{1}{q_1}}$$

$$\le \left(\sum_{k=-\infty}^{\infty} \left(|a_k| \bar{k}^{(\frac{1}{p_0}-\frac{1}{q_0})} \right)^{q_1} \right)^{\frac{1}{q_1}} \le \left(\sum_{k=-\infty}^{\infty} \left(|a_k| \bar{k}^{(\frac{1}{p_0}-\frac{1}{q_0})} \right)^{q_0} \right)^{\frac{1}{q_0}} = \left\|a\right\|_{\lambda_{p_0,q_0}}.$$

Thus,

$$\lambda_{p_0,q_0} \hookrightarrow \lambda_{p_1,q_1}.$$

On the other hand, let $m \in \mathbb{N}$ and we consider the sequence $a = \{a_k\}_{k=-\infty}^{\infty}$:

$$a_k = \begin{cases} 1, & k = m \\ 0, & \text{in otherwise.} \end{cases}$$

Then according to embedding (2.1),

$$\bar{m}^{\left(\frac{1}{p_1}-\frac{1}{q_1}\right)} = \left\|a\right\|_{\lambda_{p_1,q_1}} \le c \left\|a\right\|_{\lambda_{p_0,q_0}} = c\bar{m}^{\left(\frac{1}{p_0}-\frac{1}{q_0}\right)}.$$

Since m is arbitrary, which is possible if only if $\frac{1}{q_0} - \frac{1}{q_1} \le \frac{1}{p_0} - \frac{1}{p_1}$.

Let us pass to a case $q_1 < q_0$. Let $\frac{1}{p_0} - \frac{1}{p_1} > 0$. Denote $\varepsilon = \frac{1}{p_0} - \frac{1}{p_1} - \frac{1}{q_0} + \frac{1}{q_1}$. Further applying the Hölder inequality with the following parameters r_1 and r_2 such that $\frac{1}{r_1} = \frac{1}{q_0}$, $\frac{1}{r_2} = \frac{1}{q_1} - \frac{1}{q_0}$, $\left(\frac{1}{r_1} + \frac{1}{r_2} = \frac{1}{q_1}\right)$, we get

$$\|a\|_{\lambda_{p_1,q_1}} = \left(\sum_{k=-\infty}^{\infty} \left(|a_k| \bar{k}^{\frac{1}{p_0} - \frac{1}{q_0} - \varepsilon}\right)^{q_1}\right)^{\frac{1}{q_1}} = \left(\sum_{\bar{k}=-\infty}^{\infty} \left(|a_k| \bar{k}^{\left(\frac{1}{p_0} - \frac{1}{q_0}\right)} \bar{k}^{-\varepsilon}\right)^{q_1}\right)^{\frac{1}{q_1}}$$

$$\leq \|a\|_{\lambda_{p_0,q_0}} \left(\sum_{k=-\infty}^{\infty} \bar{k}^{-\varepsilon r_2}\right)^{\frac{1}{r_2}}.$$

Moreover $\varepsilon r_2 = \left[\left(\frac{1}{p_0} - \frac{1}{p_1}\right) + \left(\frac{1}{q_1} - \frac{1}{q_0}\right)\right] \left(\frac{1}{q_1} - \frac{1}{q_0}\right)^{-1} > 1$. We have $\sum\limits_{k=-\infty}^{\infty} \bar{k}^{-\varepsilon r_2} < \infty$.

To prove the necessity, we suppose that the embedding $\lambda_{p_0,q_0} \hookrightarrow \lambda_{p_1,q_1}$ and $m \in \mathbb{N}$ holds. We consider the sequence $a = \{a_k\}_{k=-\infty}^{\infty}$ such that

$$a_k = \begin{cases} k^{\alpha}, & 0 \leq k \leq m \\ 0, & \text{in otherwise.} \end{cases}$$

.

We have $\|a\|_{\lambda_{p_1,q_1}} = \left(\sum\limits_{i=1}^{m} \bar{i}^{\left(\frac{q_1}{p_1} + \alpha q_1 - 1\right)}\right)^{\frac{1}{q_1}} = c_1 \bar{m}^{\left(\frac{1}{p_1} + \alpha\right)}$,

$\|a\|_{\lambda_{p_0,q_0}} = \left(\sum\limits_{i=1}^{m} \bar{i}^{\left(\frac{q_0}{p_0} + \alpha q_0 - 1\right)}\right)^{\frac{1}{q_0}} = c_2 \bar{m}^{\left(\frac{1}{p_0} + \alpha\right)}$.

Therefore, since $\lambda_{p_0,q_0} \hookrightarrow \lambda_{p_1,q_1}$, and m is arbitrary we have that $\frac{1}{p_0} \geq \frac{1}{p_1}$.

In the case $\frac{1}{p_0} = \frac{1}{p_1}$. We consider the sequence $a = \{a_k\}_{k=-\infty}^{\infty}$: when

$$a_k = \begin{cases} \bar{k}^{\frac{1}{p_0}} \ln^{-\frac{1}{q_0 - \varepsilon}} \bar{k}, & 2 \leq k \\ 0, & \text{in otherwise,} \end{cases}$$

where ε is chosen so that $q_1 < q_0 - \varepsilon$. Then since $\frac{q_0}{q_0 - \varepsilon} > 1$, we get

$$\|a\|_{\lambda_{p_0,q_0}} = \left(\sum_{k=2}^{\infty} \left(\bar{k}^{-\frac{1}{p_0}} \ln^{-\frac{1}{q_0 - \varepsilon}} \bar{k}\right)^{q_0} \bar{k}^{\left(\frac{q_0}{p_0} - 1\right)}\right)^{\frac{1}{q_0}} = \left(\sum_{k=2}^{\infty} \frac{\ln^{-\frac{q_0}{q_0 - \varepsilon}} \bar{k}}{\bar{k}}\right)^{\frac{1}{q_0}} < \infty.$$

On the other hand since $\frac{q_1}{q_0 - \varepsilon} < 1$, we have

$$\|a\|_{\lambda_{p_1,q_1}} = \left(\sum_{k=2}^{\infty} \frac{1}{\bar{k} \ln^{\frac{q_1}{q_0 - \varepsilon}} \bar{k}}\right)^{\frac{1}{q_1}} = \infty.$$

Therefore, the condition $\frac{1}{p_0} > \frac{1}{p_1}$ is necessary.

Lemma 2.2 Let $0 < p_0, q_0, p_1, q_1 < \infty$, $\quad 0 < \theta < 1$, then

$$(\lambda_{p_0,q_0}, \lambda_{p_1,q_1})_{\theta,q} = \lambda_{p,q},$$

$$\frac{1}{q} = \frac{1 - \theta}{q_0} + \frac{\theta}{q_1}, \quad \frac{1}{p} = \frac{1 - \theta}{p_0} + \frac{\theta}{p_1}.$$

Proof. By the well-known theorem of powers (see [24, Th. 3.11.6]), we have

$$\left((\lambda_{p_0,q_0})^{q_0}, (\lambda_{p_1,q_1})^{q_1}\right)_{\eta,1} = \left((\lambda_{p_0,q_0}, \lambda_{p_1,q_1})_{\theta,q}\right)^q,$$

where $\eta = \frac{\theta q}{q_1}$.

The norm of the element x in the space $\left((\lambda_{p_0,q_0})^{q_0}, (\lambda_{p_1,q_1})^{q_1}\right)_{\eta,1}$ is equal to

$$\int_0^\infty t^{-\eta} \inf_{x=x^0+x^1} \left(\sum_{k=-\infty}^{\infty} |x_k^0|^{q_0} \bar{k}^{\left(\frac{q_0}{p_o}-1\right)} + t \sum_{k=-\infty}^{\infty} |x_k^1|^{q_1} \bar{k}^{\left(\frac{q_1}{p_1}-1\right)} \right) \frac{dt}{t}$$

$$= \sum_k \int_0^\infty t^{-\eta} |x_k|^{q_0} \bar{k}^{\frac{q_0}{p_0}-1} \inf_{x_k=x_k^0+x_k^1} \left(\left|\frac{x_k{}^0}{x_k}\right|^{q_0} + t \left|\frac{x_k{}^1}{x_k}\right|^{q_1} |x_k|^{q_1-q_0} \bar{k}^{\frac{q_1}{p_1}-\frac{q_0}{p_0}} \right) \frac{dt}{t}$$

$$= \sum_{k=-\infty}^{\infty} \int_0^\infty \left(s^{-1} |x_k|^{q_1-q_0} \bar{k}^{\frac{q_1}{p_1}-\frac{q_0}{p_0}} \right)^\eta |x_k|^{q_0} \bar{k}^{\frac{q_0}{p_0}-1} \inf_{1=y_k^0+y_k^1} \left(|y_k^0|^{q_0} + s|y_k^1|^{q_1} \right) \frac{ds}{s}.$$

Considering that $\inf\limits_{1=y_k^0+y_k^1} \left(|y_k^0|^{q_0} + s|y_k^1|^{q_1} \right) \sim \min(1,s)$ and $\eta = \frac{\theta q}{q_1}$ the last expression is equal to $c \sum\limits_{k=-\infty}^{\infty} |x_k|^q \bar{k}^{\left(\frac{q}{p}-1\right)}$, whence the statement of the lemma follows.

Lemma 2.3 Let $0 < q < \infty,\ 0 < s \le \infty,\ \ 0 < \theta < 1$, then

$$[\lambda_{q,s}, \lambda_{q,\infty}]_\theta = \lambda_{q,t},$$

where $\frac{1}{t} = \frac{1-\theta}{s}$.

Proof. The interpolation theorem (see [25], p. 142) concerning to the complex interpolation method is known

$$[l_{p_0}(A_k), l_\infty(B_k)]_\theta = l_p([A_k, B_k]_\theta)$$

here $1 \le p_0 < \infty,\ \ 0 < \theta < 1,\ \ \frac{1}{p} = \frac{1-\theta}{p_0}$. In our case, the spaces $\lambda_{q,s}, \lambda_{q,\infty}$ can be represented as follows

$$\lambda_{q,s} = l_s(A_k), \qquad \lambda_{q,\infty} = l_\infty(B_k),$$

where $\|\cdot\|_{A_k} = |\cdot|\bar{k}^{\frac{1}{q}-\frac{1}{s}}, \qquad \|\cdot\|_{B_k} = |\cdot|\bar{k}^{\frac{1}{q}}$.

Therefore, we have

$$[\lambda_{q,s}, \lambda_{q,\infty}]_\theta = [l_s(A_k), l_\infty(B_k)]_\theta = l_t(C_k),$$

here $\frac{1}{t} = \frac{1-\theta}{s}$,

$$\|\cdot\|_{C_k} = |\cdot|\bar{k}^{(1-\theta)(\frac{1}{q}-\frac{1}{s})+\frac{\theta}{q}} = |\cdot|\bar{k}^{\frac{1}{q}-\frac{1-\theta}{s}} = |\cdot|\bar{k}^{\frac{1}{q}-\frac{1}{t}}$$

i.e. $l_t(C_k) = \lambda_{q,t}$

Let X be a linear normed space of numerical sequences. We define the dual space X' as a set of sequences $a = \{a_k\}_{k\in\mathbb{Z}}$ for which

$$\|a\|_{X'} := \sup_{\|b\|_X=1} \sum_{k\in\mathbb{Z}} a_k b_k.$$

Lemma 2.4 Let $1 < p < \infty,\ 1 \le q \le \infty,\ \ \frac{1}{p} + \frac{1}{p'} = \frac{1}{q} + \frac{1}{q'} = 1$, then

$$(\lambda_{p,q})' = \lambda_{p',q'}.$$

Proof. The statement of Lemma follows from equality

$$\|a\|_{\lambda_{p',q'}} = \sup_{\|b\|_{\lambda_{p,q}=1}} \sum_{k\in\mathbb{Z}} a_k b_k \tag{2.2}$$

which could be proved using the Hölder inequality.

3. Convolution in the spaces $\lambda_{p,q}$

Let $a=\{a_k\}_{k=-\infty}^{\infty}$, $b=\{b_k\}_{k=-\infty}^{\infty}$ be such that

$$\sum_{k=-\infty}^{\infty} a_k b_{k-m} < \infty\ , \quad m\in\mathbb{Z}.$$

The sequence

$$\left\{\sum_{k=-\infty}^{\infty} a_k b_{k-m}\right\}_{m=-\infty}^{\infty}$$

will be called a convolution and denoted by $a*b$.

Lemma 3.1 Let $1<r,p,q<\infty$ and $\frac{1}{q}+1=\frac{1}{r}+\frac{1}{p}$.

Then

$$\|a*b\|_{\lambda_{q,\infty}} \le c\|a\|_{\lambda_{r,\infty}}\|b\|_{\lambda_{p,\infty}}.$$

Proof. By the definition of nonconforming transformations, we have

$$\left\|a*b\right\|_{\lambda_{q,\infty}} = \sup_k \left|\left(\sum_{m=-\infty}^{\infty} a_m b_{m-k}\right)\right| \bar{k}^{\frac{1}{q}} \le \sup_m |a_m|\bar{m}^{\frac{1}{r}} \sup_k \sum_{m=-\infty}^{\infty} |b_{k-m}|\bar{m}^{-\frac{1}{r}}\bar{k}^{\frac{1}{q}}$$

$$= \left\|a\right\|_{\lambda_{r,\infty}} \sup_k \bar{k}^{\frac{1}{q}} \sum_{m=-\infty}^{\infty} |b_m|(\overline{m-k})^{-\frac{1}{r}} \le \left\|a\right\|_{\lambda_{r,\infty}} \left\|b\right\|_{\lambda_{p,\infty}} \sup_k \bar{k}^{\frac{1}{q}} \sum_{m=-\infty}^{\infty} (\overline{m-k})^{-\frac{1}{r}}\bar{m}^{-\frac{1}{p}}.$$

Note that for $\bar{k}\ne\bar{0}$

$$\sum_{m=-\infty}^{\infty} (\overline{m-k})^{-\frac{1}{r}}\bar{m}^{-\frac{1}{p}} \asymp \int_{\mathbb{R}} \frac{dx}{|x-k|^{\frac{1}{r}}|x|^{\frac{1}{p}}} = k^{1-1/p-1/r}\int_{\mathbb{R}} \frac{dx}{|x-1|^{\frac{1}{r}}|x|^{\frac{1}{p}}} = ck^{-\frac{1}{q}}.$$

Therefore, we have

$$\left\|a*b\right\|_{\lambda_{q,\infty}} \le c\left\|a\right\|_{\lambda_{r,\infty}}\left\|b\right\|_{\lambda_{p,\infty}}.$$

Lemma 3.2 Let one of the following conditions be fulfilled:
either $0<s\le 1$, $0<s<p,r<q<\infty$, $\frac{1}{s}+\frac{1}{q}=\frac{1}{p}+\frac{1}{r}$
or $1<s\le\infty$, $1<p,r<q<\infty$, $1+\frac{1}{q}=\frac{1}{p}+\frac{1}{r}$.
Then the following inequalities hold:

$$\left\|a*b\right\|_{\lambda_{q,s}} \le c\left\|a\right\|_{\lambda_{r,s}}\left\|b\right\|_{\lambda_{p,\infty}},$$

$$\left\|a*b\right\|_{\lambda_{q,s}} \le c\left\|a\right\|_{\lambda_{r,\infty}}\left\|b\right\|_{\lambda_{p,s}}.$$

Proof. Let $0<s\le 1$. According to the Jensen's inequality we have

$$\|a*b\|_{\lambda_{q,s}} = \left(\sum_{k=-\infty}^{\infty}\left|\sum_{m=-\infty}^{\infty} a_m b_{k-m}\right|^s \bar{k}^{\left(\frac{s}{q}-1\right)}\right)^{\frac{1}{s}} \le \left(\sum_{k=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} |a_m b_{m-k}|^s\,\bar{k}^{\left(\frac{s}{q}-1\right)}\right)^{\frac{1}{s}}$$

$$= \left(\sum_{m=-\infty}^{\infty}|a_m|^s\sum_{k=-\infty}^{\infty}|b_{k-m}|^s\bar{k}^{\left(\frac{s}{q}-1\right)}\right)^{\frac{1}{s}} \le \|b\|_{\lambda_{r,\infty}}\left(\sum_{m=-\infty}^{\infty}|a_m|^s\sum_{k=-\infty}^{\infty}(\overline{k-m})^{-\frac{s}{p}}\bar{k}^{\left(\frac{s}{q}-1\right)}\right)^{\frac{1}{s}}.$$

Considering that

$$\sum_{k=-\infty}^{\infty} (\overline{k-m})^{-\frac{s}{p}} \bar{k}^{\left(\frac{s}{q}-1\right)} \asymp \bar{m}^{-\frac{s}{p}+\frac{s}{q}} = \bar{m}^{\frac{s}{r}-1},$$

we have $0 < s \leq 1$, $\frac{1}{s}+\frac{1}{q}=\frac{1}{p}+\frac{1}{r}$

$$\left\|a*b\right\|_{\lambda_{q,s}} \leq c \left\|a\right\|_{\lambda_{r,\infty}} \left\|b\right\|_{\lambda_{p,s}}$$

and in particular

$$\left\|a*b\right\|_{\lambda_{q,1}} \leq c \left\|a\right\|_{\lambda_{r,\infty}} \left\|b\right\|_{\lambda_{p,1}}.$$

Using Lemma 3.1 we have

$$\left\|a*b\right\|_{\lambda_{q,\infty}} \leq c \left\|a\right\|_{\lambda_{r,\infty}} \left\|b\right\|_{\lambda_{p,\infty}}.$$

Applying the bilinear interpolation theorem [2; Theorem 4.4.1] we obtain

$$\left\|a*b\right\|_{[\lambda_{q,1},\lambda_{q,\infty}]_\theta} \leq c\|a\|_{[\lambda_{r,\infty},\lambda_{r,\infty}]_\theta} \left\|b\right\|_{[\lambda_{p,1},\lambda_{p,\infty}]_\theta}.$$

Moreover, by Lemma 3.1 we obtain

$$\left\|a*b\right\|_{\lambda_{q,s}} \leq c \left\|a\right\|_{\lambda_{r,\infty}} \left\|b\right\|_{\lambda_{p,s}},$$

where $\frac{1}{s}=1-\theta$, $\theta\in[0,1]$, $1+\frac{1}{q}=\frac{1}{p}+\frac{1}{r}$.

The second inequality is proved symmetrically.

For the proof of further results we need next statement also of independent interest.

Theorem 3.1 Let $0 < s, t_1, t_2 \leq \infty$, $\frac{1}{s}=\frac{1}{t_1}+\frac{1}{t_2}$. Let one of the following conditions be fulfilled: either $0 < s \leq 1$, $0 < s < p, r < q < \infty$, $\frac{1}{s}+\frac{1}{q}=\frac{1}{p}+\frac{1}{r}$ or $1 < s \leq \infty$, $1 < p, r < q < \infty$, $1+\frac{1}{q}=\frac{1}{p}+\frac{1}{r}$. Then

$$\left\|a*b\right\|_{\lambda_{q,s}} \leq c \left\|a\right\|_{\lambda_{r,t_1}} \left\|b\right\|_{\lambda_{p,t_2}}.$$

Proof. According to the Lemma 3.2 the following basic inequalities are known

$$\left\|a*b\right\|_{\lambda_{q,s}} \leq c \left\|a\right\|_{\lambda_{r,s}} \left\|b\right\|_{\lambda_{p,\infty}}.$$

$$\left\|a*b\right\|_{\lambda_{q,s}} \leq c \left\|a\right\|_{\lambda_{r,\infty}} \left\|b\right\|_{\lambda_{p,s}}.$$

Applying the bilinear interpolation theorem and using Lemma 3.1, we obtain the desired statement.

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Тригонометриялық Фурье қатарының көбейткіштері класының $\lambda_{p,q}$ кеңістігіндегі есебі

Мақалада $a = \{a_k\}_{k=1}^{\infty}$ тізбектің жиыны ретінде анықталатын $\lambda_{p,q}$ сандық тізбектерінің салмақты кеңістігі қарастырылды, олар үшін норма

$$\|a\|_{\lambda_{p,q}} := \left(\sum_{k=1}^{\infty} |a_k|^q k^{\frac{q}{p}-1} \right)^{\frac{1}{q}} < \infty$$

шектеулі. Өспейтін тізбектер болған жағдайда $\lambda_{p,q}$ кеңістігінің нормасы классикалық $l_{p,q}$ Лоренц кеңістігінің нормасына сәйкес келеді. $\lambda_{p,q}$ кеңістігінің λ_{p_1,q_1} кеңістігіне енгізу үшін қажетті және жеткілікті шарттары алынды. Нақты интерполяция әдісіне қатысты осы кеңістіктердің интерполяциялық қасиеттері зерттелген. $\lambda_{p,q}$ кеңістіктерінің шкаласы нақты интерполяция әдісіне қатысты, сондай-ақ біріктірілген интерполяция әдісіне қатысты тұйық екендігі көрсетілген. Қосарланған кеңістіктің $\lambda_{p,q}$ салмақты кеңістігіне сипаттама алынды. Атап айтқанда, кеңістік рефлексивті, мұндағы p', q' параметрлері p және q параметрлеріне түйіндес болып келеді. Сонымен қатар осы кеңістіктерде үйірткі операторларының қасиеттері зерттелді. Бұл жұмыстың негізгі нәтижесі О'Нейл типті теңсіздігі болып табылады. Алынған теңсіздік классикалық Юнг-О'Нейл теңсіздігін жалпылайды. Зерттеу әдісі $\lambda_{p,q}$ кеңістіктері үшін дәлелденген интерполяциялық теоремаларға негізделген.

Кілт сөздер: тригонометриялық Фурье коэффициенттері, О'Нейл теңсіздігі, үйірткі операторы, $M_{p_0,q_0}^{p_1,q_1}$ класы.

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Задача классов множителей тригонометрических рядов Фурье в пространствах $\lambda_{p,q}$

В статье рассмотрены весовые пространства числовых последовательностей $\lambda_{p,q}$, которые определяются как множества последовательностей $a = \{a_k\}_{k=1}^{\infty}$, для которых конечна норма

$$\|a\|_{\lambda_{p,q}} := \left(\sum_{k=1}^{\infty} |a_k|^q k^{\frac{q}{p}-1} \right)^{\frac{1}{q}} < \infty.$$

В случае невозрастающих последовательностей норма пространства $\lambda_{p,q}$ совпадает с нормой классического пространтва Лоренца $l_{p,q}$. Получены необходимые и достаточные условия для вложений пространства $\lambda_{p,q}$ в пространство λ_{p_1,q_1}. Исследованы интерполяционные свойства этих пространств относительно вещественного интерполяционного метода. Показано, что шкала пространств $\lambda_{p,q}$ замкнута относительно вещественного интерполяционного метода, а также относительно комплексного интерполяционного метода. Получено описание двойственного пространства к весовому пространству $\lambda_{p,q}$, а именно: пространство рефлексивно, где p', q' сопряжены к параметрам p и q. Кроме того в статье изучены свойства оператора свертки в данных пространствах. Основным результатом данной работы является неравенство типа О'Нейла. Полученное неравенство обобщает классическое неравенство Юнга-О'Нейла. Метод исследования опирается на доказанные в этой работе интерполяционные теоремы для пространств $\lambda_{p,q}$.

Ключевые слова: тригонометрические коэффициенты Фурье, неравенство О'Нейла, оператор свертки, $M_{p_0,q_0}^{p_1,q_1}$ класс.

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