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One approach to solve a nonlinear boundary value problem for the Fredholm integro-differential equation

A quasilinear boundary value problem for a Fredholm integro-differential equation is considered. The interval is divided into N parts and the values of the solution to the equation at the left end points of the subintervals are introduced as additional parameters. New unknown functions are introduced on the subintervals and a special Cauchy problem with parameters is solved with respect to a system of such functions. By means of the solution to this problem, a new general solution to the quasilinear Fredholm integro-differential equation is constructed. The conditions of the existence of a unique new general solution to the equation under consideration are obtained. A new general solution is used to create a system of nonlinear algebraic equations in parameters introduced. The conditions for the existence of a unique solution to this system are established. This ensures the existence of a unique solution to original problem

Keywords: quasilinear Fredholm integro-differential equation, quasilinear boundary value problem, a new general solution, iterative process.

Introduction

Integro-differential equations (IDEs) are often encountered in the applications as mathematical models of real processes [1–6] The solvability of different problems for IDEs and methods for solving them have been studied by many authors [1, 4–20] General solutions play an important role in investigating qualitative properties of problems for IDEs and solving them. However, the classical general solution exists not for all Fredholm integro-differential equations (FIDEs) (see [7, 10]). Therefore, a new concept of general solution to FIDE is proposed in [11]. Employing parametrization's method [21] and choosing a regular partition Δ_N of the interval $[0,T]$ (see [9, 10]), a Δ_N general solution $x(\Delta_N,t,\lambda)$ to the linear FIDE is introduced. In contradistinction the classical general solution, $x(\Delta_N,t,\lambda)$ exists for all linear FIDEs and depends on a parameter $\lambda \in R^{nN}$. The paper [22] introduces the concept of the Δ_N general solution to a nonlinear ordinary differential equation. In [12], the concept of the Δ_N general solution is extended to FIDEs with nonlinear differential parts. The use of such a solution allows one to reduce a nonlinear boundary value problem (BVP) to a system of nonlinear algebraic equations in parameters λ_r, $r=\overline{1,N}$.

We consider the quasilinear FIDE

$$\frac{dx}{dt} = A(t)x + \sum_{k=1}^{m} \varphi_k(t) \int_0^T \psi_k(\tau)x(\tau)d\tau + f_0(t) + \varepsilon f(t,x), \quad t \in [0,T], \quad x \in \mathbb{R}^n, \tag{1}$$

subject to the boundary condition

$$Bx(0) + Cx(T) = d, \quad d \in \mathbb{R}^n, \tag{2}$$

where $\varepsilon > 0$, the $n\times n$ matrices $A(t)$, $\varphi_k(t)$, $\psi_k(\tau)$, $k=\overline{1,m}$, and the n vector $f_0(t)$ are continuous on $[0,T]$, $f:[0,T]\times R^n \to R^n$ is continuous, $\|x\| = \max\limits_{i=\overline{1,n}} |x_i|$.

The aim of this paper is to construct the Δ_N general solution to equation (1) by using analogous solution to a linear FIDE and solve BVP (1), (2).

Let us denote by $C\big([0,T],\ R^n\big)$ the space of all continuous functions $x:[0,T]\to R^n$ with the norm $\|x\|_1 = \max\limits_{t\in[0,T]} \|x(t)\|$. A solution to problem (1), (2) is a continuously differentiable on $[0,T]$ function $x(t)$ satisfying equation (1) and boundary condition (2). Here and below in the article, we assume that the functions observed at the end-points of the intervals have one-sided derivatives.

1 The Δ_N general solution to equation (1)

Let Δ_N be a partition of the interval $[0,T]$ with the points: $t_0=0<t_1<\ldots<t_N=T$.

We introduce the space $C\big([0,T],\Delta_N,R^{nN}\big)$ consisting of all function systems $x[t]=\big(x_1(t),x_2(t),\ldots,x_N(t)\big)$, where functions $x_r:[t_{r-1},t_r)\to R^n$, $r=\overline{1,N}$, are continuous and have finite left-sided limits $\lim\limits_{t\to t_r-0}x_r(t)$, with the norm $\big\|x[\cdot]\big\|_2=\max\limits_{r=\overline{1,N}}\sup\limits_{t\in[t_{r-1},t_r)}\|x_r(t)\|$.

First, we set $\varepsilon=0$ in equation (1) and consider the linear FIDE

$$\frac{dy}{dt}=A(t)y+\sum_{k=1}^{m}\varphi_k(t)\int_0^T\psi_k(\tau)y(\tau)d\tau+f_0(t),\quad t\in[0,T],\quad y\in\mathbb{R}^n. \tag{3}$$

Applying parametrization's method (see [10; 345]) to equation (3), for the partition Δ_N, we get the special Cauchy problem for the system of IDEs with parameters

$$\frac{dv_r}{dt}=A(t)(v_r+\lambda_r)+\sum_{k=1}^{m}\varphi_k(t)\sum_{j=1}^{N}\int_{t_{j-1}}^{t_j}\psi_k(\tau)[v_j(\tau)+\lambda_j]d\tau+f_0(t),\quad t\in[t_{r-1},t_r), \tag{4}$$

$$v_r(t_{r-1})=0,\quad r=\overline{1,N}. \tag{5}$$

A solution to problem (4), (5) for a fixed parameter $\lambda=\lambda^*=(\lambda_1^*,\lambda_2^*,\ldots,\lambda_N^*)\in R^{nN}$ is a function system $v[t,\lambda^*]=\big(v_1(t,\lambda^*),v_2(t,\lambda^*),\ldots,v_N(t,\lambda^*)\big)\in C\big([0,T],\Delta_N,R^{nN}\big)$, where $v_r(t,\lambda^*)$, $r=\overline{1,N}$, are continuously differentiable with respect to t on their domains, satisfy the system (4) for $\lambda_r=\lambda_r^*$, $r=\overline{1,N}$, and initial conditions (5).

We construct the $nm\times nm$ matrix $G(\Delta_N)=\big(G_{p,k}(\Delta_N)\big)$ with the elements

$$G_{p,k}(\Delta_N)=\sum_{r=1}^{N}\int\limits_{t_{r-1}}^{t_r}\psi_p(\tau)X_r(\tau)\int\limits_{t_{r-1}}^{\tau}X_r^{-1}(\tau_1)\varphi_k(\tau_1)d\tau_1d\tau,\quad p,k=\overline{1,m},$$

where $X_r(t)$ is the fundamental matrix of differential equation $\dfrac{dx}{dt}=A(t)x$ on the interval $[t_{r-1},t_r]$.

Assume that the matrix $\big[I-G(\Delta_N)\big]$ is invertible and its inverse is represented in the form

$$\big[I-G(\Delta_N)\big]^{-1}=\big(\mathcal{R}_{k,p}(\Delta_N)\big),\qquad k,p=\overline{1,m},$$

where I is the identity matrix of dimension nm, $\mathcal{R}_{k,p}(\Delta_N)$ are square matrices of dimension n.

The invertibility of the matrix $I-G(\Delta_N)$ provides the existence of a function system $v[t,\lambda]=\big(v_1(t,\lambda), v_2(t,\lambda),\ldots,v_N(t,\lambda)\big)\in C\big([0,T],\Delta_N,R^{nN}\big)$, a unique solution to the special Cauchy problem (4), (5) for any $\lambda=(\lambda_1,\lambda_2,\ldots,\lambda_N)\in R^{nN}$ and $f_0(t)\in C\big([0,T],R^n\big)$. Moreover, the following inequality is valid

$$\big\|v[\cdot,\lambda]\big\|_2\le\chi\big\|F_0[\cdot,\lambda]\big\|_2,$$

where χ is a constant independent of $\lambda\in R^{nN}$ and $f_0(t)\in C\big([0,T],R^n\big)$, and $F_0[t,\lambda]=\big(F_{0,1}(t,\lambda),F_{0,2}(t,\lambda),\ldots,F_{0,N}(t,\lambda)\big)\in C\big([0,T],\Delta_N,R^{nN}\big)$, with elements $F_{0,r}(t,\ \lambda)=A(t)\lambda_r+\sum\limits_{k=1}^{m}\varphi_k(t)\sum\limits_{j=1}^{N}\int_{t_{j-1}}^{t_j}\psi_k(\tau)d\tau\lambda_j+f_0(t),\quad t\in[t_{r-1},t_r)\ r=\overline{1,N}$.

The number χ is called a well-posedness constant of the special Cauchy problem (4), (5). Since $I-G(\Delta_N)$ is invertible then, by results obtained in [11], there exists a unique Δ_N general solution $y(\Delta_N,t,\lambda)$ to equation (3) and

$$y(\Delta_N,t,\lambda)=\lambda_r+\sum_{j=1}^{N}d_{r,j}(\Delta_N,t)\lambda_j+b_r(\Delta_N,t),\quad t\in[t_{r-1},t_r),\quad r=\overline{1,N},$$

$$y(\Delta_N,T,\lambda)=\lambda_N+\sum_{j=1}^{N}d_{N,j}(\Delta_N,T)\lambda_j+b_N(\Delta_N,T),$$

with the following coefficients and right-hand sides

$$d_{r,j}(\Delta_N,t)=X_r(t)\int\limits_{t_{r-1}}^{t}X_r^{-1}(\tau)\sum_{k=1}^{m}\varphi_k(\tau)\Big[\sum_{p=1}^{m}\mathcal{R}_{k,p}(\Delta_N)V_{p,j}(\Delta_N)+$$

$$+\int\limits_{t_{j-1}}^{t_j}\psi_k(\tau_1)d\tau_1\Big]d\tau,\qquad t\in[t_{r-1},t_r],\quad j\neq r,\quad r,j=\overline{1,N},$$

$$d_{r,r}(\Delta_N,t)=X_r(t)\int\limits_{t_{r-1}}^{t}X_r^{-1}(\tau)\Big\{\sum_{k=1}^{m}\varphi_k(\tau)\Big[\sum_{p=1}^{m}\mathcal{R}_{k,p}(\Delta_N)V_{p,r}(\Delta_N)+\int\limits_{t_{r-1}}^{t_r}\psi_k(\tau_1)d\tau_1\Big]+A(\tau)\Big\}d\tau,$$

$$b_r(\Delta_N,t)=X_r(t)\int\limits_{t_{r-1}}^{t}X_r^{-1}(\tau)\Big[\sum_{k=1}^{m}\varphi_k(\tau)\sum_{p=1}^{m}\mathcal{R}_{k,p}(\Delta_N)g_p(\Delta_N,f_0)+f_0(\tau)\Big]d\tau,\quad r=\overline{1,N},$$

$$V_{p,r}(\Delta_N)=\int\limits_{t_{r-1}}^{t_r}\psi_p(\tau)X_r(\tau)\int\limits_{t_{r-1}}^{\tau}X_r^{-1}(\tau_1)A(\tau_1)d\tau_1d\tau+$$

$$+\sum_{j=1}^{N}\sum_{k=1}^{m}\int\limits_{t_{j-1}}^{t_j}\psi_p(\tau)X_r(\tau)\int\limits_{t_{j-1}}^{\tau}X_r^{-1}(\tau_1)\varphi_k(\tau_1)d\tau_1d\tau\int\limits_{t_{r-1}}^{t_r}\psi_k(\tau_2)d\tau_2,$$

$$g_p(\Delta_N,f)=\sum_{r=1}^{N}\int\limits_{t_{r-1}}^{t_r}\psi_p(\tau)X_r(\tau)\int\limits_{t_{r-1}}^{\tau}X_r^{-1}(\tau_1)f(\tau_1)d\tau_1d\tau.$$

Given a vector $\lambda^{(0)}=\left(\lambda_1^{(0)},\lambda_2^{(0)},\ldots,\lambda_N^{(0)}\right)\in R^{nN}$ and numbers $\rho_\lambda>0$, $\rho>\rho_\lambda$, $\rho_u=\rho-\rho_\lambda$, we choose the piecewise continuous on $[0,T]$ function $y^{(0)}(t)=y\big(\Delta_N,t,\lambda^{(0)}\big)$, the function system $v^{(0)}[t]=\big(v_1^{(0)}(t),v_2^{(0)}(t),\ldots,v_N^{(0)}(t)\big)$ with elements $v_r^{(0)}(t)=y^{(0)}(t)-\lambda_r^{(0)}$, $t\in[t_{r-1},t_r)$, $r=\overline{1,N}$, and compose the following sets

$G^0(\rho)=\Big\{(t,x):t\in[0,T],\big\|x-y^{(0)}(t)\big\|<\rho\Big\}$,

$S\big(\lambda^{(0)},\rho_\lambda\big)=\Big\{\lambda=(\lambda_1,\lambda_2,\ldots,\lambda_N)\in R^{nN}:\big\|\lambda_r-\lambda_r^{(0)}\big\|<\rho_\lambda,\quad r=\overline{1,N}\Big\}$,

$S\big(v^{(0)}[t],\rho_u\big)=\Big\{u[t]\in C\big([0,T],\Delta_N,R^{nN}\big):\|u[\cdot]-v^{(0)}[\cdot]\|_2<\rho_u\Big\}$,

$G_p^0(\rho)=\Big\{(t,x):t\in[t_{p-1},t_p),\|x-y^{(0)}(t)\|<\rho\Big\},\quad p=\overline{1,N-1}$,

$G_N^0(\rho)=\Big\{(t,x):t\in[t_{N-1},t_N],\|x-y^{(0)}(t)\|<\rho\Big\}$, and $G^0(\Delta_N,\rho)=\bigcup\limits_{r=1}^{N}G_r^0(\rho)$.

In order to construct the Δ_N general solution to equation (1), we employ again the parametrization's method.

If a function $x(t)$ satisfies equation (1) and $(t,x(t))\in G^0(\Delta_N,\rho)$, then the functions $x_r(t)$, $r=\overline{1,N}$, as the restrictions of $x(t)$ to $[t_{r-1},t_r)$, satisfy the system of nonlinear IDEs

$$\frac{dx_r}{dt}=A(t)x_r+\sum_{k=1}^{m}\varphi_k(t)\sum_{j=1}^{N}\int_{t_{j-1}}^{t_j}\psi_k(\tau)x_j(\tau)d\tau+f_0(t)+\varepsilon f(t,x_r),\ t\in[t_{r-1},t_r),$$

and $(t,x_r(t))\in G_r^0(\rho)$, $r=\overline{1,N}$. Introducing the parameters $\lambda_r\hat{=}x_r(t_{r-1})$ and making the substitutions $u_r(t)=x_r(t)-\lambda_r$, $t\in[t_{r-1},t_r)$, $r=\overline{1,N}$, we obtain the system of nonlinear IDEs with parameters

$$\frac{du_r}{dt}=A(t)(u_r+\lambda_r)+\sum_{k=1}^{m}\varphi_k(t)\sum_{j=1}^{N}\int_{t_{j-1}}^{t_j}\psi_k(\tau)[u_j(\tau)+\lambda_j]d\tau+f_0(t)+\varepsilon f(t,u_r+\lambda_r),\quad t\in[t_{r-1},t_r),\quad (6)$$

subject to the initial conditions

$$u_r(t_{r-1}) = 0, \quad r = \overline{1,N}. \tag{7}$$

Problem (6), (7) is the special Cauchy problem for the system of nonlinear IDEs with parameters.

We represent problem (6), (7) as an operator equation and apply an iterative process for finding its solution. Set $X = \Big\{u[t] = \big(u_1(t), u_2(t), \ldots, u_N(t)\big) \in C\big([0,T], \Delta_N, R^{nN}\big) : u_r(t_{r-1}) = 0,\ r = \overline{1,N}\Big\}$, $Y = C\big([0,T], \Delta_N, R^{nN}\big)$, and introduce the linear operator $H : X \to Y$ in the following way:

$$Hu[t] = \Big(w_1^{(1)}(t), w_2^{(1)}(t), \ldots, w_N^{(1)}(t)\Big),$$

with $w_r^{(1)}(t) = \dot{u}_r(t) - A(t)u_r - \sum\limits_{k=1}^{m} \varphi_k(t) \sum\limits_{j=1}^{N} \int_{t_{j-1}}^{t_j} \psi_k(\tau) u_j(\tau) d\tau, \quad t \in [t_{r-1}, t_r), \quad r = \overline{1,N}.$

The domain of H is $D(H) = \Big\{u[t] = \big(u_1(t), u_2(t), \ldots, u_N(t)\big) \in X$, where $u_r(t)$ is continuously differentiable on $[t_{r-1}, t_r)$, $r = \overline{1,N}\Big\}$. It is easily seen that H is a closed unbounded linear operator.

Now, we can write down the special Cauchy problem (6), (7) as a nonlinear operator equation

$$Hu[t] = \varepsilon F\big(u[t], \lambda\big) + F_0[t, \lambda], \tag{8}$$

with $F\big(u[t], \lambda\big) = \big(w_1^{(2)}(t), w_2^{(2)}(t), \ldots, w_N^{(2)}(t)\big), \quad w_r^{(2)}(t) = f(t, u_r(t) + \lambda_r), \quad t \in [t_{r-1}, t_r),\ r = \overline{1,N}.$

Let $L(Y, X)$ be the space of linear bounded operators $\Lambda : Y \to X$ with the induced norm. Our assumption that the special Cauchy problem (4), (5) is well-posed with the constant χ leads to the invertibility of the operator $H : X \to Y$ and the estimate $\|H^{-1}\|_{L(Y,X)} \leq \chi$.

Theorem 1. Let the special Cauchy problem (4), (5) be well-posed with a constant χ *and the following inequalities be valid:*
(i) $\|f(t, x') - f(t, x'')\| \leq L\|x' - x''\|$, *L is a constant,* $(t, x'), (t, x'') \in G^0(\rho)$;
(ii) $q_\varepsilon = \varepsilon \chi L < 1$;
(iii) $\frac{1}{1 - q_\varepsilon} \varepsilon \chi \max\limits_{r=\overline{1,N}} \sup\limits_{t \in [t_{r-1}, t_r)} \big\|f\big(t, v_r(t, \lambda) + \lambda_r\big)\big\| < \rho_u$ *for all* $\lambda \in S(\lambda^{(0)}, \rho_\lambda)$.

Then for any $\lambda \in S(\lambda^{(0)}, \rho_\lambda)$, *there exists a unique function system* $u[t, \lambda, \varepsilon] = (u_1(t, \lambda, \varepsilon), u_2(t, \lambda, \varepsilon), \ldots, u_N(t, \lambda, \varepsilon))$, *the solution to the special Cauchy problem (6), (7) belonging to* $S\big(v^{(0)}[t], \rho_u\big)$, *and the following inequality is true*

$$\big\|u[\cdot, \lambda, \varepsilon] - v[\cdot, \lambda]\big\|_2 \leq \frac{1}{1 - q_\varepsilon} \varepsilon \chi \max_{r=\overline{1,N}} \sup_{t \in [t_{r-1}, t_r)} \big\|f\big(t, v_r(t, \lambda) + \lambda_r\big)\big\|. \tag{9}$$

Proof. Since the operator H has a bounded inverse, equation (8) is equivalent to the next operator equation:

$$u[t] = \varepsilon H^{-1} F\big(u[t], \lambda\big) + H^{-1} F_0[t, \lambda]. \tag{10}$$

For any fixed $\lambda \in S(\lambda^{(0)}, \rho_\lambda)$, the solution to equation (10) we find by the iterative process

$$u^{(0)}[t, \lambda, \varepsilon] = v[t, \lambda],$$

$$u^{(\nu+1)}[t, \lambda, \varepsilon] = \varepsilon H^{-1} F\big(u^{(\nu)}[t, \lambda, \varepsilon], \lambda\big) + H^{-1} F_0[t, \lambda], \quad \nu = 0, 1, 2, \ldots, \tag{11}$$

Using our assumptions, we obtain the following inequalities:

$$\big\|u^{(1)}[\cdot, \lambda, \varepsilon] - v[\cdot, \lambda]\big\|_2 = \varepsilon \big\|H^{-1} F\big(v[\cdot, \lambda], \lambda\big)\big\|_2 \leq \varepsilon \chi \max_{r=\overline{1,N}} \sup_{t \in [t_{r-1}, t_r)} \big\|f\big(t, v_r(t, \lambda) + \lambda_r\big)\big\|, \tag{12}$$

$$\Big\|u^{(\nu+1)}[\cdot, \lambda, \varepsilon] - u^{(\nu)}[\cdot, \lambda, \varepsilon]\Big\|_2 \leq \varepsilon \big\|H^{-1}\big\|_{L(Y,X)} \Big\|F\big(u^{(\nu)}[\cdot, \lambda, \varepsilon]\big) - F\big(u^{(\nu-1)}[\cdot, \lambda, \varepsilon]\big)\Big\|_2 \leq$$

$$\leq \varepsilon \chi \max_{r=\overline{1,N}} \sup_{t \in [t_{r-1}, t_r)} \Big\|f\big(t, u_r^{(\nu)}(t, \lambda, \varepsilon) + \lambda_r\big) - f\big(t, u_r^{(\nu-1)}(t, \lambda, \varepsilon) + \lambda_r\big)\Big\| \leq \tag{13}$$

$$\leq \varepsilon\chi L\big\|u^{(\nu)}[\cdot,\lambda,\varepsilon]-u^{(\nu-1)}[\cdot,\lambda,\varepsilon]\big\|_2, \quad \nu=1,2,\ldots,$$

$$\big\|u^{(\nu+1)}[\cdot,\lambda,\varepsilon]-v[\cdot,\lambda]\big\|_2<\frac{1}{1-q_\varepsilon}\varepsilon\chi\max_{r=\overline{1,N}}\sup_{t\in[t_{r-1},t_r)}\big\|f\big(t,v_r(t,\lambda)+\lambda_r\big)\big\|. \tag{14}$$

The inequalities (12)–(14) and condition (iii) of Theorem 1 provide the convergence of the iterative process (11) to the function system $u[t,\lambda,\varepsilon]$, a unique solution to equation (8) in $S(v[t,\lambda],\rho_u)$, and validity of estimate (9). □

Definition 1. Let a function system $u[t,\lambda,\varepsilon]=\big(u_1(t,\lambda,\varepsilon),u_2(t,\lambda,\varepsilon),\ldots,u_N(t,\lambda,\varepsilon)\big)\in S(v[t,\lambda],\rho_u)$ *be a unique solution to the special Cauchy problem (6), (7) with parameters* $\lambda=(\lambda_1,\lambda_2,\ldots,\lambda_N)\in S(\lambda^{(0)},\rho_\lambda)$. *Then the function* $x(\Delta_N,t,\lambda,\varepsilon)$ *given by the equalities:* $x(\Delta_N,t,\lambda,\varepsilon)=\lambda_r+u_r(t,\lambda,\varepsilon)$ *for* $t\in[t_{r-1},t_r)$, $r=\overline{1,N}$, *and* $x(\Delta_N,T,\lambda,\varepsilon)=\lambda_N+\lim\limits_{t\to T-0}u_N(t,\lambda,\varepsilon)$, *is called a* Δ_N *general solution to equation (1) in* $G^0(\Delta_N,\rho)$.

Definition 1 and Theorem 1 imply the following assertion.

Theorem 2. Under conditions of Theorem 1, there exists a function $x(\Delta_N,t,\lambda,\varepsilon)$, *which is a unique* Δ_N *general solution to equation (1) in* $G^0(\Delta_N,\rho)$, *and this function can be represented in the form*

$$x(\Delta_N,t,\lambda,\varepsilon)=y(\Delta_N,t,\lambda)+\Delta x(\Delta_N,t,\lambda,\varepsilon),$$

where the function $\Delta x(\Delta_N,t,\lambda,\varepsilon)$ is compiled by the equalities $\Delta x(\Delta_N,t,\lambda,\varepsilon)=u_r(t,\lambda,\varepsilon)-v_r(t,\lambda)$, for $t\in[t_{r-1},t_r)$, $r=\overline{1,N}$, $\Delta x(\Delta_N,T,\lambda,\varepsilon)=\lim\limits_{t\to T-0}u_N(t,\lambda,\varepsilon)-\lim\limits_{t\to T-0}v_N(t,\lambda)$. Moreover, the following estimate is valid

$$\sup_{t\in[0,T]}\|\Delta x(\Delta_N,t,\lambda,\varepsilon)\|\leq\frac{1}{1-q_\varepsilon}\varepsilon\chi\max_{r=\overline{1,N}}\sup_{t\in[t_{r-1},t_r)}\big\|f\big(t,v_r(t,\lambda)+\lambda_r\big)\big\|.$$

2 The solvability of problem (1)–(2)

In this Section, we investigate the solvability of quasilinear BVP (1)–(2). We first consider a linear BVP for equation (3) with the boundary condition (2).

Substituting the Δ_N general solution to equation (3) into the boundary condition (2) and the continuity conditions at the interior points of the partition, we obtain the system of linear algebraic equations in parameters

$$B\lambda_1+C\lambda_N+C\sum_{j=1}^{N}d_{N,j}(\Delta_N,T)\lambda_j=d-Cb_N(\Delta_N,T), \tag{15}$$

$$\lambda_p+\sum_{j=1}^{N}d_{p,j}(\Delta_N,t_p)\lambda_j-\lambda_{p+1}=-b_p(\Delta_N,t_p), \quad p=\overline{1,N-1}. \tag{16}$$

We rewrite equations (15), (16) in the form

$$Q_*(\Delta_N)\lambda=-F_*(\Delta_N).$$

In accordance with Theorem 2.2 in [10], the invertibility of the matrix $Q_*(\Delta_N):R^{nN}\to R^{nN}$ is equivalent to the unique solvability of linear BVP (3), (2).

Now, we study the solvability of quasilinear BVP (1), (2). If $x(t)$ is a solution to equation (1), and $x[t]=(x_1(t),x_2(t),...,x_N(t))$ is a function system of its restrictions to the subintervals $[t_{r-1},t_r)$, $r=\overline{1,N}$, then the equations

$$\lim_{t\to t_p-0}x_p(t)=x_{p+1}(t_p), \qquad p=\overline{1,N-1}, \tag{17}$$

hold. Equations (17) are the continuity conditions for solutions to equation (1) at the interior points of partition Δ_N.

Let $x(\Delta_N,t,\lambda,\varepsilon)$ be a Δ_N general solution to equation (1) in $G^0(\Delta_N,\rho)$. Substituting the corresponding expressions of $x(\Delta_N,t,\lambda,\varepsilon)$ into boundary condition (2) and continuity conditions (17), we get the system of nonlinear algebraic equations

$$B\lambda_1+C\lambda_N+C\sum_{j=1}^{N}d_{N,j}(\Delta_N,T)\lambda_j+C\Delta x(\Delta_N,T,\lambda,\varepsilon)=d-Cb_N(\Delta_N,T), \tag{18}$$

$$\lambda_p + \sum_{j=1}^{N} d_{p,j}(\Delta_N, t_p)\lambda_j - \lambda_{p+1} + \Delta x(\Delta_N, t_p, \lambda, \varepsilon) = -b_p(\Delta_N, t_p), \quad p = \overline{1, N-1}. \tag{19}$$

We rewrite system (18), (19) in the form:

$$Q_*(\Delta_N)\lambda = -F_*(\Delta_N) - \Delta Q_*(\Delta_N, \lambda, \varepsilon), \tag{20}$$

where

$$\Delta Q_*(\Delta_N, \lambda, \varepsilon) = \begin{pmatrix} C\Delta x(\Delta_N, T, \lambda, \varepsilon) \\ \Delta x(\Delta_N, t_1, \lambda, \varepsilon) \\ \dots \\ \Delta x(\Delta_N, t_{N-1}, \lambda, \varepsilon) \end{pmatrix}.$$

As proved in Theorem 3.2 [12; 31] the solvability of problem (1), (2) is equivalent to that of system of nonlinear algebraic equations (20). The conditions of the solvability of (20) are established in the following statement.

Theorem 3. Let the conditions of Theorem 1 are met and the following assumptions hold:
(i) $Q_*(\Delta_N)$ *is invertible and* $\|[Q_*(\Delta_N)]^{-1}\| \leq \gamma$;
(ii) $\sigma_\varepsilon = q_\varepsilon\Big(\dfrac{\chi}{1-q_\varepsilon}\big(\alpha + K_0 + \varepsilon L\big) + 1\Big) < 1$,
where $\alpha = \max\limits_{t\in[0,T]} \max\limits_{i=\overline{1,n}} \sum\limits_{j=1}^{n} \|a_{ij}(t)\|$, $K_0 = \sum\limits_{k=1}^{m} \max\limits_{t\in[0,T]} \|\varphi_k(t)\| \sum\limits_{j=1}^{N} \int\limits_{t_{j-1}}^{t_j} \|\psi_k(\tau)\|d\tau$;
(iii) $\dfrac{1}{1-\sigma_\varepsilon} \cdot \dfrac{\varepsilon\chi}{1-q_\varepsilon}\gamma \max\big(1, \|C\|\big) \max\limits_{r=\overline{1,N}} \sup\limits_{t\in[t_{r-1},t_r)} \big\|f\big(t, v_r(t, \lambda^{(0)}) + \lambda_r^{(0)}\big)\big\| < \rho_\lambda$.

Then system of nonlinear algebraic equations (20) has a unique solution $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_N) \in S(\lambda^{(0)}, \rho_\lambda)$.

Proof. A solution to equation (20) is found by the iterative process

$$\lambda^{(0)} = [Q_*(\Delta_N)]^{-1} \cdot F_*(\Delta_N),$$

$$\lambda^{(\nu+1)} = -[Q_*(\Delta_N)]^{-1}\Big\{F_*(\Delta_N) + \Delta Q_*\big(\Delta_N, \lambda^{(\nu)}, \varepsilon\big)\Big\}. \tag{21}$$

Under conditions of Theorem the following inequalities hold:

$$\big\|\lambda^{(1)} - \lambda^{(0)}\big\| \leq \gamma \cdot \Big\|\Delta Q_*(\Delta_N, \lambda^{(0)}, \varepsilon)\Big\| \leq \gamma \cdot \max\big(1, \|C\|\big) \max_{r=\overline{1,N}} \Big\|\Delta x(\Delta_N, t_r, \lambda^{(0)}, \varepsilon)\Big\| \leq$$

$$\leq \gamma \cdot \max\big(1, \|C\|\big) \frac{\varepsilon\chi}{1-q_\varepsilon} \max_{r=\overline{1,N}} \sup_{t\in[t_{r-1},t_r)} \big\|f\big(t, v_r(t, \lambda^{(0)}) + \lambda_r^{(0)}\big)\big\|,$$

$$\big\|\lambda^{(\nu+1)} - \lambda^{(\nu)}\big\| \leq q_\varepsilon\Big\{\frac{\chi}{1-q_\varepsilon}(\alpha + K_0 + \varepsilon L) + 1\Big\}\big\|\lambda^{(\nu)} - \lambda^{(\nu-1)}\big\|, \quad \nu = 1, 2, \dots,$$

$$\big\|\lambda^{(\nu+1)} - \lambda^{(0)}\big\| \leq \frac{1}{1-\sigma_\varepsilon} \cdot \frac{\varepsilon\chi}{1-q_\varepsilon}\gamma \max\big(1, \|C\|\big) \max_{r=\overline{1,N}} \sup_{t\in[t_{r-1},t_r)} \big\|f\big(t, v_r(t, \lambda^{(0)}) + \lambda_r^{(0)}\big)\big\|.$$

Similarly to the proof of Theorem 1, the iterative process (21) converges to the vector $\lambda \in S(\lambda^{(0)}, \rho_\lambda)$, a unique solution to equation (20). □

From Theorem 1 and Theorem 3.2 in [12], we obtain

Theorem 4. Let the conditions of Theorem 3 be fulfilled. Then quasilinear BVP (1), (2) has a unique solution $x^*(t)$ *such that* $(t, x^*(t)) \in G^0(\Delta_N, \rho)$.

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References

1 Быков Я.В. О некоторых задачах теории интегро-дифференциальных уравнений / Я.В. Быков. — Фрунзе: Киргиз. гос. ун-т, 1957. — 327 с.

2 Dehghan M. Solution of a partial integro-differential equation arising from viscoelasticity / M. Dehghan // Jnt. J. Comput. Math. Jssud. — 2006. — Vol. 83. — P. 123–129.

3 Dehghan M. Rational pseudospectral approximation to the solution of a nonlinear integro-differential equation arising in modeling of the population growth / M. Dehghan, M. Shahini // Applied Mathematical Modelling. — 2015. — Vol. 39, Iss. 18. — P. 5521–5530. DOI: https://doi.org/10.1016/j.apm.2015.01.001

4 Lakshmikantham V. Theory of Integro-Differential Equations / V. Lakshmikantham, M.R.M. Rao // — London: Gordon and Breach, 1995. — 375 p.

5 Pruss J. Evolutionary Integral equations and Applications / J. Pruss. — Basel etc.: Birkh$\ddot{a}$user Verlag, 1993. — 366 p.

6 Wazwaz A.M. Linear and Nonlinear Integral Equations [Methods and Applications] / A.M. Wazwaz. — Higher Equacation Press, Beijing and Springer-Verlag Berlin Heidelberg, 2011. — 639 p.

7 Boichuk A.A. Generalized inverse operators and Fredholm boundary value problems / A.A. Boichuk, A.M. Samoilenko // — Boston: SPV, Utrecht, 2004. — 333 p.

8 Dehghan M. The numerical solution of the non-linear integro-differential equations based on the meshless method / M. Dehghan, R. Salehi // J. Comput. Appl. Math. — 2012. — Vol. 236, Iss. 9. — P. 2367–2377. DOI: https://doi.org/10.1016/j.cam.2011.11.022

9 Dzhumabaev D.S. Necessary and Sufficient Conditions for the Solvability of Linear Boundary-Value Problems for the Fredholm Integro-differential Equations / D.S. Dzhumabaev // Ukrainian Mathematical Journal. — 2015. — Vol. 66, No. 8. — P. 1200–1219. DOI: https://doi.org/10.1007/s11253-015-1003-6

10 Dzhumabaev D.S. On one approach to solve the linear boundary value problems for Fredholm integro-differential equations / D.S. Dzhumabaev // J. Comput. Appl. Math. — 2016. — Vol. 294. — P. 342–357. DOI: https://doi.org/10.1016/j.cam.2015.08.023

11 Dzhumabaev D.S. New general solutions to linear Fredholm integro-differential equations and their applications on solving the boundary value problems / D.S. Dzhumabaev // J. Comput. Appl. Math. — 2018. — Vol. 327. — P. 79–108. DOI: https://doi.org/10.1016/j.cam.2017.06.010

12 Dzhumabaev D.S. New general solution to a nonlinear Fredholm integro-differential equation / D.S. Dzhumabaev, S.T. Mynbayeva // Eurasian Mathematical Journal. — 2019. — Vol. 10. No.4. — P. 23–32. DOI: https://doi.org/10.32523/2077-9879-2019-10-4-24-33

13 Kheybari S. A semi-analytical algorithm to solve systems of integro-differential equations under mixed boundary conditions / S. Kheybari, M.T. Darvishi, A.M. Wazwaz // J. Comput. Appl. Math. — 2017. — Vol. 317. — P. 72–89. DOI: https://doi.org/10.1016/j.cam.2016.11.029

14 Кривошеин Л.Е. Приближенные методы решения обыкновенных линейных интегро-дифференциальных уравнений / Л.Е. Кривошеин. — Фрунзе: АН КиргизССР; Ин-т физики, математики и механики, 1962. — 184 с.

15 Maleknejad K. Hybrid Legendre polynomials and Block-Pulse functions approach for nonlinear Volterra–Fredholm integro-differential equations / K. Maleknejad, B. Basirat, E. Hashemizadeh // Comput. Math. Appl. — 2013. — Vol. 61. — P. 2821–2828. DOI: https://doi.org/10.1016/j.camwa.2011.03.055

16 Pedas A. A discrete collocation method for Fredholm integro-differential equations with weakly singular kernels / A. Pedas , E. Tamme // Applied Numerical Mathematics. — 2011. — Vol. 61, Iss. 6. — P. 738–751. DOI: https://doi.org/10.1016/j.apnum.2011.01.006

17 Shakeri F. A high order finite volume element method for solving elliptic partial integro-differential equations / F. Shakeri, M. Dehghan // Applied Numerical Mathematics. — 2013. — Vol. 65. — P. 105-118. DOI: https://doi.org/10.1016/j.apnum.2012.10.002

18 Susahab D.N. Efficient quadrature rules for solving nonlinear fractional integro-differential equations of the Hammerstein type / D.N. Susahab, S. Shahmorad, M. Jahanshahi // Applied Mathematical Modelling. — 2015. — Vol. 39, Iss. 18, 5452–5458. DOI: https://doi.org/10.1016/j.apm.2015.01.008

19 Turkyilmazoglu M. High-order nonlinear Volterra–Fredholm-Hammerstein integro-differential equations

and their effective computation / M. Turkyilmazoglu // Appl. Math. Comput. — 2014. — Vol. 247. — P. 410–416. DOI: https://doi.org/10.1016/j.amc.2014.08.074

20 Yulan M. New algorithm for second order boundary value problems of integro-differential equation / W. Yulan, T. Chaolu, P. Ting // J. Comput. Appl. Math. — 2009. — Vol. 229. — P. 1–6. DOI: https://doi.org/10.1016/j.cam.2008.10.007

21 Dzhumabaev D.S. Criteria for the unique solvability of a linear boundary-value problem for an ordinary differential equation / D.S. Dzhumabaev // Comput. Math. Math. Phys. — 1989. — No. 29. — P. 34–46.

22 Dzhumabaev D.S. New general solutions to ordinary differential equations and methods for solving boundary value problems / D.S. Dzhumabaev // Ukrainian Mathematical Journal. — 2019. — Vol. 71, No. 7. — P. 1006–1031. DOI: https://doi.org/10.1007/s11253-019-01694-9

Д.С. Джумабаев, С.Т. Мынбаева

Фредгольм интегралдық-дифференциалдық теңдеуі үшін сызықты емес шеттік есепті шешудің бір тәсілі

Фредгольм интегралдық-дифференциалдық теңдеуі үшін квазисызықты шеттік есеп қарастырылды. Интервал N бөлікке бөлінеді және қарастырылып отырған теңдеу шешімінің ішкі интервалдардың сол жақ шеткі нүктелеріндегі мәндері қосымша параметрлер ретінде енгізіледі. Ішкі интервалдарға белгісіз функциялар енгізіледі және осы функциялар жүйесіне қатысты параметрлері бар арнайы Коши есебі шешіледі. Арнайы Коши есебінің табылған шешімі арқылы квазисызықты Фредгольм интегралдық-дифференциалдық теңдеуінің жаңа жалпы шешімі құрылады. Қарастырылып отырған теңдеудің жалғыз жаңа шешімі бар болу шарттары алынған. Жаңа жалпы шешімнің көмегімен енгізілген параметрлерге қатысты сызықты емес алгебралық теңдеулер жүйесі құрылады. Осы жүйенің жалғыз шешімі бар болу шарттары тағайындалған, бұл шарттар квазисызықты шеттік есептің жалғыз шешімінің бар болуын қамтамасыз етеді.

Кілт сөздер: квазисызықты Фредгольм интегралдық-дифференциалдық теңдеуі, квазисызықты шеттік есеп, жаңа жалпы шешім, итерациялық процесс.

Д.С. Джумабаев, С.Т. Мынбаева

Один подход к решению нелинейной краевой задачи для интегро-дифференциального уравнения Фредгольма

Рассмотрена квазилинейная краевая задача для интегро-дифференциального уравнения Фредгольма. Интервал поделен на N частей, а значения решения рассматриваемого уравнения в левых конечных точках подинтервалов введены в качестве дополнительных параметров. На подинтервалах введены новые неизвестные функции и относительно этой системы функций решена специальная задача Коши с параметрами. Новое общее решение квазилинейного интегро-дифференциального уравнения Фредгольма построено через найденное решение специальной задачи Коши. Получены условия существования единственного нового общего решения рассматриваемого уравнения. С помощью нового общего решения составлена система нелинейных алгебраических уравнений относительно введенных параметров. Установлены условия существования единственного решения этой системы, обеспечивающие наличие единственного решения квазилинейной краевой задачи.

Ключевые слова: квазилинейное интегро-дифференциальное уравнение Фредгольма, квазилинейная краевая задача, новое общее решение, итерационный процесс.

References

1 Bykov, Ya.V. (1957). *O nekotorykh zadachakh teorii intehro-differentsialnykh uravnenii [On some problems in the theory of integro-differential equations].* Frunze: Kirhizskii hosudarstvennyi Universitet [in Russian].

2 Dehghan M. (2006). Solution of a partial integro-differential equation arising from viscoelasticity. *Jnt. J. Comput. Math. Jssud., Vol.83,* 123–129.

3 Dehghan M., Shahini M. (2015). Rational pseudospectral approximation to the solution of a nonlinear integro-differential equation arising in modeling of the population growth. *Applied Mathematical Modelling, Vol,.9,* Iss.18, 5521–5530. DOI: https://doi.org/10.1016/j.apm.2015.01.001.

4 Lakshmikantham, V., Rao, M.R.M. (1995). *Theory of Integro-Differential Equations.* London: Gordon and Breach.

5 Pruss, J. (1993). *Evolutionary Integral equations and Applications.* Basel etc.: Birkhäuser Verlag.

6 Wazwaz, A.M. (2011). *Linear and Nonlinear Integral Equations [Methods and Applications].* Higher Equacation Press, Beijing and Springer-Verlag Berlin Heidelberg.

7 Boichuk, A.A., Samoilenko, A.M. (2004). *Generalized inverse operators and Fredholm boundary value problems.* Boston: SPV, Utrecht.

8 Dehghan M., Salehi R. (2012). The numerical solution of the non-linear integro-differential equations based on the meshless method. *J. Comput. Appl. Math., Vol.236,* Iss.9, 2367–2377. DOI: https://doi.org/10.1016/j.cam.2011.11.022.

9 Dzhumabaev, D.S. (2015). Necessary and Sufficient Conditions for the Solvability of Linear Boundary-Value Problems for the Fredholm Integro-differential Equations. *Ukrainian Mathematical Journal., Vol. 66,* No.8, 1200–1219. DOI: https://doi.org/10.1007/s11253-015-1003-6.

10 Dzhumabaev, D.S. (2016). On one approach to solve the linear boundary value problems for Fredholm integro-differential equations. *J. Comput. Appl. Math., Vol. 294,* 342–357. DOI: https://doi.org/10.1016/j.cam.2015.08.023.

11 Dzhumabaev, D.S. (2018). New general solutions to linear Fredholm integro-differential equations and their applications on solving the boundary value problems. *J. Comput. Appl. Math., Vol. 327,* 79–108. DOI: https://doi.org/10.1016/j.cam.2017.06.010.

12 Dzhumabaev, D.S., Mynbayeva S.T. (2019). New general solution to a nonlinear Fredholm integro-differential equation. *Eurasian Mathematical Journal, Vol. 10,* No.4, 23–32. DOI: https://doi.org/10.32523/2077-9879-2019-10-4-23-32.

13 Kheybari, S., Darvishi, M.T., & Wazwaz, A.M. (2017). A semi-analytical algorithm to solve systems of integro-differential equations under mixed boundary conditions. *J.Comput.Appl.Math., Vol. 317,* 72–89. DOI: https://doi.org/10.1016/j.cam.2016.11.029.

14 Krivoshein, L.E. (1962). *Priblizhennye metody resheniia obyknovennykh lineinykh intehro-differentsialnykh uravnenii [Approximate methods for solving linear ordinary integro-differential equations.* Frunze: AN KirhizSSR; Institut fiziki, matematiki i mekhaniki [in Russian].

15 Maleknejad, K., Basirat, B., & Hashemizadeh, E. (2011). Hybrid Legendre polynomials and Block-Pulse functions approach for nonlinear Volterra–Fredholm integro-differential equations. *Comput. Math. Appl., Vol. 61,* 2821–2828. DOI: https://doi.org/10.1016/j.camwa.2011.03.055.

16 Pedas A., Tamme E. (2011). A discrete collocation method for Fredholm integro-differential equations with weakly singular kernels. *Applied Numerical Mathematics, Vol. 61,* Iss.6, 738–751. DOI: https://doi.org/10.1016/j.apnum.2011.01.006.

17 Shakeri F., Dehghan M. (2013). A high order finite volume element method for solving elliptic partial integro-differential equations. *Applied Numerical Mathematics, Vol. 65,* 105-118. DOI: https://doi.org/10.1016/j.apnum.2012.10.002.

18 Susahab D.N., Shahmorad S., & Jahanshahi M. (2015). Efficient quadrature rules for solving nonlinear fractional integro-differential equations of the Hammerstein type. *Applied Mathematical Modelling, Vol. 39,* Iss.18, 5452–5458. DOI: https://doi.org/10.1016/j.apm.2015.01.008.

19 Turkyilmazoglu, M. (2014). High-order nonlinear Volterra–Fredholm-Hammerstein integro-differential equations and their effective computation. *Appl. Math. Comput., Vol. 247,* 410–416. DOI: https://doi.org/10.1016/j.amc.2014.08.074.

20 Yulan, W., Chaolu, T., & Ting, P. (2009). New algorithm for second order boundary value problems of integro-differential equation. *J. Comput. Appl. Math., Vol. 229,* 1–6. DOI: https://doi.org/10.1016/j.cam.2008.10.007.

21 Dzhumabaev, D.S. (1989). Criteria for the unique solvability of a linear boundary-value problem for an ordinary differential equation. *Comput. Math. Math. Phys., Vol. 29*, 34–46.

22 Dzhumabaev, D.S. (2019). New general solutions to ordinary differential equations and methods for solving boundary value problems. *Ukrainian Mathematical Journal, Vol. 71,* No.7, 1006–1031. DOI: https://doi.org/10.1007/s11253-019-01694-9.