DOI 10.31489/2019M2/51-59

UDC 517.5

K.K. Sadykova[1,2], N.T. Tleukhanova[1]

[1] *L.N. Gumilyov Eurasian National University, Nur-Sultan, Kazakhstan;*
[2] *S. Seifullin Kazakh Agro Technical University, Nur-Sultan, Kazakhstan*
(E-mail: sadkelbet@gmail.com)

Estimates of the norm of the convolution operator in anisotropic Besov spaces with the dominated mixed derivative

In this paper, we investigate the boundedness of the norm of the convolution operator in Sobolev spaces with the dominated mixed derivative and anisotropic Nikolsky-Besov spaces. For Sobolev spaces with the dominated mixed derivatives, an analogue of Young's inequality is obtained, namely, relations of the form

$$W_{\mathbf{p}}^{\boldsymbol{\gamma}} * W_{\mathbf{r}}^{\boldsymbol{\beta}} \hookrightarrow W_{\mathbf{q}}^{\boldsymbol{\alpha}} \qquad (1)$$

are proved when the corresponding conditions on the parameters are satisfied. The main goal of the paper is to solve the following problems. Let f and g be functions from some classes of the Nikolsky-Besov space scale. We would like to find the Nikolsky-Besov space such that the convolution $f * g$ belongs to this space. Using relation (1) and the Nursultanov interpolation theorem for anisotropic spaces, an analogue of the O'Neil theorem was obtained for the Nikolsky-Besov space scale $B_{\mathbf{pq}}^{\boldsymbol{\alpha}}$, where $\boldsymbol{\alpha}$, $\mathbf{p}$, $\mathbf{q}$ are vector parameters. Relations of the form $B_{\mathbf{ps_1}}^{\boldsymbol{\gamma}} * B_{\mathbf{rs_2}}^{\boldsymbol{\beta}} \hookrightarrow B_{\mathbf{qs}}^{\boldsymbol{\alpha}}$ are obtained, with the corresponding ratios of vector parameters. The theorems obtained in this paper complement the results of Batyrov and Burenkov, where similar problems were considered in isotropic Nikolsky-Besov spaces, that is, in spaces where the parameters are scalars.

Keywords: convolution operator, anisotropic Sobolev and Besov spaces, interpolation.

Introduction

Let I be either a n-dimensional torus $\mathbb{T}^n = [0,1)^n$, or the Euclidean space $\mathbb{R}^n$. Let $f(x)$ and $g(x)$ be measurable functions on I with respect to the n-dimensional Lebesgue measure such that for almost all $x \in I$ there exists an integral

$$\int_I f(x-y)g(y)dy.$$

In this case, it is said that the convolution of these functions is defined

$$(f*g)(x) = \int_I f(x-y)g(y)dy. \qquad (1.1)$$

The classical Young's inequality [1; 199] has the following form. Let

$$1 \le p, r, q \le \infty, \quad \frac{1}{q} + 1 = \frac{1}{p} + \frac{1}{r}. \qquad (1.2)$$

If $f \in L_p(I)$, $g \in L_r(I)$, then there exists almost everywhere on I the convolution $f * g$, belonging to the space $L_q(I)$ and the following inequality holds

$$\|f*g\|_{L_q(I)} \le \|f\|_{L_p(I)} \|g\|_{L_r(I)}. \qquad (1.3)$$

We will write this statement as follows

$$L_p(I) * L_r(I) \hookrightarrow L_q(I).$$

This inequality plays an important role in harmonic analysis and in the theory of partial differential equations [1–3].

Note that if

$$1 < p, r, q < \infty, \quad \frac{1}{q} + 1 = \frac{1}{p} + \frac{1}{r}, \tag{1.4}$$

then for $g_0(x) = \dfrac{1}{|x|^{\frac{n}{r}}}$ the inequality holds

$$\|f * g_0\|_{L_q(I)} \le C\|f\|_{L_p(I)}.$$

This inequality is called the Hardy-Littlewood-Sobolev inequality. It does not follow from Young's inequality, since $\|g_0\|_{L_r(I)} = \infty$. A generalization of inequality (1.3) obtained by O'Neil [4] (see also [5, 6]).

If (1.4) is true and $0 < s_1, s_2, s \le \infty$, $\dfrac{1}{s} = \dfrac{1}{s_1} + \dfrac{1}{s_2}$, then

$$L_{ps_1} * L_{rs_2} \hookrightarrow L_{qs} \tag{1.5}$$

and in particular

$$L_p * L_{r\infty} \hookrightarrow L_q, \tag{1.6}$$

where L_{ps} is Lorentz space.

Note that in relation (1.5), condition (1.4) is essential. The limiting cases of the O'Neil inequality with condition (1.2) were considered in [7].

The O'Neil inequality for anisotropic Lorentz spaces was studied in [8–10]. In the case of $n \ge 2$ these results are extend the inequality (1.6). In the one-dimensional case, the O'Neil inequality was extended in [11, 12].

There are generalizations of the Young and O'Neil inequalities for various functional spaces: weighted L_p spaces, classical and Lorentz weighted spaces, Hardy spaces, Wiener spaces, Orlicz spaces; [5, 6, 8, 13–18], and references therein.

Convolution operators were studied in various spaces of smooth functions in [19–22].

V.I. Burenkov and B.E. Batyrov in [21] proved the following statement: let $-\infty < l_1, l_2, l_3 < \infty$, $0 < p_1, p_2, p_3 \le \infty$, $0 < \theta_1, \theta_2, \theta_3 \le \infty$. For any $f_1 \in B^{l_1}_{p_1\theta_1}(\mathbb{R}^n)$, $f_2 \in B^{l_2}_{p_2\theta_2}(\mathbb{R}^n)$ such that Ff_1 and Ff_2 are regular generalized functions and their (pointwise) product $Ff_1 \cdot Ff_2 \in S(\mathbb{R}^n)$, there exists a number $c_3 > 0$ such that

$$\|f_1 * f_2\|_{B^{l_3}_{p_3\theta_3}(\mathbb{R}^n)} \le c_3 \|f_1\|_{B^{l_1}_{p_1\theta_1}(\mathbb{R}^n)} \|f_2\|_{B^{l_2}_{p_2\theta_2}(\mathbb{R}^n)}, \tag{1.7}$$

holds if and only if the following conditions hold:

1) $p_3 \ge p_1$, $p_3 \ge p_2$;

2) $\dfrac{1}{p_1} + \dfrac{1}{p_2} - \dfrac{1}{p_3} - 1 \ge 0$;

and one of the conditions

3a) $l_3 < l_1 + l_2 - n\left(\dfrac{1}{p_1} + \dfrac{1}{p_2} - \dfrac{1}{p_3} - 1\right)$

or

3b) $l_3 = l_1 + l_2 - n\left(\dfrac{1}{p_1} + \dfrac{1}{p_2} - \dfrac{1}{p_3} - 1\right)$ и $\dfrac{1}{\theta_3} \le \dfrac{1}{\theta_1} + \dfrac{1}{\theta_2}$,

where Ff is the Fourier transform of the function f:

$$(Ff)(x) = (2\pi)^{-\frac{n}{2}} \int\limits_{\mathbb{R}^n} e^{-ix\xi} f(\xi) d\xi.$$

For $p_2 = p_3$, $\theta_2 = \theta_3$, $0 < l_2 < l_3 < \infty$ inequality (1.7) and some of its generalizations follow from the results obtained in the works of K.K. Golovkin and V.A. Solonnikov [19, 20], and [23].

In this paper, we investigate the boundedness of the convolution operator in anisotropic Besov spaces with the dominated mixed derivative.

2. Anisotropic Besov spaces with dominated mixed derivative

Let $\boldsymbol{\alpha} \in \mathbb{R}^n$, $\mathbf{1} < \mathbf{p}= (p_1, \ldots, p_n) <\boldsymbol{\infty}$, $\mathbf{0} < \mathbf{q}= (q_1, \ldots, q_n) \leq\boldsymbol{\infty}$. Following [24–26], we define the space $B^{\boldsymbol{\alpha}}_{\mathbf{pq}}(\mathbb{T}^n)$ as the set of series $f = \sum\limits_{m\in\mathbb{Z}^n} a_m e^{2\pi i(m,x)}$ (generally speaking, divergent) for which

$$\|f\|_{B^{\boldsymbol{\alpha}}_{\mathbf{pq}}(\mathbb{T}_n)} = \left(\sum_{k_n=0}^{\infty} \ldots \left(\sum_{k_1=0}^{\infty}\left(2^{\sum\limits_{j=1}^{n}\alpha_j k_j}\|\Delta_k(f)\|_{L_{\mathbf{p}}(\mathbb{T}^n)}\right)^{q_1}\right)^{\frac{q_2}{q_1}} \ldots\right)^{\frac{1}{q_n}} < \infty$$

is finite, where $\Delta_k(f)(x) = \sum\limits_{\substack{2^{k_j-1}\leq|m_j|<2^{k_j}\\ j=1,\ldots,n}} a_m e^{2\pi i(m,x)}$, $k \in \mathbb{Z}^n_+$, $(m,x) = \sum\limits_{i=1}^{n} m_i x_i$.

For $q = \infty$ the values of $\left(\sum\limits_{k\in\mathbb{Z}} b_k^q\right)^{\frac{1}{q}}$, $\left(\int\limits_{\mathbb{T}} f^q\right)^{\frac{1}{q}}$ are understood respectively as $\sup\limits_{k\in\mathbb{Z}}|b_k|$, $ess\sup\limits_{x\in\mathbb{T}}|f(x)|$.

We say that the series $f = \sum\limits_{k\in\mathbb{Z}^n} a_k e^{2\pi i(k,x)}$ is an element of the space $W^{\boldsymbol{\alpha}}_{\mathbf{p}}(\mathbb{T}^n)$ [24] if there is a function $f^{\boldsymbol{\alpha}} \in L_{\mathbf{p}}(\mathbb{T}^n)$ Fourier series of which coincides with the series $\sum\limits_{k\in\mathbb{Z}^n} \bar{k}^{\alpha} a_k e^{2\pi i(k,x)}$, here $\bar{k}^{\alpha} = \prod\limits_{j=1}^{n}\bar{k}_j^{\alpha}$, $\bar{k}_j = \max\{|k_j|, 1\}$, $j = 1, \ldots, n$,

$$\|f\|_{W^{\boldsymbol{\alpha}}_{\mathbf{p}}(\mathbb{T}^n)} \overset{def}{=} \|f^{\alpha}\|_{L_{\mathbf{p}}(\mathbb{T}^n)}.$$

We define the concept of convolution for the elements of these spaces.

Let $f = \sum\limits_{k\in\mathbb{Z}^n} a_k e^{2\pi i(k,x)}$ and $g = \sum\limits_{k\in\mathbb{Z}^n} b_k e^{2\pi i(k,x)}$ be trigonometric series. By the convolution of these series we mean the series

$$f * g = \sum_{k\in\mathbb{Z}^n} a_k b_k e^{2\pi i(k,x)}. \tag{2.1}$$

Note that for the «good» functions f and g, the convolution defined by equality (2.1) coincides with the classical definition (1.1). If the functions f and g from the corresponding spaces in (1.3), then $f(x) \overset{L_{\mathbf{p}}}{=} \sum\limits_{k\in\mathbb{Z}^n} \hat{f}(k) e^{2\pi i(k,x)}$ and $g(x) \overset{L_{\mathbf{r}}}{=} \sum\limits_{k\in\mathbb{Z}^n} \hat{g}(k) e^{2\pi i(k,x)}$ and $(f * g)(x) = \int\limits_{\mathbb{T}^n} f(x-y)g(y)dy =^{L_{\mathbf{q}}} \sum_{k\in\mathbb{Z}^n} \hat{f}(k)\hat{g}(k)e^{2\pi i(k,x)}$. Here, equalities are understood in the sense of the corresponding metrics.

We will need interpolation properties of anisotropic Sobolev and Besov spaces [24, 27]. Let $0 < \boldsymbol{\theta} = (\theta_1, \ldots, \theta_n) < 1$, $E = \{\boldsymbol{\varepsilon} = (\varepsilon_1 \ldots, \varepsilon_n) : \varepsilon_j \in \{0, 1\}, j = 1, \ldots, n\}$ be the vertices of the n-dimensional unit cube, $\{A_{\boldsymbol{\varepsilon}}\}_{\boldsymbol{\varepsilon}\in E}$ be Banach spaces that are subspaces of some linear Hausdorff space. For the element $a \in \sum\limits_{\boldsymbol{\varepsilon}\in E} A_{\boldsymbol{\varepsilon}}$, we define the functional

$$K(t, a; A_{\boldsymbol{\varepsilon}}, \boldsymbol{\varepsilon} \in E) = \inf_{a=\sum\limits_{\boldsymbol{\varepsilon}\in E} a_{\boldsymbol{\varepsilon}}} \sum_{\boldsymbol{\varepsilon}\in E} t^{\boldsymbol{\varepsilon}}\|a_{\boldsymbol{\varepsilon}}\|_{A_{\boldsymbol{\varepsilon}}},$$

where $t^{\boldsymbol{\varepsilon}} = t_1^{\varepsilon_1} \ldots t_n^{\varepsilon_n}$.

By $A_{\boldsymbol{\theta}\mathbf{q}} = (A_{\boldsymbol{\varepsilon}};\, \boldsymbol{\varepsilon} \in E)_{\boldsymbol{\theta}\mathbf{q}}$ we denote a linear subset of $\sum\limits_{\boldsymbol{\varepsilon}\in E} A_{\boldsymbol{\varepsilon}}$, for elements of which

$$\|a\|_{A_{\boldsymbol{\theta}\mathbf{q}}} = \left(\int_0^{\infty} \ldots \left(\int_0^{\infty} |t_1^{\frac{1}{\theta_1}-1} \ldots t_n^{\frac{1}{\theta_n}-1} K(t, a; A_{\varepsilon}, \varepsilon \in E)|^{q_1} \frac{dt_1}{t_1}\right)^{\frac{q_2}{q_1}} \ldots \frac{dt_n}{t_n}\right)^{\frac{1}{q_n}} < \infty$$

is true.

Lemma 2.1 [28] Let T be a linear operator such that

$$T:\, A_{\boldsymbol{\varepsilon}} \to B_{\boldsymbol{\varepsilon}} \text{ with norm } M_{\boldsymbol{\varepsilon}},\, \boldsymbol{\varepsilon} \in E.$$

Then

$$T:\, (A_{\boldsymbol{\varepsilon}};\, \boldsymbol{\varepsilon} \in E)_{\boldsymbol{\theta}\mathbf{q}} \to (B_{\boldsymbol{\varepsilon}},\, \boldsymbol{\varepsilon} \in E)_{\boldsymbol{\theta}\mathbf{q}}$$

with the norm $\|T\| \le \max\limits_{\boldsymbol{\varepsilon}\in E} M_{\boldsymbol{\varepsilon}}$.

Theorem 2.1 ([24]) Let $\mathbf{1} \le \mathbf{p} = (p_1,\ldots,p_n) < \boldsymbol{\infty}$, $\mathbf{0} < \mathbf{r} = (r_1,\ldots,r_n)$, $\mathbf{q} = (q_1,\ldots,q_n) \le \boldsymbol{\infty}$, $\boldsymbol{\varepsilon} \in E$, $\boldsymbol{\alpha}_0 = (\alpha_1^0,\ldots,\alpha_n^0)$, $\boldsymbol{\alpha}_1 = (\alpha_1^1,\ldots,\alpha_n^1) \in \mathbb{R}^n$. Then

$$\left(B_{\mathbf{pr}}^{\boldsymbol{\alpha}_{\boldsymbol{\varepsilon}}}(\mathbb{T}^n);\, \boldsymbol{\varepsilon} \in E\right)_{\boldsymbol{\theta}\mathbf{q}} = B_{\mathbf{pq}}^{\boldsymbol{\alpha}}(\mathbb{T}^n),$$

$$\left(W_{\mathbf{p}}^{\boldsymbol{\alpha}_{\boldsymbol{\varepsilon}}}(\mathbb{T}^n);\, \boldsymbol{\varepsilon} \in E\right)_{\boldsymbol{\theta}\mathbf{q}} = B_{\mathbf{pq}}^{\boldsymbol{\alpha}}(\mathbb{T}^n),$$

where $\boldsymbol{\alpha}_{\boldsymbol{\varepsilon}} = (\alpha_1^{\varepsilon},\ldots,\alpha_n^{\varepsilon})$, $\mathbf{0} < \boldsymbol{\theta} = (\theta_1,\ldots,\theta_n) < \mathbf{1}$, $\boldsymbol{\alpha} = (1-\boldsymbol{\theta})\boldsymbol{\alpha}_0 + \boldsymbol{\theta}\boldsymbol{\alpha}_1$.

3. Main result

Lemma 3.1 Let $\mathbf{1} \le \mathbf{q},\, \mathbf{p},\, \mathbf{r} < \infty$, $\dfrac{\mathbf{1}}{\mathbf{q}} + \mathbf{1} = \dfrac{\mathbf{1}}{\mathbf{p}} + \dfrac{\mathbf{1}}{\mathbf{r}}$, $\boldsymbol{\alpha},\, \boldsymbol{\beta},\, \boldsymbol{\gamma} \in \mathbb{R}^n$, $\boldsymbol{\alpha} = \boldsymbol{\beta} + \boldsymbol{\gamma}$. Suppose that $f \in W_{\mathbf{p}}^{\boldsymbol{\beta}}(\mathbb{T}^n)$, $g \in W_{\mathbf{r}}^{\boldsymbol{\gamma}}(\mathbb{T}^n)$. Then $f * g \in W_{\mathbf{q}}^{\boldsymbol{\alpha}}(\mathbb{T}^n)$ and

$$\|f * g\|_{W_{\mathbf{q}}^{\boldsymbol{\alpha}}(\mathbb{T}^n)} \le \|f\|_{W_{\mathbf{p}}^{\boldsymbol{\beta}}(\mathbb{T}^n)} \|g\|_{W_{\mathbf{r}}^{\boldsymbol{\gamma}}(\mathbb{T}^n)}.$$

Proof. Let $f = \sum\limits_{k\in\mathbb{Z}^n} a_k e^{2\pi i(k,x)} \in W_{\mathbf{p}}^{\boldsymbol{\beta}}(\mathbb{T}^n)$, $g = \sum_{k\in\mathbb{Z}^n} b_k e^{2\pi i(k,x)} \in W_{\mathbf{r}}^{\boldsymbol{\gamma}}(\mathbb{T}^n)$. According to the definition, there are functions $f^{\boldsymbol{\beta}} \in L_{\mathbf{p}}(\mathbb{T}^n)$, $g^{\boldsymbol{\gamma}} \in L_{\mathbf{r}}(\mathbb{T}^n)$ Fourier series of which coincide, respectively, with the $\sum\limits_{k\in\mathbb{Z}^n} \bar{k}^{\boldsymbol{\beta}} a_k e^{2\pi i(k,x)}$, $\sum\limits_{k\in\mathbb{Z}^n} \bar{k}^{\boldsymbol{\gamma}} b_k e^{2\pi i(k,x)}$.

From Young's inequality for Lebesgue spaces with mixed metric [29; 25] $(f^{\boldsymbol{\beta}} * g^{\boldsymbol{\gamma}}) \in L_{\mathbf{q}}(\mathbb{T}^n)$ and has the inequality

$$\|f^{\boldsymbol{\beta}} * g^{\boldsymbol{\gamma}}\|_{L_{\mathbf{q}}(\mathbb{T}^n)} \le \|f^{\boldsymbol{\beta}}\|_{L_{\mathbf{p}}(\mathbb{T}^n)} \|g^{\boldsymbol{\gamma}}\|_{L_{\mathbf{r}}(\mathbb{T}^n)}.$$

Now we note that

$$(f^{\boldsymbol{\beta}} * g^{\boldsymbol{\gamma}})(x) \overset{L_{\mathbf{q}}}{=} \sum_{k\in Z^n} \bar{k}^{\boldsymbol{\beta}+\boldsymbol{\gamma}} a_k b_k e^{2\pi i(k,x)} = (f * g)^{\boldsymbol{\alpha}}(x),$$

which means that $(f * g) \in W_{\mathbf{q}}^{\boldsymbol{\alpha}}(\mathbb{T}^n)$ and the inequality

$$\|f * g\|_{W_{\mathbf{q}}^{\boldsymbol{\alpha}}(\mathbb{T}^n)} \le \|f\|_{W_{\mathbf{p}}^{\boldsymbol{\beta}}(\mathbb{T}^n)} \|g\|_{W_{\mathbf{r}}^{\boldsymbol{\gamma}}(\mathbb{T}^n)}$$

holds.

Theorem 3.1 Let $\boldsymbol{\alpha},\, \boldsymbol{\beta},\, \boldsymbol{\gamma} \in \mathbb{R}^n$, $\boldsymbol{\alpha} = \boldsymbol{\beta} + \boldsymbol{\gamma}$, $\mathbf{1} \le \mathbf{q},\, \mathbf{p},\, \mathbf{r} < \boldsymbol{\infty}$, $\mathbf{1} + \dfrac{\mathbf{1}}{\mathbf{q}} = \dfrac{\mathbf{1}}{\mathbf{p}} + \dfrac{\mathbf{1}}{\mathbf{r}}$, $\mathbf{0} < \mathbf{h}, \boldsymbol{\eta}, \boldsymbol{\xi} \le \boldsymbol{\infty}$, $\dfrac{\mathbf{1}}{\mathbf{h}} = \dfrac{\mathbf{1}}{\boldsymbol{\eta}} + \dfrac{\mathbf{1}}{\boldsymbol{\xi}}$. Suppose that $f \in B_{\mathbf{p}\boldsymbol{\eta}}^{\boldsymbol{\beta}}(\mathbb{T}^n)$, $g \in B_{\mathbf{r}\boldsymbol{\xi}}^{\boldsymbol{\gamma}}(\mathbb{T}^n)$. Then $f * g \in B_{\mathbf{qh}}^{\boldsymbol{\alpha}}(\mathbb{T}^n)$ and

$$\|f * g\|_{B_{\mathbf{qh}}^{\boldsymbol{\alpha}}(\mathbb{T}^n)} \le C\|f\|_{B_{\mathbf{p}\boldsymbol{\eta}}^{\boldsymbol{\beta}}(\mathbb{T}^n)} \|g\|_{B_{\mathbf{r}\boldsymbol{\xi}}^{\boldsymbol{\gamma}}(\mathbb{T}^n)}.$$

Proof. Let $f \in W_{\mathbf{p}}^{\boldsymbol{\beta}}(\mathbb{T}^n)$, $g \in W_{\mathbf{r}}^{\boldsymbol{\gamma}}(\mathbb{T}^n)$, then from Lemma 3.1 it follows that $(f * g) \in W_{\mathbf{q}}^{\boldsymbol{\alpha}}(\mathbb{T}^n)$ and the following inequality

$$\|f * g\|_{W_{\mathbf{q}}^{\boldsymbol{\alpha}}(\mathbb{T}^n)} \le \|f\|_{W_{\mathbf{p}}^{\boldsymbol{\beta}}(\mathbb{T}^n)} \|g\|_{W_{\mathbf{r}}^{\boldsymbol{\gamma}}(\mathbb{T}^n)} \tag{3.1}$$

holds true.

Let $\boldsymbol{\alpha}_0 = (\alpha_1^0,\ldots,\alpha_n^0)$, $\boldsymbol{\alpha}_1 = (\alpha_1^1,\ldots,\alpha_n^1)$, $\boldsymbol{\beta}_0 = (\beta_1^0,\ldots,\beta_n^0)$, $\boldsymbol{\beta}_1 = (\beta_1^1,\ldots,\beta_n^1) \in \mathbb{R}^n$, $\alpha_i^0 \ne \alpha_i^1$, $\beta_i^0 \ne \beta_i^1$, $i = \overline{1,n}$. Let $\boldsymbol{\alpha}_{\boldsymbol{\varepsilon}} = (\alpha_1^{\varepsilon_1},\ldots,\alpha_n^{\varepsilon_n})$, $\boldsymbol{\beta}_{\boldsymbol{\varepsilon}} = (\beta_1^{\varepsilon_1},\ldots,\beta_n^{\varepsilon_n})$, $\boldsymbol{\varepsilon} \in E$ such that $\boldsymbol{\alpha}_{\boldsymbol{\varepsilon}} - \boldsymbol{\beta}_{\boldsymbol{\varepsilon}} = \boldsymbol{\gamma}$.

We rewrite inequality (3.1) for the $\boldsymbol{\alpha}_{\varepsilon}$ and $\boldsymbol{\beta}_{\boldsymbol{\varepsilon}}$ parameters

$$\|f * g\|_{W_{\mathbf{q}}^{\boldsymbol{\alpha}_{\boldsymbol{\varepsilon}}}(\mathbb{T}^n)} \le \|f\|_{W_{\mathbf{p}}^{\boldsymbol{\beta}_{\boldsymbol{\varepsilon}}}(\mathbb{T}^n)} \|g\|_{W_{\mathbf{r}}^{\boldsymbol{\gamma}}(\mathbb{T}^n)},\ \ \boldsymbol{\alpha}_{\boldsymbol{\varepsilon}} = \boldsymbol{\beta}_{\boldsymbol{\varepsilon}} + \boldsymbol{\gamma},\, \boldsymbol{\varepsilon} \in E.$$

For a fixed $g \in W_{\mathbf{r}}^{\boldsymbol{\gamma}}(\mathbb{T}^n)$ operator $A_g f = f * g$ acts boundedly from $W_{\mathbf{p}}^{\boldsymbol{\beta}_{\boldsymbol{\varepsilon}}}(\mathbb{T}^n)$ to $W_{\mathbf{q}}^{\boldsymbol{\alpha}_{\boldsymbol{\varepsilon}}}(\mathbb{T}^n)$. Then, using the anisotoropic interpolation theorem (Theorem 2.1)

$$A_g : \left(W_{\mathbf{p}}^{\boldsymbol{\beta}_{\boldsymbol{\varepsilon}}}(\mathbb{T}^n),\, \boldsymbol{\varepsilon} \in E\right)_{\boldsymbol{\theta\xi}} \to \left(W_{\mathbf{q}}^{\boldsymbol{\alpha}_{\boldsymbol{\varepsilon}}}(\mathbb{T}^n),\, \boldsymbol{\varepsilon} \in E\right)_{\boldsymbol{\theta\xi}},$$

we obtain that the operator acts boundedly

$$A_g : B_{\mathbf{p}\boldsymbol{\xi}}^{\boldsymbol{\beta}}(\mathbb{T}^n) \to B_{\mathbf{q}\boldsymbol{\xi}}^{\boldsymbol{\alpha}}(\mathbb{T}^n)$$

and

$$\|A_g\| \leq C\|g\|_{W_{\mathbf{r}}^{\boldsymbol{\gamma}}(\mathbb{T}^n)},$$

where $\boldsymbol{\alpha} = (1-\boldsymbol{\theta})\boldsymbol{\alpha}_0 + \boldsymbol{\theta}\boldsymbol{\alpha}_1$, $\boldsymbol{\beta} = (1-\boldsymbol{\theta})\boldsymbol{\beta}_0 + \boldsymbol{\theta}\boldsymbol{\beta}_1$ for any $\mathbf{0} < \boldsymbol{\theta} = (\theta_1, \ldots, \theta_n) < \mathbf{1}$. Thus, we have obtained the inequality:

$$\|f * g\|_{B_{\mathbf{q}\boldsymbol{\xi}}^{\boldsymbol{\alpha}}(\mathbb{T}^n)} \leq C\|f\|_{B_{\mathbf{p}\boldsymbol{\xi}}^{\boldsymbol{\beta}}(\mathbb{T}^n)}\|g\|_{W_{\mathbf{r}}^{\boldsymbol{\gamma}}(\mathbb{T}^n)}, \tag{3.2}$$

where $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\gamma}$, $\mathbf{p}$, $\mathbf{q}$, $\mathbf{r}$ satisfy the conditions of the theorem.

In inequality (3.2) we assume that $\boldsymbol{\xi} = \infty$, then we have

$$\|f * g\|_{B_{\mathbf{q}\infty}^{\boldsymbol{\alpha}}(\mathbb{T}^n)} \leq C\|f\|_{B_{\mathbf{p}\infty}^{\boldsymbol{\beta}}(\mathbb{T}^n)}\|g\|_{W_{\mathbf{r}}^{\boldsymbol{\gamma}}(\mathbb{T}^n)}.$$

We fix $\boldsymbol{\beta} \in \mathbb{R}^n$. Let $\boldsymbol{\alpha}_0$, $\boldsymbol{\alpha}_1$, $\boldsymbol{\gamma}_0$, $\boldsymbol{\gamma}_1 \in \mathbb{R}^n$ be arbitrary vector parameters satisfying the conditions $\boldsymbol{\alpha}_i = \boldsymbol{\beta} + \boldsymbol{\gamma}_i$, $i = 0, 1$ и $\alpha_j^0 \neq \alpha_j^1$, $\gamma_j^0 \neq \gamma_j^1$, $j = \overline{1, n}$. Then for the parameters $\boldsymbol{\alpha}_{\boldsymbol{\varepsilon}} = (\alpha_1^{\varepsilon_1}, \ldots \alpha_n^{\varepsilon_n})$, $\boldsymbol{\gamma}_{\boldsymbol{\varepsilon}} = (\gamma_1^{\varepsilon_1}, \ldots \gamma_n^{\varepsilon_n})$, $\boldsymbol{\varepsilon} \in E$ the inequality

$$\|f * g\|_{B_{\mathbf{q}\infty}^{\boldsymbol{\alpha}_{\boldsymbol{\varepsilon}}}(\mathbb{T}^n)} \leq C\|f\|_{B_{\mathbf{p}\infty}^{\boldsymbol{\beta}}(\mathbb{T}^n)}\|g\|_{W_{\mathbf{r}}^{\boldsymbol{\gamma}_{\boldsymbol{\varepsilon}}}(\mathbb{T}^n)}.$$

holds.

Now for a fixed $f(x)$ we define a linear operator $B_f g = f * g$. Then B_f acts boundedly from $W_{\mathbf{r}}^{\boldsymbol{\gamma}_{\boldsymbol{\varepsilon}}}(\mathbb{T}^n)$ in $B_{\mathbf{q}\infty}^{\boldsymbol{\alpha}_{\boldsymbol{\varepsilon}}}(\mathbb{T}^n)$ with an estimate of the norm

$$\|B_f\|_{W_{\mathbf{r}}^{\boldsymbol{\gamma}_{\boldsymbol{\varepsilon}}}(\mathbb{T}^n) \to B_{\mathbf{q}\infty}^{\boldsymbol{\alpha}_{\boldsymbol{\varepsilon}}}(\mathbb{T}^n)} \leq C\|f\|_{B_{\mathbf{p}\infty}^{\boldsymbol{\beta}}(\mathbb{T}^n)}.$$

Further, using Theorem 2.1 and Lemma 2.1, we have that the operator B_f is bounded from $B_{\mathbf{r}\boldsymbol{\xi}}^{\boldsymbol{\gamma}}(\mathbb{T}^n)$ to $B_{\mathbf{q}\boldsymbol{\xi}}^{\boldsymbol{\alpha}}(\mathbb{T}^n)$ and

$$\|f * g\|_{B_{\mathbf{q}\boldsymbol{\xi}}^{\boldsymbol{\alpha}}(\mathbb{T}^n)} \leq C\|f\|_{B_{\mathbf{p}\infty}^{\boldsymbol{\beta}}(\mathbb{T}^n)}\|g\|_{B_{\mathbf{r}\boldsymbol{\xi}}^{\boldsymbol{\gamma}}(\mathbb{T}^n)},$$

where $\boldsymbol{\alpha} = (1-\boldsymbol{\theta})\boldsymbol{\alpha}_0 + \boldsymbol{\theta}\boldsymbol{\alpha}_1$, $\boldsymbol{\gamma} = (1-\boldsymbol{\theta})\boldsymbol{\gamma}_0 + \boldsymbol{\theta}\boldsymbol{\gamma}_1$ for any $\mathbf{0} < \boldsymbol{\theta} = (\theta_0, \ldots, \theta_n) < \mathbf{1}$.

Similarly, we can obtain the inequality

$$\|f * g\|_{B_{\mathbf{q}\boldsymbol{\eta}}^{\boldsymbol{\alpha}}(\mathbb{T}^n)} \leq C\|f\|_{B_{\mathbf{p}\boldsymbol{\eta}}^{\boldsymbol{\beta}}(\mathbb{T}^n)}\|g\|_{B_{\mathbf{r}\infty}^{\boldsymbol{\gamma}}(\mathbb{T}^n)}.$$

Thus, for the bilinear convolution operator $T(f, g) = f * g$, we have

$$T : B_{\mathbf{p}\infty}^{\boldsymbol{\beta}_0}(\mathbb{T}^n) \times B_{\mathbf{r}\boldsymbol{\xi}}^{\boldsymbol{\gamma}_0}(\mathbb{T}^n) \to B_{\mathbf{q}\boldsymbol{\xi}}^{\boldsymbol{\alpha}_0}(\mathbb{T}^n),$$

$$T : B_{\mathbf{p}\boldsymbol{\eta}}^{\boldsymbol{\beta}_1}(\mathbb{T}^n) \times B_{\mathbf{r}\infty}^{\boldsymbol{\gamma}_1}(\mathbb{T}^n) \to B_{\mathbf{q}\boldsymbol{\eta}}^{\boldsymbol{\alpha}_1}(\mathbb{T}^n),$$

where the corresponding parameters satisfy the conditions of the theorem.

Next, applying the bilinear interpolation theorem (Theorem 4.4.1, [30, 125]), we have

$$T : \left(B_{\mathbf{p}\infty}^{\boldsymbol{\beta}_0}(\mathbb{T}^n), B_{\mathbf{p}\boldsymbol{\eta}}^{\boldsymbol{\beta}_1}(\mathbb{T}^n)\right)_{[\theta]} \times \left(B_{\mathbf{r}\boldsymbol{\xi}}^{\boldsymbol{\gamma}_0}(\mathbb{T}^n), B_{\mathbf{r}\infty}^{\boldsymbol{\gamma}_1}(\mathbb{T}^n)\right)_{[\theta]} \to \left(B_{\mathbf{q}\boldsymbol{\xi}}^{\boldsymbol{\alpha}_0}(\mathbb{T}^n), B_{\mathbf{q}\boldsymbol{\eta}}^{\boldsymbol{\alpha}_1}(\mathbb{T}^n)\right)_{[\theta]}.$$

Since the space $B_{\mathbf{pq}}^{\mathbf{s}}(\mathbb{T}^n)$ is a retract of $l_{\mathbf{q}}^{\mathbf{s}}(L_{\mathbf{p}})(\mathbb{T}^n)$, we have

$$\left(B_{\mathbf{p}\mathbf{s}_0}^{\boldsymbol{\beta}_0}(\mathbb{T}^n), B_{\mathbf{p}\mathbf{s}_1}^{\boldsymbol{\beta}_1}(\mathbb{T}^n)\right)_{[\theta]} = B_{\mathbf{p}\mathbf{s}}^{\boldsymbol{\beta}}(\mathbb{T}^n),$$

where $\boldsymbol{\beta} = \boldsymbol{\beta}_0(1-\boldsymbol{\theta}) + \boldsymbol{\beta}_1\boldsymbol{\theta}$, $\dfrac{1}{\mathbf{s}} = \dfrac{1-\boldsymbol{\theta}}{\mathbf{s_0}} + \dfrac{\boldsymbol{\theta}}{\mathbf{s}_1}$.

This implies

$$T:\ B^{\boldsymbol{\beta}}_{\mathbf{p}\boldsymbol{\eta}}(\mathbb{T}^n)\times B^{\boldsymbol{\gamma}}_{\mathbf{r}\boldsymbol{\xi}}(\mathbb{T}^n)\to B^{\boldsymbol{\alpha}}_{\mathbf{qh}}(\mathbb{T}^n).$$

Finally,

$$\|f*g\|_{B^{\boldsymbol{\alpha}}_{\mathbf{qh}}(\mathbb{T}^n)}\le C\|f\|_{B^{\boldsymbol{\beta}}_{\mathbf{p}\boldsymbol{\eta}}(\mathbb{T}^n)}\|g\|_{B^{\boldsymbol{\gamma}}_{\mathbf{r}\boldsymbol{\xi}}(\mathbb{T}^n)},$$

where

$$\frac{\mathbf{1}}{\mathbf{h}}=\frac{\mathbf{1}}{\boldsymbol{\eta}}+\frac{\mathbf{1}}{\boldsymbol{\xi}},\ \mathbf{1}+\frac{\mathbf{1}}{\mathbf{q}}=\frac{\mathbf{1}}{\mathbf{p}}+\frac{\mathbf{1}}{\mathbf{r}},\ \boldsymbol{\alpha}=\boldsymbol{\beta}+\boldsymbol{\gamma}.$$

Taking into account the embeddings of spaces ([24], Theorem 4), we can obtain the following theorem.

Theorem 3.2 Let $\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}\in\mathbb{R}^n$, $\boldsymbol{\alpha}\le\boldsymbol{\beta}+\boldsymbol{\gamma}$, $1\le\mathbf{q},\mathbf{p},\mathbf{r}<\infty$, $\boldsymbol{\alpha}=\boldsymbol{\beta}+\boldsymbol{\gamma}+\mathbf{1}+\frac{\mathbf{1}}{\mathbf{q}}-\frac{\mathbf{1}}{\mathbf{p}}-\frac{\mathbf{1}}{\mathbf{r}}$, $\mathbf{0}<\mathbf{h},\boldsymbol{\eta},\boldsymbol{\xi}\le\boldsymbol{\infty}$. Suppose that $f(x)$ and $g(x)$ are measurable functions on $\mathbb{T}^n$ such that $f\in B^{\boldsymbol{\beta}}_{\mathbf{p}\boldsymbol{\eta}}(\mathbb{T}^n)$, $g\in B^{\boldsymbol{\gamma}}_{\mathbf{r}\boldsymbol{\xi}}(\mathbb{T}^n)$. Then $f*g\in B^{\boldsymbol{\alpha}}_{\mathbf{qh}}(\mathbb{T}^n)$ and

$$\|f*g\|_{B^{\boldsymbol{\alpha}}_{\mathbf{qh}}(\mathbb{T}^n)}\le C\|f\|_{B^{\boldsymbol{\beta}}_{\mathbf{p}\boldsymbol{\eta}}(\mathbb{T}^n)}\|g\|_{B^{\boldsymbol{\gamma}}_{\mathbf{r}\boldsymbol{\xi}}(\mathbb{T}^n)},$$

where $\frac{1}{\mathbf{h}}\le\frac{1}{\boldsymbol{\eta}}+\frac{1}{\boldsymbol{\xi}}$.

References

1 Bennett C. Interpolation of Operators / C. Bennett, R. Sharpley. Pure and Applied Mathematics 129, Boston, MA, Academic Press, INC. — 1988. — 469 p.

2 Brézis H. A note on limiting cases of Sobolev embeddings and convolution inequalities / H. Brézis, S. Wainger // Comm. Partial Differential Equations. — 1980. — 5. — No. 7. — P. 773–789.

3 Hörmander L. The analysis of linear partial differential operators I. Distribution theory and Fourier analysis. Reprint of the second edition / L. Hörmander. — Berlin: Classics in Mathematics, Springer-Verlag. — 2013. — 440 p.

4 O'Neil R. Convolution operators and $L(p,q)$ spaces / R. O'Neil // Duke Math. J. — 1963. — 30. — P. 129–142.

5 Yap, L.Y.H. Some remarks on convolution operators and $l(p,q)$ spaces / L.Y.H. Yap // Duke Math J. — 1969. — 36. — P. 647–658.

6 Hunt R.A. On $L(p,q)$ spaces / R.A. Hunt // Enseignement Math. — 1966. — 12. — No. 2. — P. 249-276.

7 Nursultanov E. Convolution inequalities in Lorentz spaces / E. Nursultanov, S. Tikhonov // J. Fourier Anal. Appl. — 2011. — 17. — P. 486–505.

8 Blozinski A.P. On a convolution theorem for $L(p,q)$ spaces / A.P. Blozinski // Trans. Amer. Math. Soc. — 1972. — 164. — P. 255–265.

9 Нурсултанов Е.Д. Мультипликаторы кратных рядов Фурье / Е.Д. Нурсултанов, Н.Т. Тлеуханова // Тр. МИ РАН им. В.А. Стеклова. — 1999. — 227. — P. 231–236.

10 Tleukhanova N.T. The O'Neil-type inequalities for anisotropic Lorentz spaces $L(\bar{p},\bar{q})$ / N.T. Tleukhanova, K.K. Sadykova // Eurasian Mathematical Journal. В печати.

11 Nursultanov E. Norm inequalities for convolution operators / E. Nursultanov, S. Tikhonov, N. Tleukhanova // C. R. Acad. Sci. Paris. — 2009. — I. — No. 347. — P. 1385–1388.

12 Nursultanov E. Norm convolution inequalities in Lebesgue spaces / E. Nursultanov, S. Tikhonov, N. Tleukhanova // Rev. Mat. Iberoam. — 2018. — 34. — No. 2. — P. 811–838.

13 Heil C. An introduction to weighted Wiener amalgams. In Wavelets and their applications / C. Heil // Allied Publishers, New Delphi. — 2003. — P. 183–216.

14 Kamińska A. On convolution operator in Orlicz spaces / A. Kamińska, J. Musielak // Rev. Mat. Univ. Complutense. — 1989. — 2. — P. 157–178.

15 Kerman R.A. Convolution theorems with weights / R.A. Kerman // Trans. Amer. Math. Soc. — 1983. — 280. — No. 1. — P. 207–219.

16 Kerman R. Convolution algebras with weighted rearrangement-invariant norm / R. Kerman, E. Sawyer // Studia Math. — 1994. — 108. — No. 2. — P. 103–126.

17 Nursultanov E. Weighted norm inequalities for convolution and Riesz potential / E. Nursultanov, S. Tikhonov // Potential Analysis. — 2015. — 42. — No. 2. — P. 435–456.

18 Sampson G. (L_p, L_q) mapping properties of convolution transforms / G. Sampson, A. Naparstek, V. Drobot // Studia Math. — 1976. — 55. — No. 1. — P. 41–70.

19 Головкин К.К. Оценки интегральных операторов в трансляционно-инвариантных нормах / К.К. Головкин, В.А. Солонников // Тр. МИ АН СССР. — 1964. — 70. — С. 47–58.

20 Головкин К.К. Оценки интегральных операторов в трансляционно-инвариантных нормах. II / К.К. Головкин, В.А. Солонников // Тр. МИ АН СССР. — 1966. — 92. — С. 5–30.

21 Батыров Б.Е. Об оценках сверток в пространствах Никольского-Бесова / Б.Е. Батыров, В.И. Буренков // Док. АН СССР. — 1993. — 330. — № 1. — С. 9–11.

22 Bui H. Weighted Young's inequality and convolution theorems on weighted Besov spaces / H. Bui // Math. Nachr. — 1994. — 170. — P. 25–37.

23 Головкин К.К. Об оценках операторов свертки / К.К. Головкин, В.А. Солонников // Зап. науч. сем. ЛОМИ. — 1968. — 7. — P. 6–86.

24 Nursultanov E.D. Interpolation theorems for anisotropic function spaces and their applications / E.D. Nursultanov // Dokl. Math. — 2004. — 69. — No. 1. — P. 16-–19.

25 Бекмаганбетов К.А. Теоремы вложения анизотропных пространств Бесова $B_{\mathbf{pr}}^{\alpha\mathbf{q}}([0, 2\pi)^n)$ / К.А. Бекмаганбетов, Е.Д. Нурсултанов // Изв. РАН. Сер. Матем. — 2009. — 73. — № 4. — P. 3–16.

26 Bekmaganbetov K.A. Order of the orthoprojection widths of the anisotropic Nikolsky–Besov classes in the anisotropic Lorentz space / K.A. Bekmaganbetov, Ye. Toleugazy // Eurasian Math. J. — 2016. — 7. — No. 3. — P. 8–16.

27 Nursultanov E.D. Application of Interpolational Methods to the Study of Properties of Functions of Several Variables / E.D. Nursultanov // Mathematical Notes. — 2004. — 75. — No. 3. — P. 341–351.

28 Nursultanov E.D. On the coefficients of multiple Fourier series in L_p spaces / E.D. Nursultanov // Izvestiia RAN: Ser. Mat. — 2000. — 64. — No. 1. — P. 95–122.

29 Бесов О.В. Интегральные представления функций и теоремы вложения / О.В. Бесов, В.П. Ильин, С.М. Никольский. — М.: Наука, 1975. — 480 с.

30 Берг Й. Интерполяционные пространства. Введение / Й. Берг, Й. Лефстрем. — М.: Мир, 1980. — 264 с.

К.К. Садыкова, Н.Т. Тлеуханова

Аралас туындысы басым анизотропты Бесов кеңiстiгiндегi үйiрткi операторының нормаларын бағалау

Мақалада үйiрткi операторы нормасының аралас туындысы басым Соболев және анизотропты Никольский-Бесов кеңiстiктерiндегi шенелуi зерттелдi. Аралас туындысы басым Соболев кеңiстiгi үшiн Юнг теңсiздiгiнiң аналогы алынды, атап айтқанда, $W_p^\gamma * W_r^\beta \hookrightarrow W_q^\alpha$ түрiндiгi қатынас дәлелдендi, мұнда катынас параметрлерiне сәйкесiнше шарттар қойылған. Жұмыстың негiзгi мақсаты келесi есеп болып табылады: айталық, f және g — Никольский-Бесов кеңiстiгi шкаласының қандай да бiр функциялар класы болсын. Олардың $f * g$ үйiрткiсi қай кеңiстiкке жататынын анықтау қажет. (1) қатынас пен Нұрсұлтановтың анизотропты кеңiстiктерге арналған интерполяциялық теоремаларын қолдана отырып, $B_{pq}^{\boldsymbol{\alpha}}$ Никольский-Бесов кеңiстiгiнiң шкаласы үшiн O'Нейл теоремасының аналогы алынды, мұнда α, p, q — векторлық параметрлер. $B_{ps_1}^\gamma * B_{rs_2}^\beta \hookrightarrow B_{qs}^\alpha$ түрiндiгi қатынас дәлелдендi, мұнда қатынас параметрлерiне сәйкесiнше шарттар қойылған. Осы жұмыста алынған теоремалар изотропты Никольский-Бесов кеңiстiгiндегi, яғни скаляр параметрлi кеңiстiктегi, Батыров пен Буренковтың ұқсас есептер қарастырган нәтижелерiн толықтырады.

Кiлт сөздер: үйiрткi операторы, анизотропты Бесов кеңiстiгi, анизотропты Соболев кеңiстiгi, интерполяция.

К.К. Садыкова, Н.Т. Тлеуханова

Оценки нормы оператора свертки в анизотропных пространствах Бесова с доминирующей смешанной производной

В статье исследована ограниченность нормы оператора свертки в пространствах Соболева, с доминирующей смешанной производной, и анизотропных пространствах Никольского-Бесова. Для пространств Соболева с доминирующей смешанной производной получен аналог неравенства Юнга, а именно доказаны соотношения вида $W_p^\gamma * W_r^\beta \hookrightarrow W_q^\alpha$ при выполнении соответствующих условий на параметры. Основной целью работы является решение следующей задачи: пусть f и g — функции из некоторых классов шкалы пространств Никольского-Бесова. Нужно определить, к какому пространству принадлежит их свертка $f * g$. Используя соотношение (1) и интерполяционные теоремы Нурсултанова для анизотропных пространств, получен аналог теоремы О'Нейла для шкалы пространств Никольского-Бесова B_{pq}^α, где α, p, q — векторные параметры. Получены соотношения вида $B_{ps_1}^\gamma * B_{rs_2}^\beta \hookrightarrow B_{qs}^\alpha$, при соответствующих соотношениях векторных параметров. Полученные в данной работе теоремы дополняют результаты Батырова и Буренкова, где рассматривались подобные задачи в изотропных пространствах Никольского-Бесова, т.е. в пространствах, где параметры являются скалярами.

Ключевые слова: оператор свертки, анизотропные пространства Бесова, анизотропные пространства Соболева, интерполяция.

References

1 Bennett, C. & Sharpley, R. (1988). *Interpolation of Operators.* Pure and Applied Mathematics 129, Boston, MA, Academic Press, INC.

2 Brézis, H. & Wainger, S. (1980). A note on limiting cases of Sobolev embeddings and convolution inequalities. *Comm. Partial Differential Equations, 5, 7,* 773-789.

3 Hörmander, L. (2013). *The analysis of linear partial differential operators I. Distribution theory and Fourier analysis. Reprint of the second edition.* Berlin: Classics in Mathematics, Springer-Verlag.

4 O'Neil, R. (1963). Convolution operators and $L(p, q)$ spaces. *Duke Math. J., 30,* 129–142.

5 Yap, L.Y.H. (1969). Some remarks on convolution operators and $l(p, q)$ spaces. *Duke Math J., 36,* 647–658.

6 Hunt, R.A. (1966). On $L(p, q)$ spaces. *Enseignement Math., 12, 2,* 249–276.

7 Nursultanov, E. & Tikhonov, S. (2011). Convolution inequalities in Lorentz spaces. *J. Fourier Anal. Appl., 17,* 486-505.

8 Blozinski, A.P. (1972). On a convolution theorem for $L(p, q)$ spaces. *Trans. Amer. Math. Soc., 164,* 255–265.

9 Nursultanov, E.D. & Tleukhanova, N.T. (1999). Multiplikatory kratnykh riadov Fure [Multipliers of multiple Fourier series]. *Trudy MI RAN imeni V.A. Steklova – Proceedings of the Steklov MI RAS, 227,* 231–236 [in Russian].

10 Tleukhanova, N.T. & Sadykova, K.K. The O'Neil-type inequalities for anisotropic Lorentz spaces $L(\bar{p}, \bar{q})$. *Eurasian Mathematical Journal.*

11 Nursultanov, E. & Tikhonov, S. & Tleukhanova, N. (2009). Norm inequalities for convolution operators. *C. R. Acad. Sci. Paris, I, 347,* 1385–1388.

12 Nursultanov, E. & Tikhonov, S. & Tleukhanova, N. (2018). Norm convolution inequalities in Lebesgue spaces. *Rev. Mat. Iberoam., 34, 2,* 811–838.

13 Heil, C. (2003). An introduction to weighted Wiener amalgams. In Wavelets and their applications. *Allied Publishers, New Delphi,* 183–216.

14 Kamińska, A. & Musielak, J. (1989). On convolution operator in Orlicz spaces. *Rev. Mat. Univ. Complutense, 2,* 157–178.

15 Kerman, R.A. (1983). Convolution theorems with weights. *Trans. Amer. Math. Soc., 280, 1,* 207–219.

16 Kerman, R. & Sawyer, E. (1994). Convolution algebras with weighted rearrangement-invariant norm. *Studia Math., 108, 2,* 103–126.

17 Nursultanov, E. & Tikhonov, S. (2015). Weighted norm inequalities for convolution and Riesz potential. *Potential Analysis, 42, 2,* 435–456.

18 Sampson, G. & Naparstek, A. & Drobot, V. (1976). (L_p, L_q) mapping properties of convolution transforms. *Studia Math., 55, 1,* 41-70.

19 Golovkin, K.K. & Solonnikov, V.A. (1964). Otsenki intehralnykh operatorov v transliatsionno-invariantnykh normakh [Estimates of integral operators in translation-invariant norms]. *Trudy MI AN SSSR – Proceedings of MI AN USSR, 70,* 47–58 [in Russian].

20 Golovkin, K.K. & Solonnikov, V.A. (1966). Otsenki intehralnykh operatorov v transliatsionno-invariantnykh normakh. II [Estimates of integral operators in translation-invariant norms. II]. *Trudy MI AN SSSR – Proceedings of MI AN USSR, 92,* 5–30 [in Russian].

21 Batyrov, B.E. & Burenkov, V.I. (1993). Ob otsenkakh svertok v prostranstvakh Nikolskoho-Besova [On estimates of convolutions in Nikolsky-Besov spaces]. *Doklady AN SSSR – Reports of the AS USSR, 330, 1,* 9–11 [in Russian].

22 Bui, H. (1994). Weighted Young's inequality and convolution theorems on weighted Besov spaces. *Math. Nachr., 170,* 25–37.

23 Golovkin, K.K., & Solonnikov, V.A. (1968). Ob otsenkakh operatorov svertki [On estimates of convolution operators]. *Zapiski nauchnoho seminara LOMI – Notes of scientific seminars of the LBMI, 7,* 6–86 [in Russian].

24 Nursultanov, E.D. (2004). Interpolation theorems for anisotropic function spaces and their applications. *Dokl. Math., 69, 1,* 16–19.

25 Bekmaganbetov, K.A. & Nursultanov, E.D. (2009). Teoremy vlozheniia anizotropnykh prostranstv Besova $B_{\mathbf{pr}}^{\alpha\mathbf{q}}([0,2\pi)^n)$ [Embedding theorems for anisotropic Besov spaces $B_{\mathbf{pr}}^{\alpha\mathbf{q}}([0,2\pi)^n)$]. *Izvestiia RAN SSSR. Seriia Matematila – Proceedings of the RAS USSR. Ser. Math., 73, 4,* 3–16 [in Russian].

26 Bekmaganbetov, K.A. & Toleugazy, Ye. (2016). Order of the orthoprojection widths of the anisotropic Nikolsky–Besov classes in the anisotropic Lorentz space. *Eurasian Math. J., 7, 3,* 8–16.

27 Nursultanov, E.D. (2004). Application of Interpolational Methods to the Study of Properties of Functions of Several Variables. *Mathematical Notes, 75, 3,* 341–351.

28 Nursultanov, E.D. (2000). On the coefficients of multiple Fourier series in L_p spaces. *Izvestiia RAN: Ser. Mat., 64, 1,* 95–122.

29 Besov, O.V. & Il'in, V.P. & Nikolskii, S.M. (1975). *Intehralnye predstavleniia funktsii i teoremy vlozheniia [Integral presentation of functions and theorem of embanding].* Moscow: Nauka [in Russian].

30 Berg, I. & Löfstrem, I. (1980) *Interpoliatsionnye prostranstva. Vvedenie [Interpolation spaces. Introduction].* Moscow: Mir [in Russian].