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Best trigonometric approximation and modulus of smoothness of functions in weighted grand Lebesgue spaces

In this work, first of all, $L_{\omega}^{p),\theta}(\mathbb{T})$ weighted grand Lebesgue spaces and Muckenhoupt weights is defined. The information about properties of these spaces is given. Let T_n be the trigonometric polynomial of best approximation. The approximation of the functions in grand Lebesgue spaces have been investigated by many authors. In this work the relation between fractional derivatives of a T_n trigonımetric polynomial and the best approximation of the function is investigated in weighted grand Lebesgue spaces. In that regard, the neccessary and sufficient condition is expressed in Theorem 1. In addition, in this work in weighted grand Lebesgue spaces a specific operator is defined. Later on, with the help of this operator the fractional modules of smoothness of order r of function f is defined. Also, in this work, using the properties of modulus of smoothness of function, the relationship between the fractional modulus of smoothness of the function and $n-th$ partial and de la Vallée-Poussin sums of its Fourier series in subspace of weighted grand Lebesgue spaces are studied. These results are expressed in Theorem 2.

Keywords: heneralized grand Lebesgue spaces, fractional derivative, fractional moduli of smoothness, $n-th$ partial sums, de la Vallée-Poussin sums, best approximation by trigonometric polynomials.

Introduction and the main results

Let $\mathbb{T}$ denote the interval $[-\pi,\pi]$. We denote by $L^p(\mathbb{T})$, $1 \le p \le \infty$, the Lebesgue space of all measurable functions f on $\mathbb{T}$, that is, the space of all such functions for which

$$\|f\|_p = \left(\int_{\mathbb{T}} |f(x)|^p \, dx \right)^{1/p} < \infty.$$

A function ω is called a *weight* on $\mathbb{T}$ if $\omega : \mathbb{T} \to [0,\infty]$ is measurable and $\omega^{-1}(\{0,\infty\})$ has measure zero (with respect to Lebesgue measure).

Let ω be a 2π periodic weight function. We denote by $L_{\omega}^p(\mathbb{T})$, $1 < p < \infty$, the weighted Lebesgue space of all measurable functions on $\mathbb{T}$ for which the norm

$$\|f\|_p = \left(\int_{\mathbb{T}} |f(x)|^p \, \omega dx \right)^{1/p} < \infty.$$

We define a class $L_{\omega}^{p),\theta}(\mathbb{T})$, $\theta > 0$ of 2π periodic measurable functions on $\mathbb{T}$ satisfying the condition

$$\sup_{0<\varepsilon<p-1} \left\{ \frac{\varepsilon^{\theta}}{2\pi} \int_{\mathbb{T}} |f(x)|^{p-\varepsilon} \, \omega(x) dx \right\}^{1/(p-\varepsilon)} < \infty.$$

The class $L_{\omega}^{p),\theta}(\mathbb{T})$, $\theta > 0$, is a Banach space with respect to the norm

$$\|f\|_{L_{\omega}^{p),\theta}(\mathbb{T})} := \sup_{0<\varepsilon<p-1} \left\{ \varepsilon^{\theta} \frac{1}{2\pi} \int_{\mathbb{T}} |f(x)|^{p-\varepsilon} \, \omega(x) dx \right\}^{1/(p-\varepsilon)}. \tag{1}$$

The class $L_{\omega}^{p),\theta}(\mathbb{T})$ with the norm (1) is called as the weighted generalized grand Lebesgue space. Note that non- weighted grand Lebesgue space $L^{p)}(\mathbb{T})$ was introduced by Iwaniec and Sbordone [1]. Information about properties of these spaces can be found in [2–4]. The embeddings

$$L^p(\mathbb{T}) \subset L^{p)}(\mathbb{T}) \subset L^{p-\varepsilon},$$

hold. According to [2] $L^p(\mathbb{T})$ is not dense in $L^{p)}(\mathbb{T})$. Also, if $\theta_1 < \theta_2$ and $1 < p < \infty$, for weighted generalized grand Lebesgue space, the following relations hold:

$$L_{\omega}^{p}(\mathbb{T}) \subset L_{\omega}^{p),\theta_1}(\mathbb{T}) \subset L_{\omega}^{p),\theta_2}(\mathbb{T}) \subset L_{\omega}^{p-\varepsilon,}(\mathbb{T}).$$

The closure of the space $L^p(\mathbb{T})$ by the norm of $L_{\omega}^{p),\theta}(\mathbb{T})$, $\theta > 0$, we denote by $\widetilde{L}_{\omega}^{p,\theta}(\mathbb{T})$.

Let $1 < p < \infty$ and let $A_p(\mathbb{T})$ be the collection of all weights on $\mathbb{T}$ satisfying the condition

$$\sup_I \left(\frac{1}{|I|}\int_I \omega(x)^p dx\right)^{1/p} \left(\frac{1}{|I|}\int_I [\omega(x)]^{-1/(p-1)} dx\right)^{p-1} < \infty, \qquad (2)$$

where the supremum is taken over all intervals I with length $|I| \le 2\pi$. The condition (2) is called the *Muckenhoupt* -A_p condition [5] and the weight functions which belong to $A_p(\mathbb{T})$, $(1 < p < \infty)$, are called as the *Muckenhoupt weights.*

Suppose that $f \in L_{\omega}^{p),\theta}$. We define the operator by

$$A_h f(x) := \frac{1}{h}\int\limits_{x-\frac{h}{2}}^{x+\frac{h}{2}} f(t)dt, \quad x \in \mathbb{T} \;\; 0 < h \le 1.$$

Note that $0 < p < \infty,\ \theta > 0$ and $\omega \in A_p$ then the operator A_h is bounded in $L_{\omega}^{p),\theta}$. For $f \in L_{\omega}^{p),\theta}$, we define

$$\sigma_h^r f(x) := (I - A_h)^r f(x) =$$
$$= \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(r+1)}{\Gamma(k+1)\Gamma(r-k+1)} (A_h)^k, \;\; x, \;\; h \in \mathbb{T}, \, 0 \le r,$$

where I is the identity operator and Γ is gamma function.

Let $\omega \in A_p$ and $f \in L_{\omega}^{p),\theta}$. If $0 \le r$ we can define the *fractional modules of smoothness of order* r of f as

$$\Omega_r(f,\delta)_{p),\theta,\omega} := \sup_{0 \le h_i,\ t \le \delta} \left\| \prod_{i=1}^{[r]} (I - A_{h_i})\, \sigma_t^{\{r\}} f \right\|_{p),\theta,\omega}, \;\; \delta \ge 0,$$

where $[r]$ denotes the integer part of the real number r and $\{r\} := r - [r]$.– Note that $\Omega_0(f,\delta)_{p),\theta,\omega} := \|f\|_{p),\theta,\omega}$ and $\prod_{i=1}^{0} (I - A_{h_i})\, \sigma_t^r f := \sigma_t^r$ for $0 < r < 1$. The modulus of smoothness $\Omega_r(f,\delta)_{p)\theta,\omega}$, $r \in \mathbb{R}^+$, is a nondecreasing, nonnegative function of δ, and

$$\begin{aligned} \Omega_r(f+g,\delta)_{p),\theta,\omega} &\le \Omega_\alpha(f,\delta)_{p),\theta,\omega} + \Omega_\alpha(g,\delta)_{p),\theta,\omega}, \\ \lim_{\delta\to 0} \Omega_r(f,\delta)_{p),\theta,\omega} &= 0 \end{aligned}$$

for $f, g \in L_{\omega}^{p),\theta}$.

Let

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} A_k(f,x), \;\; A_k(f,x) := a_k(f)\cos kx + b_k(f)\sin kx \qquad (3)$$

be the Fourier series of the function $f \in L^1(\mathbf{T})$, where $a_k(f)$ and $b_k(f)$ are Fourier coefficients of the function f. The nth *partial sums* and *de la Vallée-Poussin sums* of the series (3) are defined, respectively, as:

$$S_n(f,x) = \frac{a_0}{2} + \sum_{k=1}^{n} A_k(f,x),$$
$$V_n(f,x) := \frac{1}{n}\sum_{\nu=n}^{2n-1} S_\nu(f,x).$$

We denote by $E_n(f)_{p(.)}$, $(n = 0,1,2,...)$ the best approximation of $f \in L_{2\pi}^{p(.)}$ by trigomonetric polynomials of degree not exceeding n, i.e.

$$E_n(f)_{p),\theta,\omega} := \inf\left\{\|f - T_n\|_{p),\theta,\omega} : T_n \in \Pi_n\right\},$$

where Π_n denotes the class of trigonometric polynomials of degree at most n.

We use the relation $\alpha_n = O(\beta_n)$, $n = 1,2,...$, that is, there exists a constant $C > 0$ such as $\alpha_n \le C\beta_n$, $n = 1,2,....$

The approximation problems of the functions by trigonometric polynomials in grand Lebesgue spaces have been investigated by several authors [6–13].

In the present paper, in weighted generalized grand Lebesgue spaces we investigate the relation between derivatives of a polynomial of best approximation and the best approximation of the function. In addition, relationship between fractional modulus of smoothness of the function and nth partial and de la Vallée-Poussin sums of its Fourier series in subspace of weighted grand Lebesgue spaces are studied. Simillar results in different spaces have been investigated in [14–34], and [5]. Note that, in the proof of the main results we use the method as in the proof of [27, 28].

Our main results are the following:

Theorem 1. Let $T_n(f) \in \Pi_n$ be the polynomial of best approximation to f, let r, $\alpha \in \mathbb{R}^+$. In order that

$$\left\|T_n^{(r)}(f)\right\|_{p),\theta,\omega} = O(n^{r-\alpha}) \quad (r > \alpha > 0)$$

it is necessary and sufficient that

$$E_n(f)_{p),\theta,\omega} = O(n^{-\alpha}).$$

Theorem 2. Let $1 < p < \infty$, $\theta > 0$, $r \in \mathbb{R}^+$ and $\omega \in A_p$. If $f \in \widetilde{L}_\omega^{p),\theta}(\mathbb{T})$, then

1.

$$c_4\Omega_r(f,\frac{1}{n})_{p),\theta,\omega} \le n^{-2r}\left\|V_n^{(2r)}(f,\cdot)\right\|_{p),\theta,\omega} + \|f(x) - V_n(f,\cdot)\|_{p),\theta,\omega} \le$$
$$\le c_5\Omega_r(f,\frac{1}{n})_{p),\theta,\omega}, \tag{4}$$

where the constants c_4 and c_5 are dependent on p and r.

2.

$$c_6\Omega_r(f,\frac{1}{n})_{p),\theta,\omega} \le n^{-2r}\left\|S_n^{(2r)}(f,\cdot)\right\|_{p),\theta,\omega} + \|f(x) - S_n(f,\cdot)\|_{p),\theta,\omega} \le$$
$$\le c_7\Omega_r(f,\frac{1}{n})_{p),\theta,\omega}, \tag{5}$$

where the constants c_6 and c_7 are dependent on p and r.

Proofs of the main results

Proof of Theorem 1. Let us assume that

$$E_n(f)_{p),\theta,\omega} = \|f - T_n(f)\|_{p),\theta,\omega} = O(n^{-\alpha}), \quad (\alpha > 0). \tag{6}$$

is satisfied. We can write

$$T_n^{(r)}(x) = T_0^{(r)}(x) + \sum_{\nu=0}^{n-1} \left\{ T_{\nu+1}^{(r)}(x) - T_\nu^{(r)}(x) \right\}. \tag{7}$$

Using Bernstein inequality for the spaces $\widetilde{L}_\omega^{p),\theta}(\mathbb{T})$ in [35] we have

$$\left\| T_n^{(r)}(f) \right\|_{p),\theta,\omega} \le c_8 n^r \left\| T_n(f) \right\|_{p),\theta,\omega}.$$

From (6), (7) and the last relation we conclude that

$$\left\| T_n^{(r)}(f) \right\|_{p),\theta,\omega} \le c_9 c_{10} n^r n^{-\alpha} \le c_{11} n^{r-\alpha}.$$

Now we suppose that

$$\left\| T_n^{(r)}(f) \right\|_{p),\theta,\omega} = O(n^{r-\alpha}). \tag{8}$$

Use of [9] and (8) leads to

$$\begin{aligned} \left\| T_{2n}(f) - T_n(T_{2n}(f)) \right\|_{p),\theta,\omega} &\le \left\| f - T_{2n}(f) \right\|_{p),\theta,\omega} + \left\| f - T_n(T_{2n}(f)) \right\|_{p),\theta,\omega} \le \\ &\le c_{12} n^{-r} \left\| T_n^{(r)}(f) \right\|_{p),\theta,\omega} \le c_{13} n^{-\alpha}. \end{aligned} \tag{9}$$

On the other hand, since $T_n(T_{2n}(f))$ is a polynomial of order n the following inequality holds:

$$\begin{aligned} \left\| T_{2n}(f) - T_n(T_{2n}(f)) \right\|_{p),\theta,\omega} &= \left\| f - T_n(T_{2n}(f)) - (f - T_{2n}(f)) \right\|_{p),\theta,\omega} \ge \\ &\ge \left\| f - T_n(T_{2n}(f)) \right\|_{p),\theta,\omega} - \left\| f - T_{2n}(f)) \right\|_{p),\theta,\omega} \ge \\ &\ge E_n(f)_{p),\theta,\omega} - E_{2n}(f)_{p),\theta,\omega} \ge 0. \end{aligned} \tag{10}$$

Use of (9) and (10) gives us

$$0 \le E_n(f)_{p),\theta,\omega} - E_{2n}(f)_{p),\theta,\omega} \le c_{14} n^{-\alpha}. \tag{11}$$

Condition $E_n(f)_{p),\theta,\omega} \to 0$ is satisfied. Therefore, from the inequality (11) we have

$$\sum_{k=n_0}^{\infty} \left\{ E_{2^k}(f)_{p),\theta,\omega} - E_{2^{k+1}}(f)_{p),\theta,\omega} \right\} \le c_{15} \sum_{k=n_0}^{\infty} 2^{-k\alpha}.$$

Thus,

$$E_{2^{n_0}}(f)_{p),\theta,\omega} \le c_{16} 2^{-n_0 \alpha}. \tag{12}$$

By (12) we conclude that $E_n(f)_{p),\theta,\omega} \le c_{15}(n^{-\alpha})$.

This completes the proof Theorem 1.

Proof of Theorem 2. By [9] the inequality

$$\Omega_r(T_n, \frac{1}{n})_{p),\theta,\omega} \le c_{17}(p,r) n^{-2r} \left\| T_n^{(2r)} \right\|_{p),\theta,\omega}, \tag{13}$$

holds, where $T_n \in \Pi_n$. On the other hand, using the properties of modulus of smoothness $\Omega_r(f, \frac{1}{n})_{p),\theta,\omega}$ and (13), we find

$$\begin{aligned} \Omega_r\left(f, \frac{1}{n}\right)_{p),\theta,\omega} &\le \left(\Omega_r\left(f - T_n, \frac{1}{n}\right)_{p),\theta,\omega} + \Omega_r\left(T_n, \frac{1}{n}\right)_{p),\theta,\omega} \right) \le \\ &\le c_{18}(p,r) \left(\left\| f - T_n \right\|_{p),\theta,\omega} + n^{-2r} \left\| T_n^{(2r)} \right\|_{p),\theta,\omega} \right). \end{aligned}$$

Now we estimate the modulus of smoothness $\Omega_r(f,\cdot)_{p),\theta,\omega}$ from below. According to reference [10] the following inequalities hold:

$$E_n(f)_{p),\theta,\omega} \le c_{19}\,(p,r)\,\Omega_r\left(f,\frac{2\pi}{n+1}\right)_{p),\theta,\omega}; \tag{14}$$

$$n^{-2r}\left\|T_n^{(2r)}\right\|_{p),\theta,\omega} \le c_{20}\,(p,r)\,\Omega_r\left(f,\frac{2\pi}{n+1}\right)_{p),\theta,\omega}. \tag{15}$$

Let $V_n(f,x)$ be de la Vallée-Poussin sums of the series (3) and let $T_n^* \in \Pi_n$ be the polynomial of best approximation to f in $\widetilde{L}_{\omega}^{p),\theta}(\mathbb{T})$, that is $\|f-T_n^*\|_{p),\theta,\omega} = E_n(f)_{p),\theta,\omega}$. Then we get

$$\begin{aligned}\|f-V_n(f,\cdot)\|_{p),\theta,\omega} &\le \|f-T_n^*\|_{p),\theta,\omega} + \|T_n^*-V_n(f,\cdot)\|_{p),\theta,\omega} \le \\ &\le c_{21}(p)E_n(f)_{p),\theta,\omega} + \|V_n(T_n^*-f,\cdot)\|_{p),\theta,\omega} \le \\ &\le c_{22}(p)E_n(f)_{p),\theta,\omega}.\end{aligned} \tag{16}$$

Using (14), (15) and (16) we have

$$\begin{aligned}&n^{-2r}\left\|V_n^{(2r)}(f,\cdot)\right\|_{p),\theta,\omega} + \|f-V_n(f,\cdot)\|_{p),\theta,\omega} \le \\ &\le c_{23}\,(p,r)\left(\Omega_r(V_n,\frac{1}{n})_{p),\theta,\omega} + E_n(f)_{p),\theta,\omega}\right) \le \\ &\le c_{24}\,(p,r)\left(\Omega_r(f,\frac{1}{n})_{p),\theta,\omega} + \Omega_r(f-V_n,\frac{1}{n})_{p),\theta,\omega} + E_n(f)_{p),\theta,\omega}\right) \le \\ &\le c_{25}\,(p,r)\,\Omega_r(f,\frac{1}{n})_{p),\theta,\omega}.\end{aligned}$$

which completes the estimation (4) of Theorem 2.

Let T_n be the best approximation polynomial for f, i.e.,

$$E_n(f)_{p),\theta,\omega} = \|f-T_n\|_{p),\theta,\omega}.$$

By [9], Theorem 5; [10], Theorem 2.1 there exists a constant $c_{25}(p)$ such as

$$\|f-S_n(f,\cdot)\|_{p),\theta,\omega} \le \|f-T_n\|_{p),\theta,\omega} + \|S_n\,(T_n-f)\|_{p),\theta,\omega} \le c_{26}(p)E_n(f)_{p),\theta,\omega}. \tag{17}$$

Using inequality (17) and the scheme of proof of the estimation (4) we have the estimate (5).

Proof of Theorem 2 is completed.

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С.З. Джафаров

Ең жақын тригонометриялық жуықтау және Лебегтің салмақтық гранд-кеңістіктеріндегі функцияның тегістігінің модулі

Мақалада бірінші кезекте Лебегтің салмақтық гранд-кеңістіктері және Макенхаупт $L_{\omega}^{p),\theta}(\mathbb{T})$ салмақтары анықталды. Осы кеңістіктердің қасиеттері жайлы ақпарат берілді. T_n ең жақын жуықтаудың тригонометриялық полиномы болсын. Лебегтің гранд-кеңістіктеріндегі функцияны аппроксимациялау көптеген авторлармен зерттелді. Бұл жұмыста тригонометриялық полиномдағы T_n бөлшек туынды мен Лебегтің салмақтық гранд-кеңістіктеріндегі ең жақын жуықтау арасындағы байланыс қарастырылды. Осыған байланысты қажетті және жеткілікті шарттар 1-теоремада келтірілді. Автор Лебегтің салмақтық гранд-кеңістіктерінде нақты операторды анықтаған. Кейінірек бұл оператордың көмегімен f функциясының r ретті тегістігінің бөлшекті модульдері анықталды. Сонымен қоса бұл жұмыста функцияның тегістігінің модулінің қасиеттерін қолдана отырып, тегістіктің бөлшекті модулі мен Лебегтің салмақтық гранд-кеңістігінің ішкі кеңістігіндегі Фурье қатарының де Валле-Пуссеннің $n-th$ дербес қосындылары арасындағы өзара байланыс қарастырылды. Бұл нәтижелер 2-теоремада келтірілген.

Кілт сөздер: Лебегтің жалпыланған гранд-кеңістіктері, бөлшекті туынды, тегістіктің бөлшекті модульдері, $n-th$ дербес қосындылары, де Валле-Пуссен қосындылары, тригонометриялық полиномдармен ең жақын жуықтау.

С.З. Джафаров

Наилучшее тригонометрическое приближение и модуль гладкости функций в весовых гранд-пространствах Лебега

В статье, в первую очередь, определены весовые гранд-пространства Лебега и веса Макенхаупта $L_{\omega}^{p),\theta}(\mathbb{T})$. Дана информация о свойствах этих пространств. Пусть T_n будет тригонометрическим полиномом наилучшего приближения. Аппроксимация функций в гранд-пространствах Лебега исследовалась многими авторами. В этой работе изучена связь между дробными производными T_n тригонометрического полинома и наилучшим приближением функции в весовых гранд-пространствах Лебега. В связи с этим необходимое и достаточное условие выражено в теореме 1. Кроме того, в этой работе в весовых гранд-пространствах Лебега определен конкретный оператор. Позже с помощью этого оператора будут определены дробные модули гладкости порядка r функции f. Также, используя свойства модуля гладкости функции, автором изучена взаимосвязь между дробным модулем гладкости и $n-th$ частичными суммами де Валле-Пуссена ряда Фурье в подпространстве весового гранд-пространства Лебега. Эти результаты выражены в теореме 2.

Ключевые слова: обобщенные гранд-пространства Лебега, дробная производная, дробные модули гладкости, $n-th$ частичные суммы, суммы де Валле-Пуссена, наилучшее приближение тригонометрическими полиномами.