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New singular solutions for the (3+1)-D Protter problem

For the nonhomogeneous wave equation with three space and one time variables we study a boundary value problem that can be regarded as a four-dimensional analogue of the Darboux problem in $\mathbb{R}^2$. Unlike the planar Darboux problem, the $\mathbb{R}^4$-version is not well posed and has an infinite-dimensional cokernel. Therefore the problem is not Fredholm in the framework of classical solvability. On the other hand, it is known that for smooth right-hand side functions, there is a uniquely determined generalized solution that may have a strong power-type singularity at one boundary point. The singularity is isolated at the vertex of the characteristic light cone and does not propagate along the cone. In the present article we announce new singular solutions with exponential growth.

Keywords: wave equation, boundary value problems, generalized solution, singular solutions, propagation of singularities, special functions.

Introduction

In this paper we consider some boundary value problems for the wave equation with three space and one time variables that were proposed by M.H. Protter. From a historical perspective, Protter formulated these problems in connection with BVPs for mixed-type equations that describe transonic flows in fluid dynamics. The topic was extensively studied in the 1950 s and 1960 s with the development of supersonic aircrafts. In particular, the classical two-dimensional Guderley-Morawetz problem for the Gellerstedt equation of hyperbolic-elliptic type models flows around airfoils and is well studied. Regarding 2-D mixed-type boundary value problems and their transonic background we refer to the recent survey by Morawetz [1]. In 1954 Protter [2] formulated some multi-dimensional analogues of the planar Guderley-Morawetz problem. Initially, expectation was that the methods used in the $2D$ case could be applied, with minor modifications, for the problems in higher dimensions. However, the multi-dimensional case turns out to be quite different and the situation there is still not clear. Some of the difficulties and differences with the planar BVPs are illustrated by the related Protter's problems in the hyperbolic part of the domain, also formulated in [2]. In particular, for the wave equation in $\mathbb{R}^4$, with points $(x,t)=(x_1,x_2,x_3,t)$,

$$u_{x_1x_1}+u_{x_2x_2}+u_{x_3x_3}-u_{tt}=f(x,t) \tag{1}$$

the domain is

$$\Omega=\left\{(x,t):0<t<1/2, t<\sqrt{x_1^2+x_2^2+x_3^2}<1-t\right\}.$$

The boundary of Ω consists of two characteristic cones

$$\Sigma_1=\left\{(x,t):0<t<1/2, \sqrt{x_1^2+x_2^2+x_3^2}=1-t\right\},$$

$$\Sigma_2=\left\{(x,t):0<t<1/2, \sqrt{x_1^2+x_2^2+x_3^2}=t\right\}$$

and the ball

$$\Sigma_0=\left\{t=0, \sqrt{x_1^2+x_2^2+x_3^2}<1\right\}.$$

Let us point out that the origin $O:x=0,t=0$ is both the center of the non-characteristic part of the boundary Σ_0, and the vertex of the characteristic cone Σ_2. We will study the following BVPs.

Problem P1. Find a solution of the wave equation (1) in Ω which satisfies the boundary conditions

$$P1: \quad u|_{\Sigma_0}=0, \quad u|_{\Sigma_1}=0.$$

One can regard the domain Ω as a four-dimensional analogue of the characteristic triangle $D = \left\{(x_1,t) \in \mathbb{R}^2 : 0 < t < x_1 < 1-t\right\}$ for the string operator $\Box v(x_1,t) := v_{x_1x_1} - v_{tt}$ in $\mathbb{R}^2$ with points (x_1,t). The boundary of D consists of two characteristic – $l_1 = \{x_1 = 1-t, 0 < t < 1/2\}$ and $l_2 = \{x_1 = t, 0 < t < 1/2\}$, and a non-characteristic segment – $l_0 = \{t = 0, 0 < x_1 < 1\}$. In fact, the domain Ω can be constructed by revolving D in $\mathbb{R}^4$ about the t-axis. Then the segments l_0, l_1 and l_2 form Σ_0, Σ_1 and Σ_2, respectively. In this context the Protter problems $P1$ and $P1^*$ are four-dimensional variants of the classical Darboux problems for the string equation in $D \subset \mathbb{R}^2$: the data are prescribed on one of the characteristics and on the non-characteristic part of the boundary. On the other hand, unlike the planar Darboux problem, the Protter's problems in $\mathbb{R}^4$ are not well posed. Actually, the homogeneous adjoint problem $P1^*$ has smooth classical solutions and the linear space they generate is infinite dimensional (see Lemma 1 in the next section). Thus, in the frame of classical solvability the Protter problem $P1$ is not Fredholm, since it has infinite-dimensional cokernel. Naturally, a necessary condition for the existence of a classical solution for the problem $P1$ is the orthogonality of the right-hand side function f to the cokernel. Alternatively, to avoid imposing an infinite number of conditions on f, the notion of generalized solution have been introduced.

Definition 1 [3]. A function $u = u(x,t)$ is called a generalized solution of the problem $P1$ in Ω, if the following conditions are satisfied:

1) $u \in C^1\left(\overline{\Omega}\backslash O\right)$, $u|_{\Sigma_0\backslash O} = 0$, $u|_{\Sigma_1} = 0$, and

2) the identity

$$\int_\Omega (u_t w_t - u_{x_1}w_{x_1} - u_{x_2}w_{x_2} - u_{x_3}w_{x_3} - fw)\, dxdt = 0$$

holds for all $w \in C^1(\overline{\Omega})$ such that $w = 0$ on Σ_0 and in a neighborhood of Σ_2.

Notice that this definition allows the generalized solution of the problem $P1$ to have singularity on Σ_2. Now, it is known that when the right-hand side f is smooth, there exists a unique generalized solution of the problem $P1$ and it turns out that its singularity is isolated at only one point, that is, the origin O. In [4] it is shown that for each $n \in \mathbb{N}$ there is a generalized solution that behaves like $|x|^{-n}$ near O. The existence of a solution with exponential growth is announced in [5]. It is interesting that these singularities are isolated at the vertex O and do not propagate along the characteristic cone Σ_2. This differs the conventional case of propagation of singularities, like in Hörmander [6, Chapter 24.5].

In this paper we discuss for right-hand sides $f \in C^1(\overline{\Omega})$ the behavior of the generalized solution of problem $P1$ and the rate of its growth at the point O.

In the special case when the right-hand side function f is a harmonic polynomial, the exact behavior of the generalized solution of problem $P1$ is found in [3]. In [7] the semi-Fredholm solvability of problem $P1$ is discussed. A short historic survey and a comparison of various recent results for Protter problems can be found in [8–10]. Garabedian [11] proved the uniqueness of a classical solution for the problem $P1$. According to the classical and singular solutions let us mention here a series of papers by Aldashev (see [12–15]). Some other multi-dimensional versions of the planar Darboux problem for the wave equation are studied in [16–19]. For Protter problems for the wave equation but with lower order terms see [20, 21] and references therein. The existence of bounded or unbounded solutions for some other connected equations is considered in [13, 22]. Regarding results for degenerated hyperbolic equations we refer to [14, 23, 24] for Keldysh-type equations see [24, 25], and for BVPs for multi-dimensional mixed-type Lavrent'ev-Bitsadze equation see [12, 15]. For the Protter's mixed-type hyperbolic-elliptic problems, uniqueness results for quasi-regular solutions are proved in [26]. There are a recent series of results concerning existence or nonexistence of nontrivial solutions of related quasi-linear problems of mixed hyperbolic-elliptic type in the multi-dimensional case, see [27, 28].

In the present paper new singular solutions of problem $P1$ with exponential growth at the origin O are announced. The main Theorem 6 is formulated in the last section. It is based on some previous results from [10] for the existence and the behavior of the generalized solution, that will be presented and discussed in the next section.

Existence of generalized solutions

Naturally, the behavior of the generalized solution of problem $P1$ is affected by the correlations of the right-hand side function f with the solutions of the homogeneous adjoint problem $P1^*$. In order to construct the latter, we will use in $\mathbb{R}^3$ the orthonormal system of spherical functions Y_n^m ($n \in \mathbb{N} \cup \{0\}$, and $m = 1, \ldots, 2n+1$). The spherical functions are introduced commonly on the unit sphere $S^2 := \{(x_1,x_2,x_3) : x_1^2 + x_2^2 + x_3^2 = 1\}$ with spherical polar coordinates (see [29]). Expressed in Cartesian coordinates here, one can define them by

$$Y_n^{2k}(x_1,x_2,x_3)=C_{n,k}\frac{d^k}{dx_3^k}P_n(x_3)\,\mathrm{Im}\left\{(x_1+ix_2)^k\right\},\quad \text{for}\quad k=1,...,n;$$

$$Y_n^{2k+1}(x_1,x_2,x_3)=C_{n,k}\frac{d^k}{dx_3^k}P_n(x_3)\,\mathrm{Re}\left\{(x_1+ix_2)^k\right\},\quad \text{for}\quad k=0,...,n,$$

where $C_{n,k}$ are constants and P_n are the Legendre polynomials. The Legendre polynomials are given by the Rodrigues formula as

$$P_n(s):=\frac{1}{2^n n!}\frac{d^n}{ds^n}(s^2-1)^n=\sum_{k=0}^{\left[\frac{n}{2}\right]}a_{n,2k}s^{n-2k},$$

with coefficients

$$a_{n,2k}=(-1)^k\frac{(2n-2k)!}{2^n k!(n-k)!(n-2k)!}. \tag{2}$$

The constants $C_{n,m}$ are such that functions Y_n^m form a complete orthonormal system in $L_2(S^2)$. For convenience in the discussions that follow, we extend the spherical functions out of S^2 radially, keeping the same notation Y_n^m for the extended function, i.e., $Y_n^m(x):=Y_n^m(x/|x|)$ for $x\in\mathbb{R}^3\backslash O$.

Now, let us define for $n,k\in\mathbb{N}\cup\{0\}$ the functions

$$h_{n,k}(\xi,\eta)=\int_\eta^\xi s^k P_n\left(\frac{\xi\eta+s^2}{s(\xi+\eta)}\right)ds.$$

Following Lemma 1 from [10] and Lemmas 1.1 and 2.3 from [30] we can construct solutions of the homogeneous adjoint problem.

Lemma 1 [10]. The functions

$$v_{k,m}^n(x,t)=|x|^{-1}h_{n,n-2k-2}\left(\frac{|x|+t}{2},\frac{|x|-t}{2}\right)Y_n^m(x).$$

are classical solutions from $C^\infty(\Omega)\cap C(\overline{\Omega})$ of the homogeneous problem $P1^*$ for $n\in\mathbb{N}$, $m=1,\dots,2n+1$ and $k=0,1,\dots,[(n-1)/2]-2$.

Solutions for the homogenous adjoint problem were first found by Tong Kwang-Chang [31]. Some different representations of the solutions of the homogeneous problem $P1^*$ and the functions $v_{k,m}^n$ are given by Khe Kan Cher [22].

Next we will present some useful conditions from [10] for the function f that are sufficient for the existence of the generalized solution of problem $P1$.

Since the spherical functions form a complete orthonormal system in $L_2(S^2)$, generally, a smooth function $f(x,t)$ can be expanded as a harmonic series

$$f(x,t)=\sum_{n=0}^{\infty}\sum_{m=1}^{2n+1}f_n^m(|x|,t)Y_n^m(x) \tag{3}$$

with Fourier coefficients

$$f_n^m(r,t):=\int_{S(r)}f(x,t)Y_n^m(x)\,d\sigma_r\,, \tag{4}$$

where $S(r)$ is the three-dimensional sphere $S(r):=\{x=(x_1,x_2,x_3)\in\mathbb{R}^3:\ |x|=r\}$. The results from [10] ensure the existence of the generalized solution of problem $P1$ assuming that the Fourier series (3) converges fast enough. They also give a priori estimates for the singularity of the solution. In fact, the behavior of the generalized solution depends strongly on the $L_2(\Omega)$-inner product of the right-hand side function $f(x,t)$ with the functions $v_{k,m}^n(x,t)$ from Lemma 1 (see also [20, 3]). Accordingly, we denote by $\beta_{k,m}^n$ the parameters

$$\beta_{k,m}^n:=\int_\Omega v_{k,m}^n(x,t)f(x,t)\,dxdt, \tag{5}$$

where $n = 0, \dots, l$; $k = 0, \dots, \left[\frac{n-1}{2}\right]$ and $m = 1, \dots, 2n+1$. In order to formulate the general existence result, we need also to introduce for $p \geq 0$ and $k \in \mathbb{N}$ the series

$$||f; n^p; C^k|| := \left\| f_0^0(|x|, t) \right\|_{C^0(\Omega)} + \sum_{n=1}^{\infty} n^p \left\| \sum_{m=1}^{2n+1} f_n^m(|x|, t) Y_n^m(x) \right\|_{C^k(\Omega)}$$

and the power series

$$\Phi(s) := \sum_{n=1}^{\infty} \left[\sum_{m=1}^{2n+1} \sum_{k=0}^{[(n-1)/2]} \left| \beta_{k,m}^n \right| \right] s^n .$$

Apparently, the convergence of $||f; n^p; C^k||$ gives information on the rate of convergence of the Fourier series (3).

Theorem 2 [10]. Let the function $f(x,t)$ belong to $C^1(\overline{\Omega})$. Suppose that the series $||f; n^6; C^0||$ and $||f; n^4; C^1||$ are convergent and the power series $\Phi(s)$ has an infinite radius of convergence. Then there exists a unique generalized solution $u(x,t) \in C^1(\overline{\Omega} \backslash O)$ of the Protter problem $P1$ and it satisfies in $\overline{\Omega} \backslash O$ the a priori estimates

$$|u(x,t)| \leq C \left[\Phi\left(\frac{C_1}{|x|+t} \right) + |x|^{-1} ||f; n^4; C^0|| \right];$$

$$|u(x,t)| \leq C \left[\Phi\left(\frac{C_1}{|x|+t} \right) + ||f; n^6; C^0|| + ||f; n^4; C^1|| \right];$$

$$\sum_{i=1}^{3} |u_{x_i}(x,t)| + |u_t(x,t)| \leq C|x|^{-2} \left[\Phi\left(\frac{C_2}{|x|+t} \right) + ||f; n^6; C^0|| \right];$$

where the constants C, C_1 and C_2 are independent of the function $f(x,t)$.

In these estimates, the singularity of the generalized solution at the origin O is controlled by the function $\Phi(s)$, while $||f; n^p; C^k||$ bounds the «regular part» of $u(x,t)$.

Notice that the definition of $\Phi(s)$ involves parameters $\beta_{k,m}^n$ with index $k > \left[\frac{n-1}{2}\right] - 2$ also, and the corresponding functions $v_{k,m}^n$ are not classical solutions of the homogenous problem $P1^*$. Nevertheless, these functions $v_{k,m}^n$ still «control» some discontinuities of the generalized solution and cannot be omitted as seen from the following result from [7]. At the same time, Theorem 3 also suggests that there are no other linearly independent nontrivial classical solutions of the homogenous adjoint problem $P1^*$.

Theorem 3 [7]. Let the function $f(x,t)$ belong to $C^{10}(\overline{\Omega})$. Then the necessary and sufficient conditions for existence of *bounded* generalized solution $u(x,t)$ of the Protter problem $P1$ are

$$\int_{\Omega} v_{k,m}^n(x,t) f(x,t) \, dxdt \; = \; 0,$$

for all $n \in \mathbb{N}$, $k = 0, \dots, \left[\frac{n-1}{2}\right]$, $m = 1, \dots, 2n+1$. Moreover, this generalized solution $u(x,t) \in C^1(\overline{\Omega} \backslash O)$ and satisfies the a priori estimates

$$|u(x,t)| \leq C \, \|f\|_{C^{10}(\overline{\Omega})};$$

$$\sum_{i=1}^{3} |u_{x_i}(x,t)| + |u_t(x,t)| \leq C(|x|^2 + t^2)^{-1} \, \|f\|_{C^{10}(\overline{\Omega})},$$

where the constant C is independent of the function $f(x,t)$.

In practice, it is not always easy to compute all the parameters $\beta_{k,m}^n$ from (5) and therefore to construct and study the behaviour of the series $\Phi(s)$. On the other hand, notice that we have

$$\left| \beta_{k,m}^n \right| \leq C n^{1/2} \, \|f_n^m\|_{C^0(\Omega)},$$

since directly from the definition of the functions $v_{k,m}^n(x,t)$ we get the estimate $|v_{k,m}^n| \leq |Y_n^m| \leq C n^{1/2}$. This allow us to formulate the next direct corollary of Theorem 2.

Corollary 4. Let the function $f(x,t)$ belong to $C^1(\overline{\Omega})$. Suppose that the series $||f; n^6; C^0||$ and $||f; n^4; C^1||$ are convergent and the power series

$$\Phi_1(s) := \sum_{n=1}^{\infty} \left[\sum_{m=1}^{2n+1} \|f_n^m\|_{C^0(\Omega)} \right] s^n$$

has an infinite radius of convergence. Then the unique generalized solution $u(x,t) \in C^1(\overline{\Omega}\backslash O)$ of the Protter problem $P1$ satisfies near the origin the estimate

$$|u(x,t)| \leq C\Phi_1\left(\frac{C_0}{|x|+t}\right), \tag{6}$$

where the constants C and C_0 are independent of the function $f(x,t)$.

Remark. Although Corollary 4 is somewhat weaker than Theorem 2, it still gives better estimate than the previously known general a priori estimates for the singularity of the solution. In particular, Protter problems in the (2+1)-D case (two space and one time dimensions) were studied in [4]. According to [4, Theorem 5.3] the sufficient condition for the existence of a generalized solution is the convergence of the series

$$\sum_{n=1}^{\infty}\frac{1}{n}I_0\left(\frac{2n}{\varepsilon}\right)\left(\left\|f_n^1\right\|_{C^0(\Omega)}+\left\|f_n^2\right\|_{C^0(\Omega)}\right), \text{ for all } \varepsilon>0,$$

where I_0 is the modified Bessel function of first kind, and f_n^i are the Fourier coefficients for the right-hand side, and could be viewed as the analogues of the functions f_n^m given by (4). Using the inequality $I_0(s) \leq e^s$ for $s \geq 0$, one could paraphrase Theorem 5.3 from 4 in somewhat weaken form as follows. Suppose that the power series

$$\Phi_2(s) := \sum_{n=1}^{\infty}\left(\left\|f_n^1\right\|_{C^0(\Omega)}+\left\|f_n^2\right\|_{C^0(\Omega)}\right)n^{-1}s^n$$

is convergent for all s. Then for the singularity of the unique generalized solution $u(x,t)$ for the (2+1)-D Protter problem $P1$, near the origin we have the estimate

$$|u(x,t)| \leq C\Phi_2\left(\exp\left(\frac{4}{|x|+t}\right)\right). \tag{7}$$

Notice that the exponent in the argument of Φ_2 in (7) is replaced now in (6) by simply a linear function.

Evidently, Theorem 2 gives only an upper bound, but the generalized solution does not necessarily grows like $\Phi(C/|x|)$ near the origin. The paper [3] considers the special case when the right-hand side function f is a harmonic polynomial, i.e., (3) is a finite sum ($f_n^m \equiv 0$ for large n), and the function $\Phi(s)$ is simply a polynomial. In [3] the exact asymptotic formula for the generalized solution at O is found. It shows that the a priori estimate is sharp and the solution can indeed have a power-type singularity as $\Phi(C/|x|)$. On the other hand, in the general case $f(x,t) \in C^1(\overline{\Omega})$ stronger singularities are also possible. Actually, a generalized solutions with at least exponential growth at the origin was found in [5]. In the present article the existence of solutions with stronger singularities is announced.

Singular solutions with exponential growth

Regarding the possible singularities of the generalized solution of problem $P1$ the next question naturally arises. Given the function $\phi(s)$, can we find a smooth right-hand side function f such that the corresponding generalized solution grows like $\phi(1/|x|)$ at O? As a possible answer, the following result is given in [10], that provides a method for finding suitable functions f. Recall that $a_{n,2k}$ are the coefficients (2) of the Legendre polynomials.

Theorem 5 [10]. Let the function $f(x,t)$ belong to $C^1(\overline{\Omega})$, the series $||f;n^6;C^0||$, $||f;n^4;C^1||$ are convergent, and the power series $\Phi(s)$ has an infinite radius of convergence. Let the numbers $\alpha_p \geq 0$, $p=0,1,2,...$, are such that the series

$$\phi(s) := \sum_{p=0}^{\infty}\alpha_p s^p$$

is convergent for all $s \in \mathbb{R}$. Suppose that there is $x^*=(x_1^*,x_2^*,x_3^*) \in \mathbb{R}^3$ such that

$$\sum_{k=0}^{\infty}\sum_{m=1}^{2p+4k+1} p\, a_{n,2k}\beta_{m,k}^{p+2k}Y_{p+2k}^m(x^*) \geq \alpha_p \quad \text{for all } p \in \mathbb{N}\cup\{0\}. \tag{8}$$

Then there exists a number $\delta \in (0, 1/2)$ that the unique generalized solution $u(x,t)$ of problem $P1$ satisfies the estimate

$$|u(tx_1^*, tx_2^*, tx_3^*, t)| \geq \phi\left(\frac{1}{2t}\right)$$

for $t \in (0, \delta)$.

According to Theorem 5 one could try to construct a right-hand side $f(x,t) \in C^1(\overline{\Omega})$ by choosing suitable Fourier coefficients $f_n^m(r,t)$. They have to be «small enough» that the required series $||f; n^p; C^k||$ and $\Phi(s)$ are convergent, but at the same time, satisfy the inequality (8). The main result in the present paper is that it is possible to apply this procedure to build an appropriate function f such that the corresponding solution grows like $\exp(|x|^{-k})$ at O.

Theorem 6. Let $k \in \mathbb{N}$. Then there exist functions $f_k \in C^1(\overline{\Omega})$ and positive numbers $\delta_k \in (0, 1/2)$ and C_k, such that the unique generalized solutions $u_k(x,t) \equiv u_k(x_1, x_2, x_3, t) \in C^1(\overline{\Omega}\backslash O)$ of the problem $P1$ for the wave equation (1) with right-hand function f_k, satisfy the estimates

$$u_k(0,0,t,t) \geq \exp(t^{-k}) \quad \text{for} \quad t \in (0, \delta_k),$$

and

$$|u_k(x,t)| \leq C_k \exp(2|x|^{-k}) \quad \text{for} \quad (x,t) \in \Omega.$$

From [5] it is known that there is a right-hand side function $f \in C^\infty(\overline{\Omega})$ such that the generalized solution grows at least like $\exp(|x|^{-1})$. Obviously this corresponds to the case $k = 1$ in Theorem 6. Unlike [5] here we have also an estimate from above, that shows that the solution behalves «exactly» like $\exp(|x|^{-k})$ at O. On the other hand, the functions f_k are only C^1-smooth, and is not clear whether, like in [5], one could construct functions from $C^\infty(\overline{\Omega})$ with the desired property.

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Т.П. Попов

Проттер (3+1)-D есебінің жаңа сингулярлық шешімдері

Уақытқа байланысты бір айнымалысы бар, үш өлшемді кеңістікте біртекті емес толқын теңдеуі қарастырылды. Бұл теңдеу үшін R^2 кеңістігіндегі Дарбу есебінің аналогы болып табылатын төртөлшемді шеттік есеп зерттелген. Бұл есептің жазықтықтағы Дарбу есебінен мынадай айырмашылығы бар: R^4 кеңістігінде қарастырылатын бұл есеп корректілі емес және оның өлшемі ақырсыз коядросы бар. Классикалық шешімділік тұрғысынан мұндай есеп фредгольмдік емес есеп болып табылады. Екінші жағынан, теңдеудің оң жағындағы функция тегіс функция болған жағдайда теңдеудің белгілі бір жалпылама шешімі бар және де ол шешім бір шеттік нүктеде дәрежелік түрде ерекшеленген болуы мүмкін. Ерекшелік нүктесі характеристикалық конустың төбесінде оқшауланған және конус бойында таралмайды. Бұл мақалада экспонента түрінде өсетін жаңа сингулярлық шешім бар екенін анықталды.

Кілт сөздер: толқын теңдеуі, шеттік есептер, жалпылама шешім, сингулярлық шешімдер, ерекшеліктердің таралуы, арнайы функциялар.

Т.П. Попов

Новые сингулярные решения для (3+1)-D задачи Проттера

Для неоднородного волнового уравнения с тремя пространственными и одной временной переменными изучена краевая задача, которую можно рассматривать как четырехмерный аналог задачи Дарбу в $\mathbb{R}^2$. В отличие от плоской задачи Дарбу, $\mathbb{R}^4$–версия не является корректной и имеет бесконечномерное коядро. Поэтому задача не является фредгольмовой в рамках классической разрешимости. С другой стороны, известно, что для гладких правых частей уравнения есть однозначно определенное обобщенное решение, которое может иметь сильную особенность степенного типа в одной граничной точке. Особенность изолирована в вершине характеристического светового конуса и не распространяется вдоль конуса. В настоящей статье анонсированы новые сингулярные решения с экспоненциальным ростом.

Ключевые слова: волновое уравнение, краевые задачи, обобщенное решение, сингулярные решения, распространение особенностей, специальные функции.