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# Approaching of the solution of a static compressible medium to the solution of an incompressible medium

A well-known analogy of the flow of viscous incompressible fluid and incompressible elastic medium. According to this analogy, the solution of the equations of the elasticity theory with the Poisson's ratio $\nu = 0,5$ and for any fixed shear modulus $\mu$ can be interpreted as a motion of a viscous incompressible fluid with viscosity $\mu$. Thus, we can consider the usual static linear elasticity task with Hooke's law at $\lambda \to \infty$, as a mathematical model of approaching to incompressible medium. In this paper, we obtained the asymptotic $\lambda \to \infty$. Estimation of the proximity of the solution of an elastic static problem with Hooke's law to the solution of incompressible medium (Stokes problem). The final estimate allows to use well-known difference schemes and algorithms for an elastic compressible medium to solve incompressible medium. In this paper, an estimate of the proximity of the solutions of these problems is proved at $\lambda \to \infty$, i.e. $\underset{\lambda\to\infty}{u\to\overline{u}^H}$ $\underset{\lambda\to\infty}{\lambda \operatorname{div}\overline{u}\to -p}$ $\underset{\lambda\to\infty}{\sigma\to\sigma^H}$. To substantiate this fact in [1–3], various methods for the first boundary value problem were investigated. For the static problem of the theory of elasticity, there is currently a whole series of papers devoted to numerical implementation using difference schemes. In paper [4], the estimate $O(\lambda^{-\alpha})$ where $k = \frac{1}{2}$ was obtained, in the proposed paper the estimate $O(\lambda^{-1})$, and in further work we will show that this estimate is best possible in order.

*Keywords:* incompressible medium, Hooke's law, stresses, deformations, displacements, Lamé coefficients.

A well-known analogy of the flow of viscous incompressible fluid and incompressible elastic medium. According to this analogy, any solution of the equations of the elasticity theory with the Poisson's ratio $\nu = 0,5$ $\left(\nu = \frac{\lambda}{2(\lambda+\mu)}\right)$ to any shear modulus $\mu$ can be interpreted as a motion of a viscous incompressible fluid with a viscosity $\mu$ (Stokes problem) [5].

In a bounded simply connected domain $D \subset R^3$ with a sufficiently smooth boundary $\gamma$ we seek a solution to the problem of the theory of elasticity for an incompressible medium that satisfies the equilibrium equation

$$\mu\triangle\overline{u} - \nabla p + \overline{f} = 0, \;\; x \in D, \tag{1}$$

the condition of incompressibility of medium

$$\operatorname{div}\overline{u} = 0, x \in D, \tag{2}$$

by the correlation of the displacement-strain

$$\varepsilon_{ik} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right), \;\; i,k = 1,2,3, \tag{3}$$

to equations of state of medium

$$\sigma_{ik} = -\delta_{ik}p + 2\mu\varepsilon_{ik}, \tag{4}$$

where $\sigma_{ik}$ is components of the stress tensor, $\delta_{ik}$ is the Kronecker symbol, $p$ is function of pressure, and to boundary conditions

$$\sum_{k=1}^{3}\sigma_{ik}n_k = 0, \;\; x \in \gamma, \tag{5}$$

the task (1)–(4) with boundary conditions of the first kind, i.e. when

$$u = 0, \;\; x \in \gamma, \tag{6}$$

was investigated by various techniques. Its solution was considered as the limit in a certain sense at $\lambda \to \infty$ solutions $\overline{u}^{\lambda}$ the problem of the theory of elasticity for a compressible medium.

$$\mu \triangle \overline{u}^{\lambda} + (\lambda + \mu)\nabla div \overline{u}^{\lambda} + \overline{f} = 0, x \in D, \tag{7}$$

where the components of the strain and stress tensors are related to Hooke's law

$$\sigma_{ij} = \delta_{ij}\lambda\theta + 2\mu\varepsilon_{ij}, \;\; i,j = 1,2,3, \tag{8}$$

where $\theta = \sum_{i=1}^{3} \varepsilon_{ii}$, $\lambda > 0$, $\mu > 0$ the Lamé constants, the static problem of the theory of elasticity for an incompressible material (1)–(4), (6) studied in [6–7]. For it, a difference scheme is constructed from the first order of accuracy. Let us turn to the study of the behavior of the solution of problem (3), (5), (7), (8). This problem is not always solvable [8]. The conditions for its solvability are that the main vector and the main moment of the bulk forces are zero.

$$\int_D \overline{f} dx = 0, \;\; \int_D \overline{f} \times \overline{r} dx = 0, \tag{9}$$

in the case of fulfillment of conditions (9), problems (3), (5), (7), (8) are not uniquely solvable. To single out its only solution, additional conditions are needed.

$$\int_D \overline{u} dx = 0, \;\; \int_D \operatorname{rot} \overline{u} dx = 0, \tag{10}$$

further, we will assume that for solving the problem (3), (5), (7), (8) the conditions (9), (10) are fulfilled. We first carry out auxiliary arguments. Let problem be solved.

$$\operatorname{div} \overline{\mathrm{v}} = p, \;\; x \in D, \;\; \overline{\mathrm{v}} = \overline{\varphi}, \;\; x \in \gamma. \tag{11}$$

*Lemma 1.* Following [10, 11], let it be

$$\gamma \in C^{2+m}, \;\; p \in W_2^m(D), \;\; \varphi \in W_2^{m+\frac{1}{2}}(\gamma), \tag{12}$$

moreover we suppose that $(p,1)_D = 0$, $(\overline{\varphi}, \overline{n})_{\gamma} = \int_{\gamma} \overline{\varphi} \; \overline{n} \; ds$. Then there exists an additive and homogeneous operator $\mathrm{v} = \mathrm{v}(p,u)$ solving the problem (11), (12) and there is fair estimate

$$\|\mathrm{v}\|_{W_2^{m+1}(D)} \leq M_m \left( \|p\|_{W_2^m(D)} + \|\varphi\|_{W_2^{m+\frac{1}{2}}(\gamma)} \right).$$

*Lemma 2.* Let conditions (12) be fulfilled, then there exists a homogeneous operator $\mathrm{v} = \mathrm{v}(p,u)$ that the evaluation takes place

$$\|\mathrm{v}\|_{W_2^{m+1}(D)} \leq M_m \left[ \|p\|_{W_2^m(D)} + \|\varphi\|_{W_2^{m+\frac{1}{2}}(\gamma)} \right],$$

where v satisfies the following problem

$$\operatorname{div} \overline{\mathrm{v}} = p \cdot [\mu_n(D)]^{-1}, \;\; (\overline{\varphi}, \overline{n})_{\gamma}, \;\; in \;\; D,$$

$$\overline{\mathrm{v}} = \overline{\varphi} + [\mu_{n-1}(\gamma)]^{-1} \cdot (p,1)_D \cdot \overline{n}, \;\; on \;\; \gamma,$$

$\mu_n(D)$ is $n$-dimensional Lebesgue measure of the domain $D$, $\mu_{n-1}(\gamma)$ is $n-1$-dimensional Lebesgue measure of its boundary.

*Proof.* Let $\Psi_1$ - this sequel to $D$, that the assessment is made

$$\|\Psi_1\|_{W_2^{m+1}(D)} \leq M_m \|\Psi\|_{W_2^{m+\frac{1}{2}}(\gamma)}.$$

As $\Psi_1$, you can take the solution of the following problem:

$$\Psi_1 = 0, \ \ in \ \ D,$$

$$\Psi_1 = \varphi, \ \ on \ \ \gamma.$$

Let $\Psi_2$ be similar extension of $n$ to $D$. Finally, let be

$$\overline{\Psi} = \overline{\Psi}_1 + [\mu_{n-1}(\gamma)]^{-1} \cdot (p, 1)_D \cdot \overline{\Psi}_2.$$

Then the vector function $\overline{z} = \overline{\mathrm{v}} - \overline{\Psi}$ is a solution of problem

$$\operatorname{div} \overline{z} = p - \operatorname{div} \overline{\Psi} + [\mu_n(D)]^{-1} (\overline{\varphi}, \overline{n})_\gamma,$$

$$\overline{z} = 0, \ \ on \ \ \gamma.$$

Thus, for the vector function $\overline{z}(x)$, the conditions of the Lemma are already satisfied, as required. Now, for any scalar function is $p(x) \in L_z(D)$.

The formula

$$\|p\|_{W_2^{-1}(D)} = \sup_{q \in W_2^1(D)} \frac{|(p,q)_D|}{\|q\|_{W_2^1(D)}}$$

for all $q \neq 0$ defines the norm of a linear functionality over the space $W_2^1(D)$. A formula

$$\|P\|_{\overset{0}{W}{}_2^{-1}(D)} = \sup_{q \in \overset{0}{W}{}_2^{1}(D)} \frac{|(p,q)_D|}{\|q\|_{\overset{0}{W}{}_2^{1}(D)}}$$

defines the norm of a linear functionality over a space $\overset{0}{W}{}_2^{-1}(D)$. Whence it follows that

$$\|p\|_{\overset{0}{W}{}_2^{-1}(D)} \leq \|p\|_{W_2^{-1}(D)} \leq \|p\|_{L_2(D)}.$$

Consequence of Lemma 2. Let v be a solution to problem

$$\operatorname{div} \overline{\mathrm{v}} = p, \ \ in \ \ D. \tag{13}$$

$$\overline{\mathrm{v}} = [\mu_{n-1}(\gamma)]^{-1} (p, 1)_D \cdot \overline{n}, \ \ on \ \ \gamma, \tag{14}$$

For an arbitrary function $p(x) \in L_2(D)$. Then the estimate is true

$$\|\overline{\mathrm{v}}\|_{W_2^1(D)} \leq c \|p\|_{L_2(D)}.$$

Let us turn to the problem (3), (5), (7), (8). The solution of this problem satisfies the integral identity

$$E(\overline{u}, \overline{\mathrm{v}}) + \lambda \int\limits_D \operatorname{div} \overline{u} \cdot \operatorname{div} \overline{\mathrm{v}} dx = \int\limits_D \overline{f} \cdot \overline{\mathrm{v}} dx. \tag{15}$$

For all $\mathrm{v} \in W_2^1(D)$ where

$$E(\overline{u}, \overline{\mathrm{v}}) = \frac{1}{2} \mu \int\limits_D \sum_{i,j=1}^{3} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \left( \frac{\partial \mathrm{v}_i}{\partial x_j} + \frac{\partial \mathrm{v}_j}{\partial x_i} \right) dx.$$

Let in (15 ) $\overline{\mathrm{v}} = \overline{u}$, then

$$E(\overline{u}, \overline{u}) + \lambda \| \operatorname{div} \overline{u} \|^2 = \int\limits_D \overline{f} \overline{u} dx, \tag{16}$$

Consistently evaluating the right-hand side of equality (16) we will have

$$\left| \int\limits_D \overline{f} \overline{u} dx \right| \leq \|\overline{f}\|_{W_2^{-1}(D)} \cdot \|u\|_{W_2^1(D)} \leq \delta \|u\|^2_{W_2^1(D)} + c_\delta \|f\|^2_{W_2^{-1}(D)},$$

$\delta > 0$, $c_\delta > 0$ are constants. Further, taking into account the Korn inequality [2], we obtain

$$\|u\|^2_{W_2^1(D)} + \lambda \|\operatorname{div} \overline{u}\|^2 \le c\|f\|^2_{W_2^{-1}(D)}.$$

Let v from (13) be the solution of the problem (13), (14), setting that in (15) $\operatorname{div} \overline{\mathrm{v}} = p$, we have

$$\lambda \int\limits_D \operatorname{div} \overline{u} \cdot p dx \le c\|p\|_{L_2(D)} + \|f\|_{W_2^{-1}(D)} \cdot \|p\|_{L_2(D)} \le c\|p\|_{L_2(D)}, \tag{17}$$

in (17) we set $p = \operatorname{div} u$, whence it follows that

$$\lambda \|\operatorname{div} \overline{u}\|_{L_2(D)} \le c < \infty.$$

Thus, we have obtained the following estimate

$$\|u\|_{W_2^1(D)} + \lambda \|\operatorname{div} \overline{u}\|_{L_2(D)} \le c < \infty. \tag{18}$$

Let us pass in (15) to the limit $\lambda \to \infty$. Since, by virtue of estimate (18), we have the relations $\overline{u} \to \overline{u}_0$ weakly in $W_2^1(D)$ at $\lambda \to \infty$, $\lambda \operatorname{div} \overline{u} \to p$ weakly in $L_2(D)$ at $\lambda \to \infty$. From this we obtain that $\overline{u}_0$ and $p$ satisfy the integral identity

$$E(\overline{u}_0, \overline{\mathrm{v}}) - \int\limits_D p \operatorname{div} \overline{\mathrm{v}} dx = \int\limits_D \overline{f} \overline{\mathrm{v}} dx.$$

For an arbitrary vector function $\overline{\mathrm{v}} \in W_2^1(D)$. That is, we will have in the limit for $\lambda = \infty$ a generalized solution of the boundary value problem

$$\mu \triangle \overline{u}_0 - \nabla p + \overline{f} = 0, \;\; x \in D, \tag{19}$$

$$\operatorname{div} \overline{u}_0 = 0, \;\; \sigma^0_{ik} = \mu \left( \frac{\partial u_{oi}}{\partial x_k} + \frac{\partial u_{ok}}{\partial x_i} \right) - \delta_{ik} p, \;\; i, k = 1, 2, 3,$$

$$\sum_{k=1}^{3} \sigma^0_{ik} n_k = 0, \gamma.$$

Next, we estimate the rate of convergence of the solution and problem (3), (5), (7), (8) to the solution $\overline{u}_0$, $p$ of the problem (19).

Denote by $\overline{\mathrm{w}} = \overline{u} - \overline{u}_0$, $p - \lambda \operatorname{div} \overline{u} = \pi$. By virtue of (5), (7), (8) and (19) we obtain

$$E(\overline{\mathrm{w}}, \overline{\mathrm{v}}) - (\pi, \operatorname{div} \overline{\mathrm{v}})_D = 0. \tag{20}$$

Whence it follows that

$$\|\overline{\mathrm{w}}\|^2_{W_2^1(D)} \le \|\pi\|_{L_2(D)} \cdot \|\operatorname{div} \overline{\mathrm{v}}\| \le c \cdot \|\pi\|_{L_2(D)} \cdot \lambda^{-1}. \tag{21}$$

Let v be the solution of the following problem

$$\operatorname{div} \overline{\mathrm{v}} = \pi, \;\; x \in D, \tag{22}$$

$$\overline{\mathrm{v}} = \|\mu_{n-1}(\gamma)\|^{-1} (\pi, 1)_D \cdot \overline{n}, \;\; on \;\; \gamma. \tag{23}$$

And suppose that in (20) the vector function $\overline{\mathrm{v}}$ satisfies (22), (23), then using the consequence of Lemma 2 we obtain

$$\|\pi\|^2 \le c\|\mathrm{w}\|_{W_2^1(D)} \cdot \|\overline{\mathrm{v}}\|_{W_2^1(D)} \le c\|\mathrm{w}\|_{W_2^1(D)} \cdot \|\pi\|_{L_2(D)}. \tag{24}$$

As a result, taking into account (24) there is an assessment

$$\|\pi\|_{L_2(D)} \le c\|\mathrm{w}\|_{W_2^1(D)}. \tag{25}$$

Referring to the estimate (21) then to (25) as a result, we obtain

$$\|\mathrm{w}\|^2_{W_2^1(D)} + \|\pi\|^2_{L_2(D)} \le c \cdot \lambda^{-2}, \tag{26}$$

so we have proved.

*Theorem.* Let $f \in W_2^{-1}(D)$, then the estimate (26) is valid.

*Comment.* Here, in the course of the argument, the existence and uniqueness of theorem for the generalized solution of problem (24) is proved. In [11], an estimate of proximity

$$\|\pi\|_{L_2(D)} \leq c \cdot \lambda^{-1/2},$$

was obtained, here from (26) follows

$$\|\pi\|_{L_2(D)} \leq c \cdot \lambda^{-1}.$$

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## Статикалық сығылатын ортаның шешімін сығылмайтын ортаны шешуге жақындату

Тұтқыр сығылмайтын сұйықтықтың және сығылмайтын серпімді ортаның ағынының ұқсастығы белгілі. Бұл ұқсастыққа сәйкес Пуассон коэффициентіндегі серпімділік теориясының теңдеулерінің шешімі $v = 0,5$ және кез келген тіркелген модулінде $\mu$ тұтқырлықпен тұтқыр қысылмайтын сұйықтықтың қозғалысы $\mu$ ретінде түсіндірілуі мүмкін. Осылайша, Гук заңымен сызықтық серпімділіктің әдеттегі статикалық есебін $\lambda \to \infty$ сызық қысылмайтын ортаға жақындаудың математикалық моделі ретінде қарастыруға болады. Мақалада асимптотикалық $\lambda \to \infty$ алынды. Гук заңымен тығыз статикалық есептің шешілмейтін ортаны шешуге жақындығын бағалауға (Стокс есебі) негіз бар. Соңғы бағалау сығылмайтын ортаны шешу үшін серпімді қысылған ортаға арналған белгілі айырымдық схемалар мен алгоритмдерді пайдалануға мүмкіндік береді. Авторлар осы міндеттерді шешу жақындығын дәлелдеді $\lambda \to \infty$, с. с. $\underset{\lambda\to\infty}{u\to\overline{u}^H}$ $\underset{\lambda\to\infty}{\lambda\,\mathrm{div}\,\overline{u}\to -p}$ $\underset{\lambda\to\infty}{\sigma\to\sigma^H}$. Бұл фактіні негіздеу үшін [1–3] бірінші шеттік есептің әртүрлі әдістерімен зерттелгенін көруге болады. Серпімділік теориясының статикалық есебі үшін қазіргі уақытта әртүрлі схемалардың көмегімен сандық іске асыруға арналған жұмыстардың тұтас циклі бар. [4] жұмыста $k = \frac{1}{2}$ болғандағы $O(\lambda^{-\alpha})$ бағалауы алынды, онда $O(\lambda^{-1})$ бағалауы бар, алдағы уақытта да бұл баға рет-ретімен жақсартылмайтыны көрсетіледі.

*Кілт сөздер:* қысылмайтын орта, Гук заңы, кернеу, орын ауыстыру, Ламе коэффициенттері.

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# Приближение решения статической сжимаемой среды к решению несжимаемой среды

Известна аналогия течения вязкой несжимаемой жидкости и несжимаемой упругой среды. Согласно этой аналогии, решение уравнений теории упругости при коэффициенте Пуассона $v = 0,5$ и при любом фиксированном модуле сдвига $\mu$ может быть интерпретировано как движение вязкой несжимаемой жидкости с вязкостью $\mu$. Таким образом, можно рассматривать обычную статическую задачу линейной упругости с законом Гука при $\lambda \to \infty$ как математическую модель приближения к несжимаемой среде. В статье была получена асимптотическая по $\lambda \to \infty$ оценка близости решения упругой статической задачи с законом Гука к решению несжимаемой среды (задача Стокса). Конечная оценка позволяет использовать известные разностные схемы и алгоритмы для упругой сжимаемой среды для решения несжимаемой среды. В работе доказана оценка близости решений этих задач при $\lambda \to \infty$, т.е. $\underset{\lambda\to\infty}{u\to\overline{u}^H}$ $\underset{\lambda\to\infty}{\lambda\,\mathrm{div}\,\overline{u}\to -p}$ $\underset{\lambda\to\infty}{\sigma\to\sigma^H}$. Для обоснования этого факта в [1–3] были исследованы различные приемы для первой краевой задачи. Для статической задачи теории упругости в настоящее время имеется целый цикл работ, посвященных численной реализации с помощью разностных схем. В [4] получена оценка $O(\lambda^{-\alpha})$, где $k = \frac{1}{2}$, в предлагаемой работе имеет место оценка $O(\lambda^{-1})$ , и в дальнейшем будет показано, что это оценка неулучшаема по порядку.

*Ключевые слова:* несжимаемая среда, закон Гука, напряжения, деформации, перемещения, коэффициенты Ламе.

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