UDC 517.518

G.A. Yessenbayeva[1], D.N. Yesbayeva[2]

[1] *Ye.A.Buketov Karaganda State University, Kazakhstan;*
[2] *Shanghai Factory-Amigo EC Technology Co., Ltd, China*
(E-mail: esenbaevagulsima@mail.ru)

Characterizations for the family of functions classes $\left\{B^r_{p,q,N}\right\}$ and their connection with Besov's spaces

In this paper we define families of classes of functions related to best approximations in harmonic intervals. These families of function classes characterize the order of approximation of functions by trigonometric polynomials with a spectrum from harmonic intervals. The article contains an investigation of various characterizations of the indicated families of function classes, introduces imbedding theorems, and shows the connection between the introduced families of functions classes and the classical Besov spaces. The article is intended for researchers specializing in the theory of approximation and functional analysis, and all those whose interests lie in these areas.

Keywords: classes of functions, harmonic intervals, best approximation of a function, theorem of embedding.

Definition 1. Let $1 \le p, q \le \infty,\ r > 0,\ f \in L_p\left[0; 2\pi\right).$ Family of function classes $\left\{B^r_{p,q,N}\right\}_N$ is defined by the following expression

$$B^r_{p,q,N} = \left\{f : \|f\|_{B^r_{p,q,N}} < \infty\right\}, N \in \mathrm{N},$$

where

$$\|f\|_{B^r_{p,q,N}} = \left(\sum_{k=1}^{N} k^{rq-1} \left(E^N_{k-1}\left(f\right)_p\right)^q\right)^{\frac{1}{q}}.$$

Definition 2. Let two families of function classes are $\left\{A^N\right\}_N$ and $\left\{B^N\right\}_N$, $\left\{A^N\right\}_N \bigcap \left\{B^N\right\}_N = \emptyset,\ \ N \in \mathrm{N}.$ We assume that

$$\|f\|_{A^N} \sim \|f\|_{B^N},$$

if there exist the parameters $C_1,\ C_2$ such that for any $f \in A^N$ the following expression correct

$$C_1 \|f\|_{B^N} \le \|f\|_{A^N} \le C_2 \|f\|_{B^N},$$

moreover the parameters $C_1,\ C_2$ are independent of$f$ and N.

In this case we assume that the families of function classes $\left\{A^N\right\}_N$ and $\left\{B^N\right\}_N$ coincide.

$$\left\{A^N\right\}_N = \left\{B^N\right\}_N.$$

Different characterizations of families of function classes $\left\{B^r_{p,q,2^m}\right\}_N$ are shown in the theorems 1 and 2.

$E^N_k\left(f\right)_p$ is the best approximation of the function $f \in L_p[0; 2\pi),\ 1 \le p \le \infty$, in the harmonic intervals I^N_k [1] by trigonometrical polynomials with order less than or equal to k [2].

Theorem 1. Let $f \in B^r_{p,q,2^m},\ \ m \in \mathrm{N}$, then for $1 \le p,\ q \le \infty,\ \ r > 0$ we have the relation of the form

$$\|f\|_{B^r_{p,q,N}} \sim \left(\sum_{k=1}^{m} 2^{rqk} \left(E^{2m}_{2^k-1}\left(f\right)_p\right)^q\right)^{\frac{1}{q}}.$$

Proof. Using the definition

$$\|f\|_{B^r_{p,q,N}} = \left(\sum_{k=1}^{2m} k^{rq-1} \left(E^{2m}_{k-1}\left(f\right)_p\right)^q\right)^{\frac{1}{q}} =$$

$$=\left\{\sum_{k=1}^{m}\sum_{s=2^k-1}^{2^k-1}s^{rq-1}\left(E_{s-1}^{2m}(f)_p\right)^q+2^{m(rq-1)}\left(E_{2^m-1}^{2^m}(f)_p\right)^q\right\}^{\frac{1}{q}}. \tag{1}$$

We use one of the properties of best approximation in harmonic intervals

$$E_{2^k-1}^{2^m}\leq E_{2^k-2}^{2^m}\leq ...\leq E_{2^{k-1}}^{2^m}\leq E_{2^{k-1}-1}^{2^m}, 1\leq k\leq m. \tag{2}$$

Besides we have the following estimation

$$\left(2^{k-1}\right)^{rq-1}\cdot 2^{k-1}\leq\sum_{s=2^{k-1}}^{2^k-1}s^{rq-1}\leq\left(2^k-1\right)^{rq-1}\cdot 2^{k-1};$$

$$2^{rqk-rq}\leq\sum_{s=2^{k-1}}^{2^k-1}s^{rq-1}\leq 2^{rqk}. \tag{3}$$

Considering (2), (3), we get from (1)

$$2^{rq(k-1)}\left(E_{2^k-1}^{2^m}(f)_p\right)^q\leq\sum_{s=2^{k-1}}^{2^k-1}s^{rq-1}\left(E_{s-1}^{2^m}(f)_p\right)^q\leq 2^{rqk}\left(E_{2^{k-1}-1}^{2^m}(f)_p\right)^q;$$

$$2^{-rq}\sum_{k=1}^{m}2^{rqk}\left(E_{2^k-1}^{2^m}(f)_p\right)^q\leq\sum_{k=1}^{m}\sum_{s=2^{k-1}}^{2^k-1}s^{rq-1}\left(E_{s-1}^{2^m}(f)_p\right)^q+$$

$$+2^{m(rq-1)}\left(E_{2^m-1}^{2^m}(f)_p\right)^q\leq\sum_{k=1}^{m}2^{rqk}\left(E_{2^{k-1}}^{2^m}(f)_p\right)^q+2^{rqm}\left(E_{2^{m-1}-1}^{2^m}(f)_p\right)^q;$$

$$2^{-r}\left(\sum_{k=1}^{m}2^{rkq}\left(E_{2^k-1}^{2^m}(f)_p\right)^q\right)^{\frac{1}{q}}\leq\|f\|_{B_{p,q.2^m}^r}\leq$$

$$\leq\left\{\sum_{k=1}^{m}2^{rqk}\left(E_{2^{k-1}-1}^{2^m}(f)_p\right)^q+2^{rqm}\left(E_{2^{m-1}-1}^{2^m}(f)_p\right)^q\right\}^{\frac{1}{q}};$$

$$2^{-r}\left(\sum_{k=1}^{m}2^{rqk}\left(E_{2^k-1}^{2^m}(f)_p\right)^q\right)^{\frac{1}{q}}\leq\|f\|_{B_{p,q,2^m}^r}\leq$$

$$\leq\left\{\sum_{k=0}^{m-1}2^{rq(k+1)}\left(E_{2^k-1}^{2^m}(f)_p\right)^q+2^{rqm}\left(E_{2^{m-1}-1}^{2^m}(f)_p\right)^q\right\}^{\frac{1}{q}}\leq$$

$$\leq\left(2^{1+rq}\sum_{k=0}^{m-1}2^{rqk}\left(E_{2^k-1}^{2^m}(f)_p\right)^q\right)^{\frac{1}{q}}.$$

As a result, we obtain the required inequality

$$2^{-r}\left(\sum_{k=1}^{m}2^{rqk}\left(E_{2^k-1}^{2^m}(f)_p\right)^q\right)^{\frac{1}{q}}\leq\|f\|_{B_{p,q,2^m}^r}\leq 2^{\frac{1}{q}+r}\left(\sum_{k=0}^{m-1}2^{rqk}\left(E_{2^k-1}^{2^m}(f)_p\right)^q\right)^{\frac{1}{q}}.$$

The theorem is proved.

Theorem 2. Let $f\in B_{p,q,2^m}^r$, $m\in\mathrm{N}$, *then the following relation holds for* $1\leq p,\ q\leq\infty,\ \ r>0$

$$\|f\|_{B_{p,q,N}^r}\sim\left(\sum_{k=1}^{m}2^{rqk}\left(\delta_k^{2m}(f)_p\right)^q\right)^{\frac{1}{q}},$$

where

$$\delta_k^{2m}(f)_p = \left\| \sum_{s\in Z} \sum_{\tau=2^k-1}^{2^k-1} a_\tau + s\cdot 2^m \cdot e^{i(\tau+s\cdot 2^m)x} \right\|_p .$$

Proof. According to the boundedness of the partial sum over the harmonic interval [1] and Theorem 1, we have

$$\delta_k^{2m}(f)_p \le C \left\| \sum_{s\in Z} \sum_{\tau=2^k-1}^{2^m-1} a_\tau + s\cdot 2^m \cdot e^{i(\tau+s\cdot 2^m)x} \right\|_p \sim E_{2^k-1}^{2^m}(f)_p ,$$

therefore, using Theorem 1, we obtain

$$\left(\sum_{k=1}^{m} 2^{rqk} \left(\delta_k^{2m}(f)_p\right) q\right)^{\frac{1}{q}} \le C \left(\sum_{k=1}^{m} 2^{rqk} \left(E_{2^k-1}^{2^m}(f)_p\right) q\right)^{\frac{1}{q}} \sim C \|f\|_{B^r_{p,q,2^m}} .$$

Let us prove the reverse inequality. As

$$E_{2^k-1}^{2^m}(f)_p \le \sum_{\tau=k}^{m} \delta_\tau^{2^m}(f)_p ,$$

then by Theorem 1, the Holder inequality for numerical sequences the following inequality holds

$$\|f\|_{B^r_{p.q.2^m}} \le C \left(\sum_{k=1}^{m} 2^{rqk}\cdot \left(E_{2^k-1}^{2^m}(f)_p\right) q\right)^{\frac{1}{q}} \le$$

$$\le C \left(\sum_{k=1}^{m} 2^{rqk}\cdot \left(\sum_{\tau=k}^{m} \delta_\tau^{2^m}(f)_p\right)^q\right)^{\frac{1}{q}} = C\left\{\sum_{k=1}^{m} 2^{rqk} \left(\sum_{\tau=k}^{m} 2^{\lambda\tau}2^{-\lambda\tau}\delta_\tau^{2^m}(f)_p\right)^q\right\}^{\frac{1}{q}} \le$$

$$\le C\left\{\sum_{k=1}^{m} 2^{rqk}\left[\left(\sum_{\tau=k}^{m} 2^{-\lambda\tau q^/}\right)^{\frac{1}{q^/}} \cdot \left(\sum_{\tau=k}^{m}\left(2^{\lambda\tau}\delta_\tau^{2^m}(f)_p\right)^q\right)^{\frac{1}{q}}\right]^q\right\}^{\frac{1}{q}} =$$

$$= C\left\{\sum_{k=1}^{m} 2^{rqk}\left(\sum_{\tau=k}^{m} 2^{-\lambda\tau q^/}\right)^{\frac{q}{q^/}} \cdot \sum_{\tau=k}^{m}\left(2^{\lambda\tau}\delta_\tau^{2^m}(f)_p\right)^q\right\}^{\frac{1}{q}} \le C\left\{\sum_{k=1}^{m} 2^{rqk}2^{-\lambda qk}\cdot \sum_{\lambda=k}^{m}\left(2^{\lambda\tau}\delta_\tau^{2^m}(f)_p\right)^q\right\}^{\frac{1}{q}} =$$

$$= C\left\{\sum_{\lambda=1}^{m} 2^{\lambda q\tau}\left(\delta_\tau^{2^m}(f)_p\right)^q \sum_{k=1}^{m} 2^{(r-\lambda)qk}\right\}^{\frac{1}{q}} \le$$

$$\le C\left\{\sum_{\tau=1}^{m} 2^{\lambda q\tau}\left(\delta_\tau^{2^m}(f)_p\right)^q \cdot 2^{(r-\lambda)q\tau}\right\}^{\frac{1}{q}} = C\left\{\sum_{\tau=1}^{m} 2^{rq\tau}\left(\delta_\tau^{2^m}(f)_p\right)^q\right\}^{\frac{1}{q}} ,$$

$$\Rightarrow \|f\|_{B^r_{p.q.2^m}} \left(\sum_{k=1}^{m} 2^{rqk}\left(\delta_k^{2^m}(f)_p\right) q\right)^{\frac{1}{q}} .$$

So the theorem is proved.

Note that in the Theorems 1 and 2 consecutive constants are independent of f and m.

Definition 3. Let two function classes A^N and B^N dependent on the parameter N are given. We say that a class of functions A^N is embedded in a class of functions B^N and denote $A^N\, B^N$, if the following conditions hold:

1) $A^N \subset B^N$;

2) there exists the parameter C such for any $f \in A^N$ the relation holds

$$\|f\|_{B^N} \le C\|f\|_{A^N}$$

and the parameter C is independent of$f$ and N.

Theorem 3. Let $N \in \mathrm{N}$, $1 \le p, q, q_1 \le \infty$, $r > 0$, *then the following embedding holds for* $q < q_1$

$$B^r_{p,q,N} \longrightarrow B^r_{p,q_1,N}.$$

Proof. Applying Theorem 2, we obtain

$$\|f\|^q_{B^r_{p,q_1,N}} \le C \left(\sum_{k=1}^{[\log_2 N]} 2^{rq_1 k} \cdot \left(\delta_k (f)_p \right)^{q_1} \right)^{\frac{q}{q_1}},$$

where the parameter C is independent of f and N.

Since $\frac{q}{q_1} < 1$ and $2^{rq_1 k} \cdot \left(\delta_k (f)_p \right)^{q_1} \ge 0$, then, taking into account [3], that

$$\left(\sum_{k=m}^{n} a_k \right)^{\delta} \le \sum_{k=m}^{n} a_k^{\delta}$$

for $0 \le \delta \le 1$, $a_k \ge 0$, $0 \le m \le k \le n \le \infty$, we get

$$\|f\|^q_{B^r_{p,q_1,N}} \le C \sum_{k=1}^{[\log_2 N]} 2^{rqk} \cdot \left(\delta_k (f)_p \right)^q \le C \, \|f\|^q_{B^r_{p,q,N}}.$$

Thus, the theorem is proved.

Theorem 4. Let $N \in \mathrm{N}$, $1 \le p, q, q_1 \le \infty$, $r > 0$, *then for any* $\varepsilon > 0$ *the following embedding holds*

$$B^r_{p,q,N} \longrightarrow B^{r-\varepsilon}_{p,q_1,N}.$$

Proof. It suffices to show that

$$B^r_{p,\infty,N} \longrightarrow B^{r-\varepsilon}_{p,q_1,N}.$$

Let $f \in B^r_{p,\infty,N}$. Using Theorem 1, we obtain

$$\|f\|_{B^{r-\varepsilon}_{p,q_1 N}} = \left(\sum_{k=1}^{[\log_2 N]} 2^{(r-\varepsilon) q_1 k} \left(E^N_{2^k-1} (f)_p \right)^{q_1} \right)^{\frac{1}{q_1}} \le$$

$$\le \sup_{1 \le k \le [\log_2 N]} 2^{rk} \cdot E^N_{2^k-1} (f)_p \cdot \left(\sum_{k=1}^{[\log_2 N]} 2^{-\varepsilon q_1 k} \right)^{\frac{1}{q_1}} \le \|f\|_{B^r_{p,\infty,N}} \cdot C(q, \varepsilon).$$

$$\Rightarrow B^r_{p,\infty,N} \longrightarrow B^{r-\varepsilon}_{p,q_1,N}.$$

On the other hand,

$$B^r_{p,q,N} \to B^r_{p,\infty,N}$$

for any q such that $1 \le q \le \infty$. The theorem is proved.

Definition 4. Let $r > 0$, $1 \le p \le \infty$, $1 \le \theta \le \infty$. Suppose that $B^r_{p,\theta}[0; 2\pi) = B^r_{p,\theta} \left(B^r_{p,\infty} = H^r_p \right)$. We say that a function $f \in L_p [0; 2\pi)$ belongs to the Besov space $B^r_{p,\theta}$,if the norm is finite

$$\|f\|_{B^r_{p,\theta}} = \|f\|_p + \left(\sum_{s=0}^{\infty} 2^{s\theta r} \cdot E_{2^s} (f)^{\theta}_p \right)^{\frac{1}{\theta}}.$$

Here the expression $\left(\sum_{s=0}^{\infty} 2^{s\theta r} \cdot E_{2^s} (f)^{\theta}_p \right)^{\frac{1}{\theta}}$ for $\theta = \infty$ is understood as $\sup\limits_{s>0} 2^{sr} \cdot E_{2^s} (f)_p$ [4].

The following statement shows the relationship of families of functions classes $\left\{ B^r_{p,q,N} \right\}_N$ and classical Besov spaces.

Theorem 5. Let $N \in \mathrm{N}$, $1 \le p, q, \le \infty$, $r > 0$, *then we have the following relation*

$$\bigcap_{N-1}^{\infty} B^r_{p,q,N} = B^r_{p,q}.$$

Proof. By definition, we have

$$\|f\|_{\bigcap\limits_{N=1}^{\infty} B^r_{p,q,N}} = \sup_{1 \le N \le \infty} \|f\|_{B^r_{p,q,N}}.$$

Since the following inequality

$$\|f\|_{B_{p,q,N}} \le C \cdot \|f\|_{B^r_{p,q}}$$

holds for any $N \in \mathrm{N}$, then relation

$$\sup_{1 \le N \le \infty} \|f\|_{B^r_{p,q,N}} = \|f\|_{\bigcap\limits_{N=1}^{\infty} B^r_{p,q,N}} \le \|f\|_{B^r_{p,q}},$$

holds. From which it immediately follows that

$$B^r_{p,q} \longrightarrow \bigcap_{N=1}^{\infty} B^r_{p,q,N}.$$

On the other hand, for a partial sum $S_{2^m}(f)$, where $m \in \mathrm{N}$, we have

$$\|S_{2^m}(f)\|_{B^r_{p,q}} = \|S_{2^m}(f)\|_{B^r_{p,q,2^m}} \le C(p,q,r) \cdot \|f\|_{B^r_{p,q,2^m}} \le$$

$$\le C(p,q,r) \sup_{1 \le N \le \infty} \|f\|_{B^r_{p,q,N}} = C(p,q,r) \cdot \|f\|_{\bigcap\limits_{N=1}^{\infty} B^r_{p,q,N}}.$$

From the last relation by the Banach-Steinhaus theorem [4] we obtain the required inequality

$$\|f\|_{B^r_{p,q}} \le C(p,q,r) \cdot \|f\|_{\bigcap\limits_{N=1}^{\infty} B^r_{p,q,N}},$$

i.e.

$$\bigcap_{N=1}^{\infty} B^r_{p,q,N} \longrightarrow B^r_{p,q}.$$

As a result, the theorem is proved.

References

1 Нурсултанов Е.Д. Сетевые пространства и их приложения к задачам гармонического анализа: дис. ... д-ра физ.-мат. наук: 01.01.01 / Ерлан Даутбекович Нурсултанов. — СПб., 1999. — С. 171–179.

2 Есенбаева Г.А. О свойствах наилучших приближений функций тригонометрическими полиномами со спектром из гармонических интервалов / Г.А. Есенбаева // Вестн. Караганд. ун-та. Серия Математика. — 2005. – № 1(37). — С. 44–49.

3 Кокилашвили В.М. О приближении периодических функций / В.М. Кокилашвили // Тр. Тбилисского матем. ин-та. — 1968. — С. 51–81.

4 Никольский С.М. Приближение функций многих переменных и теоремы вложения / С.М. Никольский. — М.: Наука, 1977. — 456 с.

Г.А. Есенбаева, Д.Н. Есбаева

$\left\{B^r_{p,q,N}\right\}$ функция кластары үйірінің сипаттамасы және олардың Бесов кеңістіктерімен байланысы

Мақалада гармоникалық интервалдар бойынша ең жақсы жуықтаулармен байланысты функция кластарының үйірі анықталған. Функция кластарының үйірі деректері функцияның гармоникалық интервалдарда спектрі бар тригонометриялық көпмүшеліктермен жуықтау ретін сипаттайды. Авторлармен көрсетілген функция кластары үйірінің түрлі сипаттамаларының зерттемелері баяндалған, енгізу теориясы берілген және енгізілген функция кластарының үйірі мен классикалық Бесов кеңістіктері арасындағы байланыс көрсетілген.

Кілт сөздер: функция кластары, гармоникалық интервалдар, функцияның ең жақсы жуықталуы, енгізу теоремасы.

Г.А. Есенбаева, Д.Н. Есбаева

Характеризации семейства классов функций $\left\{B^r_{p,q,N}\right\}$ и их связь с пространствами Бесова

В статье определены семейства классов функций, связанные с наилучшими приближениями по гармоническим интервалам. Данные семейства классов функций характеризуют порядки приближения функций тригонометрическими полиномами со спектром из гармонических интервалов. В статье изложено исследование различных характеризаций указанных семейств классов функций, даны теоремы вложения и показана связь введенных семейств классов функций и классических пространств Бесова.

Ключевые слова: классы функций, гармонические интервалы, наилучшее приближение функции, теорема вложения.

References

1 Nursultanov, E.D. (1999). Setevye prostranstva i ikh prilozheniia k zadacham harmonicheskoho analiza [Net spaces and their applications to the problems of harmonic analysis]. *Doctor's thesis.* Saint Petersburg [in Russian].

2 Yessenbayeva, G.A. (2005). O svoistvakh nailuchshikh priblizhenii funktii trihonometricheskimi polinomami so spektrom iz harmonicheskikh intervalov [On the properties of best approximation of functions by trigonometric polynomials with a spectrum of harmonic intervals]. *Vestnik Karahandinskoho universiteta. Seriya Matematika – Bulletin of Karaganda University. Mathematics series, 1(37),* 44–49 [in Russian].

3 Kokilashvili, V.M. (1986). O priblizhenii periodicheskikh funktsii [On the approximation of periodic functions]. *Trudy Tbilisskoho matematicheskoho instituta – Proceedings of the Tbilisi Mathematical Institute,* 51–81 [in Russian].

4 Nikolsky, S.M. (1977). *Priblizhenie funktsii mnohikh peremennykh i teoremy vlozheniia [Approximation of functions of several variables and embedding theorems].* Moscow: Nauka [in Russian].