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Estimations of the best M-term approximations of functions in the Lorentz space with constructive methods

This paper considers the Lorentz space of periodic functions of many variables with the anisotropic norm, of functional Nikol'skii-Besov's class and of the best M-term approximation of function. We have established sufficient conditions for the function to belong to one of the Lorentz spaces in another. We obtain upper and lower bounds for the best M-member approximations of functions from the Nikol'skii-Besov class in the anisotropic Lorentz space To prove the upper bound, we used a new constructive method developed by V.N. Temlyakov.

Keywords: Lorentz space, Nikol'ski-Besov class, the best M-term approximations, approximation, sufficient conditions, estimate.

Introduction

Let $\bar{x} = (x_1, ..., x_m) \in \mathbb{R}^m$, $I^m = [0, 2\pi)^m$ and numbers $\theta_j, q_j \in [1, +\infty)$, $j = 1, ..., m$. Let $L^*_{\bar{q},\bar{\theta}}(I^m)$ denotes the space of Lebesgue measurable functions $f(\overline{x})$ defined on $\mathbb{R}^m$ with the period 2π with respect to each variable such that the quantity

$$\|f\|^*_{\bar{q},\bar{\theta}} = \left[\int_0^{2\pi} t_m^{\frac{\theta_m}{q_m}-1}\left[\cdots\left[\int_0^{2\pi}\left(f^{*_1,\dots,*_m}(t_1,...,t_m)\right)^{\theta_1} t_1^{\frac{\theta_1}{q_1}-1}dt_1\right]^{\frac{\theta_2}{\theta_1}}\cdots\right]^{\frac{\theta_m}{\theta_{m-1}}}dt_m\right]^{\frac{1}{\theta_m}},$$

is finite, where $f^{*_1,\dots,*_m}(t_1,...,t_m)$ is a non-increasing rearrangement of the function $|f(\overline{x})|$ in each variable x_j, whereas the other variables are fixed [1].

In case when the $q_1 = ... = q_m = \theta_1 = ... = \theta_m = q$, the space of Lorentz $L^*_{\bar{q},\bar{\theta}}(I^m)$ coincides with the space of Lebesgue $L_q(I^m)$ with the norm ([2], Ch. I, item 1.1)

$$\|f\|_q = \left[\int_0^{2\pi}...\int_0^{2\pi}|f(x_1,...,x_m)|^q dx_1...dx_m\right]^{\frac{1}{q}}.$$

Let $\overset{\circ}{L}{}^*_{\bar{q},\bar{\theta}}(I^m)$ be the set of all functions $f \in L^*_{\bar{q},\bar{\theta}}(I^m)$ such that

$$\int_0^{2\pi} f(\overline{x})\, dx_j = 0, \ \ \forall j = 1, ..., m.$$

For any function $f \in L_1(I^m) = L(I^m)$, let

$$\sum_{\overline{n}\in\mathbb{Z}^m} a_{\overline{n}}(f)\, e^{i\langle\overline{n},\overline{x}\rangle}$$

be function's Fourier series with respect to the multiple trigonometric system $\{e^{i\langle\overline{n},\overline{x}\rangle}\}_{\mathbb{Z}^m}$, where $\mathbb{Z}^m$ is the set of points in $\mathbb{R}^m$ with integer coordinates.

Suppose

$$\delta_{\overline{s}}(f,\overline{x}) = \sum_{\overline{n}\in\rho(\overline{s})} a_{\overline{n}}(f)\, e^{i\langle\overline{n},\overline{x}\rangle},$$

where $\langle \bar{y}, \bar{x} \rangle = \sum\limits_{j=1}^{m} y_j x_j$, $s_j = 1, 2, ...$ and

$$\rho(\bar{s}) = \left\{ \bar{k} = (k_1, ..., k_m) \in \mathbb{Z}^m : \quad 2^{s_j - 1} \leq |k_j| < 2^{s_j}, j = 1, ..., m \right\}.$$

For a number sequence, we will write $\{a_{\overline{n}}\}_{\overline{n} \in \mathbb{Z}^m} \in l_{\overline{p}}$ if

$$\left\| \{a_{\overline{n}}\}_{\overline{n} \in \mathbb{Z}^m} \right\|_{l_{\overline{p}}} = \left\{ \sum_{n_m = -\infty}^{\infty} \left[... \left[\sum_{n_1 = -\infty}^{\infty} |a_{\overline{n}}|^{p_1} \right]^{\frac{p_2}{p_1}} ... \right]^{\frac{p_m}{p_{m-1}}} \right\}^{\frac{1}{p_m}} < +\infty,$$

where $\overline{p} = (p_1, ..., p_m)$, $1 \leq p_j < +\infty$, $j = 1, 2, ..., m$.

The spaces $S_p^{\bar{r}} H$ and $S_{p,\theta}^{\bar{r}} B$ of functions with the dominating mixed derivative were introduced by S.M. Nikol'skii [3] and T.I. Amanov ([4] Ch.I, item 17). The spaces $S_p^{\bar{r}} H$, $S_{p,\theta}^{\bar{r}} B$ are called Nikolski-Besov's space, or, sometimes, Nikolski-Besov-Amanov's space.

P.I. Lizorkin and S.M. Nikol'skii [5] investigated a decomposition of elements of the space $S_{p,\theta}^{\bar{r}} B$. We will use their definition.

Let $\bar{r} = (r_1, ..., r_m)$, $r_j > 0, j = 1, ..., m$, $1 \leq p, \theta \leq +\infty$. Suppose $S_{p,\theta}^{\bar{r}} B$ is the space all of functions $f \in \overset{\circ}{L}{}_{\bar{q},\bar{\theta}}^{*}(I^m)$ such that

$$\|f\|_{S_{p,\theta}^{\bar{r}} B} = \left[\int_0^{2\pi} ... \int_0^{2\pi} \|\Delta_{\bar{t}}^{\bar{k}} f(\bullet)\|_p^{\theta} \prod_{j=1}^{m} \frac{dt_j}{t_j^{1+\theta r_j}} \right]^{\frac{1}{\theta}} < +\infty,$$

where $\Delta_{\bar{t}}^{\bar{k}} f(\bar{x}) = \Delta_{t_m}^{k_m}(...\Delta_{t_1}^{k_1} f(\bar{x}))$ is the mixed difference of order $\bar{k}$ with step $\bar{t} = (t_1, ..., t_m)$ and $k_j > r_j$, $j = 1, ..., m$.

In [5], it is given that the function $f \in S_{p,\theta}^{\bar{r}} B$ can be decomposed into Fourier series in the following form

$$\sum_{\overline{n} \in \mathbb{Z}^m, \prod_{j=1}^{m} n_j \neq 0} a_{\overline{n}}(f) \, e^{i\langle \overline{n}, \overline{x} \rangle}.$$

Moreover, it is known (see [5]) that $\|f\|_{S_{p,\theta}^{\bar{r}} B}$ is a norm and

$$\|f\|_{S_{p,\theta}^{\bar{r}} B} \asymp \left\{ \sum_{\bar{s} \in \mathbb{Z}_+^m} 2^{\langle \bar{s}, \bar{r} \rangle \theta} \|\delta_{\bar{s}}(f)\|_p^{\theta} \right\}^{\frac{1}{\theta}}$$

provided $1 < p < +\infty$, $1 \leq \theta \leq +\infty$.

Therefore, in the anisotropic Lorentz space $\overset{\circ}{L}{}_{\bar{p},\bar{\theta}}^{*}(I^m)$, we will consider an analogous space. Suppose $S_{\bar{p},\bar{\theta},\bar{\tau}}^{\bar{r}} B$ denotes the space of all functions $f \in \overset{\circ}{L}{}_{\bar{p},\bar{\theta}}^{*}(I^m)$ such that

$$\|f\|_{S_{\bar{p},\bar{\theta},\bar{\tau}}^{\bar{r}} B} = \left\| \left\{ 2^{\langle \bar{s}, \bar{r} \rangle} \|\delta_{\bar{s}}(f)\|_{\bar{p},\bar{\theta}}^{*} \right\}_{\bar{s} \in \mathbb{Z}_+^m} \right\|_{l_{\bar{\tau}}} < \infty,$$

where $\overline{p} = (p_1, \ldots, p_m)$, $\overline{\theta} = (\theta_1, \ldots, \theta_m)$, $\overline{\tau} = (\tau_1, \ldots, \tau_m)$, $1 < p_j < \infty$, $1 < \theta_j < \infty$, $1 \leq \tau_j \leq +\infty$, $r_j > 0$, $j = 1, \ldots, m$.

In this space, let's consider the unit ball (with keeping the notation)

$$S_{\bar{p},\bar{\theta},\bar{\tau}}^{\bar{r}} B = \left\{ f \in \overset{\circ}{L}{}_{\bar{p},\bar{\theta}}^{*}(I^m) : \|f\|_{\overset{\circ}{S}{}_{\bar{p},\bar{\theta},\bar{\tau}}^{\bar{r}} B} = \left\| \left\{ 2^{\langle \bar{s}, \bar{r} \rangle} \|\delta_{\bar{s}}(f)\|_{\bar{p},\bar{\theta}}^{*} \right\}_{\bar{s} \in \mathbb{Z}_+^m} \right\|_{l_{\bar{\tau}}} \leq 1 \right\}.$$

For a fixed vector $\bar{\gamma} = (\gamma_1, ..., \gamma_m)$, $\gamma_j > 0$, $j = 1, ..., m$, set

$$Q_n^{\bar{\gamma}} = \cup_{\langle \bar{s}, \bar{\gamma} \rangle < n} \rho(\bar{s}), \quad T(Q_n^{\bar{\gamma}}) = \{t(\bar{x}) = \sum_{\bar{k} \in Q_n^{\bar{\gamma}}} b_{\bar{k}} e^{i\langle \bar{k}, \bar{x} \rangle}\},$$

$$Y^m(\overline{\gamma}, n) = \left\{ \overline{s} = (s_1, ..., s_m) \in \mathbb{Z}_+^m : \sum_{j=1}^{m} s_j \gamma_j \geq n \right\}.$$

Let X, Y be spaces with the norm of 2π-periodic functions of several variables. For a function $f \in X$ the following quantity is called the best M-term approximation of f [6–8]:

$$e_M(f)_X = \inf_{\bar{k}^{(j)}, b_j} \|f - \sum_{j=1}^{M} b_j e^{i\langle \bar{k}^{(j)}, \bar{x} \rangle}\|_X,$$

where $\{\bar{k}^{(j)}\}_{j=1}^{M}$ is a system of vectors $\bar{k}^{(j)} = (k_1^{(j)}, ..., k_m^{(j)})$ with integer coordinates and b_j are arbitrary numbers.

For a given class F, let

$$e_M(F)_X = \sup_{f \in F} e_M(f)_X.$$

In the case $X = L_2$, the quantity $e_M(f)_{L_2}$ for a function of one variable was introduced by S.B. Stechkin [6] and it was used in a criteria for an absolute convergence of Fourier series by complete orthonormal systems. Order estimations of the quantity $e_M(F)_X$ were investigated by R.S. Ismagilov [7], B.Ye. Mayorov [8] (for $X = L_p$, one-dimensional case), E.S. Belinskii [9–11] (multi-dimensional case in the case $Y = L_q(I^m), X = L_p(I^m)$, $F = W_p^r$), V.N. Temlyakov [12] (in the case $Y = L_q(I^m), F = H_p^r$), A.S. Romanyuk [13, 14], R. De Vore and R.A. Devore [15], V.N. Temlyakov [16] (in the case $Y = L_q(I^m), F = B_{p,\theta}^r$), Dinh Dung [17]. We should note that B.S. Kashin [18] established an estimation of the quantity $e_M(f)_X$ in the case $X = L_2$ by orthonormal systems. The latest results in this direction can be found in [19–21].

In particular, the following theorem is well known.

Theorem 1 (A.S. Romanyuk [13]). Let $\overline{r} = (r_1, ..., r_m)$, $0 < r_1 = ... = r_\nu < r_{\nu+1} \leq ... \leq r_m$, $1 \leq p \leq 2 < q < +\infty$, and $1 \leq \theta \leq +\infty$.

1) If $r_1 > \frac{1}{p}$ then

$$e_M\left(S_{p,\theta}^{\overline{r}} B\right)_q \asymp M^{-\left(r_1 + \frac{1}{2} - \frac{1}{p}\right)} (\log M)^{(\nu-1)\left(r_1 - \frac{1}{p} + \frac{1}{2}\right) + \sum\limits_{j=2}^{\nu} \left(\frac{1}{2} - \frac{1}{\theta}\right)_+};$$

2) If $\frac{1}{p} - \frac{1}{q} < r_1 < \frac{1}{p}$, then

$$e_M\left(S_{p,\theta}^{\overline{r}} B\right)_q \asymp M^{-\frac{q}{2}\left(r_1 + \frac{1}{q} - \frac{1}{p}\right)} (\log M)^{(q-1)(\nu-1)\left(r_1 - \frac{1}{p} + \frac{q'}{q} \frac{1}{\theta'}\right)_+};$$

3) If $r_1 = \frac{1}{p}$, then

$$e_M\left(S_{p,\theta}^{\overline{r}} B\right)_q \asymp M^{-\frac{1}{2}} \left(\log^\nu M\right)^{\frac{1}{\theta'}},$$

where $a_+ = \max\{a, 0\}$, $\frac{1}{b} + \frac{1}{b'} = 1$.

Here $\log M$ is the logarithm with the base 2 of $M > 0$.

V.N. Temlyakov [22–24] developed a constructive method of estimation of the $M-$ term best approximations of functions of the Nikol'skii-Besov's class $S_{p,\tau}^{\overline{r}} B$ in the space $L_q(I^m)$ in the case $1 < p < q < \infty$. This method is based on greedy algorithms.

The main goal of the present paper is to find the exact order of the best M-term approximation of a function in the class $S_{\bar{p},\bar{\theta},\bar{\tau}}^{\overline{r}} B$ in the Lorentz spaces with anisotropic norm in the case $1 < p_j < q_j \leq 2$, $j = 1, ..., m$, and in the case $1 < p_j < 2 \leq q_j < \infty, j = 1, ..., m$, to give a constructive proof for an upper bound for the quantity $e_M\left(S_{\bar{p},\bar{\theta},\bar{\tau}}^{\overline{r}} B\right)_{\bar{q},\bar{\theta}}$.

Let us denote by $C(p, q, r, y)$ positive quantities which depend on the parameters in the parentheses, which are, in general, are distinct in distinct formulas. $A(y) \asymp B(y)$ means that there are positive C_1, C_2 such that $C_1 \cdot A(y) \leq B(y) \leq C_2 \cdot A(y)$.

S.1. Auxiliary results

To prove the main results, we need the following auxiliary results.

Lemma 1 [25]. Suppose $\alpha \in (0, +\infty)$ and $\overline{\gamma} = (\gamma_1, ..., \gamma_m)$, $\overline{\gamma}' = \left(\gamma_1', ..., \gamma_m'\right)$, $\bar{\theta} = (\theta_1, ..., \theta_m)$, $\theta_j \in [1, +\infty)$, $j = 1, ..., m$, $1 = \gamma_1 = ... = \gamma_\nu < \gamma_{\nu+1} \le ... \le \gamma_m$, $1 = \gamma_j' = \gamma_j$, $j = 1, ..., \nu$, and $1 = \gamma_j' < \gamma_j$, $j = \nu + 1, ..., m$. Then the following relation holds

$$\left\| \left\{ 2^{-\alpha \langle \bar{s}, \overline{\gamma} \rangle} \right\}_{\bar{s} \in Y^m\left(\overline{\gamma}', n\right)} \right\|_{l_{\bar{\theta}}} \asymp 2^{-n\alpha} n^{\sum\limits_{j=2}^{\nu} \frac{1}{\theta_j}}.$$

Remark. For the case $\theta_1 = ... = \theta_m$, Lemma 1 was proved by V.N. Temlyakov [12]. In what follows we denote by $\chi_{\varkappa(n)}(\bar{s})$ the characteristic function of the set $\varkappa(n) = \{\bar{s} = (s_1, ..., s_m) \in \mathbb{Z}_+^m : \quad \langle \bar{s}, \bar{\gamma} \rangle = n\}$.

Lemma 2 ([25], Lemma 2.3 and [26], Lemma 4). Let $\bar{\tau} = (\tau_1, ..., \tau_m)$, $1 \le \tau_j < +\infty$, and $j = 1, ..., m$. Then the following relation holds

$$\left\| \left\{ \chi_{\varkappa(n)}(\bar{s}) \right\}_{\bar{s} \in \varkappa(n)} \right\|_{l_{\bar{\tau}}} \asymp n^{\sum\limits_{j=2}^{m} \frac{1}{\tau_j}}, \quad n \in \mathbb{N}.$$

Theorem 2 [27]. Let $\overline{p} = (p_1, ..., p_m)$, $\overline{q} = (q_1, ..., q_m)$, $\overline{r} = (r_1, ..., r_m)$, $\bar{\theta} = (\theta_1, ..., \theta_m)$, $\quad \overline{\tau} = (\tau_1, ..., \tau_m)$, $1 \le p_j < 2 < q_j$, $j = 1, ..., m$, $1 \le \theta_j, \tau_j < +\infty$, $0 < r_1 + \frac{1}{q_1} - \frac{1}{p_1} = ... = r_\nu + \frac{1}{q_\nu} - \frac{1}{p_\nu} < r_{\nu+1} + \frac{1}{q_{\nu+1}} - \frac{1}{p_{\nu+1}} \le ... \le r_m + \frac{1}{q_m} - \frac{1}{p_m}$.

1) If $r_j > \frac{1}{p_j}, j = 1, ..., m$, $(r_1 - \frac{1}{p_1})\frac{1}{q_j} < (r_j - \frac{1}{p_j})\frac{1}{q_1}, j = \nu + 1, ..., m$, then

$$e_M \left(S^{\overline{r}}_{\overline{p}, \bar{\theta}, \overline{\tau}} B \right)_{\overline{q}, \bar{\theta}} \asymp M^{-\left(r_1 + \frac{1}{2} - \frac{1}{p_1}\right)} (\log M)^{(\nu-1)\left(r_1 - \frac{1}{p_1} + \frac{1}{2}\right) + \sum\limits_{j=2}^{\nu} \left(\frac{1}{2} - \frac{1}{\tau_j}\right)_+};$$

2) If $\frac{1}{p_j} - \frac{1}{q_j} < r_j < \frac{1}{p_j}, \theta_j < \tau_j, j = 1, ..., m$, $(r_1 - \frac{1}{p_1})\frac{1}{q_j} < (r_j - \frac{1}{p_j})\frac{1}{q_1}, j = \nu + 1, ..., m$, then

$$e_M \left(S^{\overline{r}}_{\overline{p}, \bar{\theta}, \overline{\tau}} B \right)_{\overline{q}, \bar{\theta}} \asymp M^{-\frac{q_1}{2}\left(r_1 + \frac{1}{q_1} - \frac{1}{p_1}\right)} (\log M)^{q_1\left(r_1 - \frac{1}{p_1}\right) \sum\limits_{j=2}^{\nu} \frac{1}{\theta_j'} + \sum\limits_{j=2}^{\nu} \frac{1}{\tau_j'}};$$

3) If $\nu \le \mu$, $r_j = \frac{1}{p_j}, j = 1, ..., \mu$ and $r_j > \frac{1}{p_j}, j = \mu + 1, ..., m$, then

$$e_M \left(S^{\overline{r}}_{\overline{p}, \bar{\theta}, \overline{\tau}} B \right)_{\overline{q}, \bar{\theta}} \asymp M^{-\frac{1}{2}} (\log M)^{\sum\limits_{j=1}^{\mu} \frac{1}{\tau_j'}}.$$

Theorem 3 [25]. *Let* $\bar{p} = (p_1, ..., p_m)$, $\quad \bar{q} = (q_1, ..., q_m)$, $\quad \bar{\theta}^{(1)} = (\theta_1^{(1)}, ..., \theta_m^{(1)})$, $\quad \bar{\theta}^{(2)} = (\theta_1^{(2)}, ..., \theta_m^{(2)})$ *and* $1 \le p_j < q_j < +\infty$, $\quad 1 \le \theta_j^{(1)}, \theta_j^{(2)} < +\infty$, $\quad j = 1, ..., m$. *If* $f \in \overset{\circ}{L}{}^*_{\bar{p}, \bar{\theta}^{(1)}}(I^m)$ *and the quantity*

$$\sigma(f) \equiv \left\{ \sum_{s_m=1}^{\infty} 2^{s_m \theta_m^{(2)} (\frac{1}{p_m} - \frac{1}{q_m})} \left[\cdots \left[\sum_{s_1=1}^{\infty} 2^{s_1 \theta_1^{(2)} (\frac{1}{p_1} - \frac{1}{q_1})} \left(\|\delta_{\bar{s}}(f)\|^*_{\bar{p}, \bar{\theta}^{(1)}} \right)^{\theta_1^{(2)}} \right]^{\frac{\theta_2^{(2)}}{\theta_1^{(2)}}} \cdots \right]^{\frac{\theta_m^{(2)}}{\theta_{m-1}^{(2)}}} \right\}^{\frac{1}{\theta_m^{(2)}}}$$

is finite, then $f \in \overset{\circ}{L}{}^*_{\bar{q}, \bar{\theta}^{(2)}}(I^m)$ *and*

$$\|f\|^*_{\bar{q}, \bar{\theta}^{(2)}} \le C(p, q, \theta) \cdot \sigma(f).$$

S. 2. Estimates of the best M−term approximations of functions

Theorem 4. Let $1 < q_j < 2 \le p_j < +\infty$, $1 < \theta_j, \lambda_j < +\infty$, $j = 1, ..., m$. *If* $f \in L^*_{\bar{q}, \bar{\theta}}(I^m)$, *then*

$$\|f\|^*_{\bar{q}, \bar{\theta}} \ge C(q, \theta, p, m) \Big\{ \sum_{s_m=1}^{\infty} 2^{s_m \theta_m \left(\frac{1}{p_m} - \frac{1}{q_m}\right)} \Big[... \Big[\sum_{s_1=1}^{\infty} 2^{s_1 \theta_1 \left(\frac{1}{p_1} - \frac{1}{q_1}\right)} \left(\|\delta_{\bar{s}}(f)\|^*_{\bar{p}, \bar{\lambda}} \right)^{\theta_1} \Big]^{\frac{\theta_2}{\theta_1}} ... \Big]^{\frac{\theta_m}{\theta_{m-1}}} \Big\}^{\frac{1}{\theta_m}}.$$

Proof. Firstly, we will prove the theorem for the case $p_j = \lambda_j^{(1)} = 2, \ \ j = 1, ...m$. Then

$$\|\delta_{\bar{s}}(f)\|_{\bar{p},\bar{\lambda}}^{*} = \|\delta_{\bar{s}}(f)\|_2, \ \ \bar{s} \in \mathbb{Z}_{+}^{m}.$$

By the conditions of the theorem $f \in \overset{\circ}{L}{}_{\bar{q},\bar{\theta}}^{*}(I^m)$, $1 < q_j, \theta_j < +\infty$, $j = 1, ..., m$. Therefore [1]

$$\|f\|_{\bar{q},\bar{\theta}}^{*} \geq \sup_{\|g\|_{\bar{q}',\bar{\theta}'}^{*} \leq 1} \int_{I^m} f(\bar{x}) g(\bar{x}) d\bar{x}, \tag{1}$$

where $\bar{q}' = (q_1', ..., q_m')$, $\bar{\theta}' = (\theta_1', ..., \theta_m')$, $\frac{1}{q_j} + \frac{1}{q_j'} = 1$, $\frac{1}{\theta_j} + \frac{1}{\theta_j'} = 1$, $j = 1, ..., m$.

From the inequality (1), we have

$$\|f\|_{\bar{q},\bar{\theta}}^{*} \geq \sup_{\|g\|_{\bar{q}',\bar{\theta}'}^{*} \leq 1} \sum_{\bar{s} \in \mathbb{Z}_{+}^{m}} \int_{I^m} \delta_{\bar{s}}(f, \bar{x}) \delta_{\bar{s}}(g, \bar{x}) d\bar{x}. \tag{2}$$

Since by the assumption of the theorem $1 < q_j < 2, j = 1, ..., m$, we have $2 < q_j^{'} < \infty, j = 1, ..., m$. Therefore, by Theorem 3 for $p_j = \theta_j^{(1)} = 2, \ \ j = 1, ...m$, we obtain

$$\|g\|_{\bar{q}',\bar{\theta}'}^{*} \leq C_0 \left\| \left\{ \prod_{j=1}^{m} 2^{s_j(\frac{1}{2} - \frac{1}{q_j'})} \|\delta_{\bar{s}}(f)\|_2 \right\}_{\bar{s} \in \mathbb{Z}_{+}^{m}} \right\|_{l_{\bar{\theta}'}}. \tag{3}$$

Let's introduce the following notation

$$U_{\bar{q}',\bar{\theta}'} = \left\{ g \in L_{\bar{q}',\bar{\theta}'}^{*} : C_0 \left\| \left\{ \prod_{j=1}^{m} 2^{s_j(\frac{1}{2} - \frac{1}{q_j'})} \|\delta_{\bar{s}}(g)\|_2 \right\}_{\bar{s} \in \mathbb{Z}_{+}^{m}} \right\|_{l_{\bar{\theta}'}} \leq 1 \right\}.$$

Then it follows from the inequality (3) that the set $U_{\bar{q}',\bar{\theta}'}$ is a subset of the unit ball of the the space $L_{\bar{q}',\bar{\theta}'}^{*}(I^m)$. Therefore, the formula (2) implies that

$$\|f\|_{\bar{q},\bar{\theta}}^{*} \geq C \sup_{g \in U_{\bar{q}',\bar{\theta}'}} \sum_{\bar{s} \in \mathbb{Z}_{+}^{m}} \int_{I^m} \delta_{\bar{s}}(f, \bar{x}) \delta_{\bar{s}}(g, \bar{x}) d\bar{x}. \tag{4}$$

If $g \in L_{\bar{q}',\bar{\theta}'}^{*}(I^m)$, such that $\|\delta_{\bar{s}}(g)\|_2 \leq b_{\bar{s}}, \ \ \bar{s} \in \mathbb{Z}_{+}^{m}$ and

$$\left\| \left\{ \prod_{j=1}^{m} 2^{s_j(\frac{1}{2} - \frac{1}{q_j'})} b_{\bar{s}} \right\}_{\bar{s} \in \mathbb{Z}_{+}^{m}} \right\|_{l_{\bar{\theta}'}} \leq 1,$$

then from inequality (4) we obtain

$$\|f\|_{\bar{q},\bar{\theta}}^{*} \geq C \sup_{\{b_{\bar{s}}\}} \sum_{\bar{s} \in \mathbb{Z}_{+}^{m}} \sup_{\|\delta_{\bar{s}}(g)\|_2 \leq b_{\bar{s}}} \int_{I^m} \delta_{\bar{s}}(f, \bar{x}) \delta_{\bar{s}}(g, \bar{x}) d\bar{x}. \tag{5}$$

Now let's prove that the following inequality holds

$$\sup_{\|\delta_{\bar{s}}(g)\|_2 \leq b_{\bar{s}}} \int_{I^m} \delta_{\bar{s}}(f, \bar{x}) \delta_{\bar{s}}(g, \bar{x}) d\bar{x} \geq \|\delta_{\bar{s}}(f)\|_2 \, b_{\bar{s}} \tag{6}$$

for each $\bar{s} \in \mathbb{Z}_{+}^{m}$. So consider the function

$$g_0(\bar{x}) = \sum_{\bar{k}\in\rho(\bar{s})} a_{\bar{k}}(g_0) e^{i\langle \bar{k},\bar{x}\rangle},$$

where

$$a_{\bar{k}}(g_0) = \|\delta_{\bar{s}}(f)\|_2^{-1} a_{\bar{k}}(f) b_{\bar{s}}.$$

Then $\|\delta_{\bar{s}}(g)\|_2 \le b_{\bar{s}}, \ \ \bar{s}\in\mathbb{Z}_+^m$. Hence, by the Perseval's equality, we have

$$\sup_{\|\delta_{\bar{s}}(g)\|_2\le b_{\bar{s}}} \int_{I^m} \delta_{\bar{s}}(f,\bar{x})\delta_{\bar{s}}(g,\bar{x})d\bar{x} \ge \int_{I^m} \delta_{\bar{s}}(f,\bar{x})\delta_{\bar{s}}(g_0,\bar{x})d\bar{x} =$$

$$= \|\delta_{\bar{s}}(f)\|_2^{-1} b_{\bar{s}} \sum_{\bar{k}\in\rho(\bar{s})} a_{\bar{k}}^2(f) = \|\delta_{\bar{s}}(f)\|_2 \, b_{\bar{s}},$$

which proves the formula (6).

Next, it follows from formulas (5) and (6) that

$$\|f\|_{\bar{q},\bar{\theta}}^{*} \ge C \sup_{\{b_{\bar{s}}\}} \sum_{\bar{s}\in\mathbb{Z}_+^m} b_{\bar{s}} \|\delta_{\bar{s}}(f)\|_2 . \qquad (7)$$

Suppose

$$\sigma_{\bar{s}_{m-j}}(f)_{\bar{\theta}_j} =$$

$$= \Big\{ \sum_{s_j=1}^{\infty} 2^{s_j\theta_j\left(\frac{1}{2}-\frac{1}{q_j}\right)} \Big[... \Big[\sum_{s_1=1}^{\infty} 2^{s_1\theta_1\left(\frac{1}{2}-\frac{1}{q_1}\right)} \|\delta_{\bar{s}}(f)\|_2^{\theta_1} \Big]^{\frac{\theta_2}{\theta_1}} ... \Big]^{\frac{\theta_j}{\theta_{j-1}}} \Big\}^{\frac{1}{\theta_j}},$$

where $\bar{s}_{m-j} = (s_{j+1},...,s_m), \ \ \bar{\theta}_j = (\theta_1,...,\theta_j), \ \ j = 1,...,m-1$.

Consider the following sequence

$$b_{\bar{s}} = \left\| \left\{ \prod_{j=1}^{m} 2^{s_j(\frac{1}{2}-\frac{1}{q_j})} \|\delta_{\bar{s}}(f)\|_2 \right\}_{\bar{s}\in\mathbb{Z}_+^m} \right\|_{l_{\bar{\theta}}}^{-\frac{\theta_m}{\theta'_m}} \prod_{j=1}^{m} (\sigma_{\bar{s}_{m-j}}(f)_{\bar{\theta}_j})^{\theta_{j+1}-\theta_j} \times$$

$$\times \|\delta_{\bar{s}}(f)\|_2^{\theta_1-1} \prod_{j=1}^{m} 2^{s_j(\frac{1}{2}-\frac{1}{q_j})\theta_j}$$

for $\bar{s}\in\mathbb{Z}_+^m$. Then

$$\left\| \left\{ \prod_{j=1}^{m} 2^{s_j(\frac{1}{2}-\frac{1}{q'_j})} b_{\bar{s}} \right\}_{\bar{s}\in\mathbb{Z}_+^m} \right\|_{l_{\bar{\theta}}} = 1$$

and

$$\sum_{\bar{s}\in\mathbb{Z}_+^m} b_{\bar{s}} \|\delta_{\bar{s}}(f)\|_2 = \left\| \left\{ \prod_{j=1}^{m} 2^{s_j(\frac{1}{2}-\frac{1}{q_j})} \|\delta_{\bar{s}}(f)\|_2 \right\}_{\bar{s}\in\mathbb{Z}_+^m} \right\|_{l_{\bar{\theta}}} .$$

Therefore, from the inequality (7), we get

$$\|f\|_{\bar{q},\bar{\theta}}^{*} \ge C \left\| \left\{ \prod_{j=1}^{m} 2^{s_j(\frac{1}{2}-\frac{1}{q_j})} \|\delta_{\bar{s}}(f)\|_2 \right\}_{\bar{s}\in\mathbb{Z}_+^m} \right\|_{l_{\bar{\theta}}} . \qquad (8)$$

Next, if $2 < p_j < \infty, \ \ j = 1,...,m$, then, by applying the inequality of different metrics for trigonometric polynomials [28, 29], we obtain from the estimate (8) the following

$$\|f\|_{\bar{q},\bar{\theta}}^{*}\geq C\left\|\left\{\prod_{j=1}^{m}2^{s_j\left(\frac{1}{p_j}-\frac{1}{q_j}\right)}\|\delta_{\bar{s}}(f)\|_{\bar{p},\bar{\lambda}}^{*}\right\}_{\bar{s}\in\mathbb{Z}_+^m}\right\|_{l_{\bar{\theta}}}$$

for $1<q_j<2\leq p_j<\infty,\quad 1<\theta_j,\lambda_j<\infty,\quad j=1,...,m$. The theorem has been proved.

Suppose $y_+=\max\{0,y\}$.

Theorem 5. Let $\overline{p}=(p_1,...,p_m)$, $\overline{q}=(q_1,...,q_m)$, $\overline{r}=(r_1,...,r_m)$, $1\leq p_j<q_j\leq 2$, $j=1,...,m$, $1\leq\theta_j^{(1)},\theta_j^{(2)}<+\infty$, $1\leq\tau_j\leq+\infty$, $0<r_1+\frac{1}{q_1}-\frac{1}{p_1}=...=r_\nu+\frac{1}{q_\nu}-\frac{1}{p_\nu}<r_{\nu+1}+\frac{1}{q_{\nu+1}}-\frac{1}{p_{\nu+1}}=...=r_m+\frac{1}{q_m}-\frac{1}{p_m}$.

1. If $\theta_j^{(2)}\leq\tau_j, j=1,...,m$, then

$$e_M\left(S_{\overline{p},\bar{\theta}^{(1)},\bar{\tau}}^{\overline{r}}B\right)_{\overline{q},\bar{\theta}^{(2)}}\asymp M^{-\left(r_1+\frac{1}{q_1}-\frac{1}{p_1}\right)}(\log M)^{(\nu-1)\left(r_1-\frac{1}{p_1}+\frac{1}{q_1}\right)+\sum\limits_{j=2}^{\nu}\left(\frac{1}{\theta_j^{(2)}}-\frac{1}{\tau_j}\right)}.$$

2. If $\theta_1^{(2)}=...=\theta_m^{(2)}=\theta>\tau=\tau_1=...=\tau_m$, then

$$e_M\left(S_{\overline{p},\bar{\theta}^{(1)},\bar{\tau}}^{\overline{r}}B\right)_{\overline{q},\theta}\leq M^{-\left(r_1+\frac{1}{q_1}-\frac{1}{p_1}\right)}\left(\log^{\nu-1}M\right)^{\left(r_1-\frac{1}{p_1}+\frac{1}{q_1}+\frac{1}{\theta}-\frac{1}{\tau}\right)_+}.$$

3. If $\tau_j\leq\theta_j^{(2)}, j=1,...,m$, then

$$e_M\left(S_{\overline{p},\bar{\theta}^{(1)},\bar{\tau}}^{\overline{r}}B\right)_{\overline{q},\bar{\theta}^{(2)}}\geq CM^{-\left(r_1+\frac{1}{q_1}-\frac{1}{p_1}\right)}(\log M)^{(\nu-1)\left(r_1-\frac{1}{p_1}+\frac{1}{q_1}\right)+\sum\limits_{j=2}^{\nu}\left(\frac{1}{\theta_j^{(2)}}-\frac{1}{\tau_j}\right)}.$$

Proof. The upper bound estimation of the quantity $e_M\left(S_{\overline{p},\bar{\theta}^{(1)},\bar{\tau}}^{\overline{r}}B\right)_{\overline{q},\bar{\theta}^{(2)}}$ was proved in [30]. Let us consider the lower bound.

For a number $M\in\mathbb{N}$ choose a natural number n such that $M\asymp 2^n n^{m-1}$ and $2^n n^{m-1}\geq 4M$.

Consider the function

$$f_0(\bar{x})=n^{-\sum\limits_{j=2}^{m}\frac{1}{\tau_j}}\sum_{\langle\bar{s},\bar{\gamma}\rangle=n}\prod_{j=1}^{m}2^{-s_j\left(r_j+1-\frac{1}{p_j}\right)}\sum_{\bar{k}\in\rho(\bar{s})}e^{i\langle\bar{k},\bar{x}\rangle}.$$

Then, by Lemma 2,

$$\left\|\left\{2^{\langle\bar{s},\bar{r}\rangle}\|\delta_{\bar{s}}(f_0)\|_{\bar{p},\bar{\lambda}}^{*}\right\}_{\langle\bar{s},\bar{r}\rangle=n}\right\|_{l_{\bar{\tau}}}=$$

$$=n^{-\sum\limits_{j=2}^{m}\frac{1}{\tau_j}}\left\|\left\{2^{\langle\bar{s},\bar{r}\rangle}\prod_{j=1}^{m}2^{-s_j\left(r_j+1-\frac{1}{p_j}\right)}\left\|\sum_{\bar{k}\in\rho(\bar{s})}e^{i\langle\bar{k},\bar{x}\rangle}\right\|_{\bar{p},\bar{\lambda}}^{*}\right\}_{\langle\bar{s},\bar{r}\rangle=n}\right\|_{l_{\bar{\tau}}}\leq$$

$$\leq Cn^{-\sum\limits_{j=2}^{m}\frac{1}{\tau_j}}\left\|\left\{2^{\langle\bar{s},\bar{r}\rangle}\prod_{j=1}^{m}2^{-s_j\left(r_j+1-\frac{1}{p_j}\right)}\prod_{j=1}^{m}2^{s_j\left(1-\frac{1}{p_j}\right)}\right\}_{\langle\bar{s},\bar{r}\rangle=n}\right\|_{l_{\bar{\tau}}}=$$

$$=Cn^{-\sum\limits_{j=2}^{m}\frac{1}{\tau_j}}\left\|\{1\}_{\langle\bar{s},\bar{r}\rangle=n}\right\|_{l_{\bar{\tau}}}\leq C_0.$$

Thus, the function $f_0\in S_{\overline{p},\bar{\theta}^{(1)},\bar{\tau}}^{\overline{r}}B$.

Let Ω_M be an arbitrary collection in M of m-dimensional vectors $\{\bar{k}^{(1)},...,\bar{k}^{(M)}\}$ with integer coordinates. For every vector $\bar{s}$, that satisfies the condition $\langle\bar{s},\bar{\gamma}\rangle=n$, we consider the set $\Omega_M\cap\rho(\bar{s})$. Then, according to the choice of the number n, the set S of vectors $\bar{s}$ such that $\langle\bar{s},\bar{\gamma}\rangle=n$ and $|\Omega_M\cap\rho(\bar{s})|\leq\frac{1}{2}|\rho(\bar{s})|$ contains at least half of all vectors $\bar{s}$ such that $\langle\bar{s},\bar{\gamma}\rangle=n$. Hence $|S|\asymp n^{m-1}$.

Let $T(\bar{x})$ be an arbitrary polynomial with a collection of harmonics from Ω_M. Then, by Theorem 4, for $p_j=\lambda_j=2,\ \ j=1,...,m$, we get

$$\|f_0-T\|^*_{\bar{q},\bar{\theta}^{(2)}}\geq C\left\|\left\{\prod_{j=1}^{m}2^{s_j(\frac{1}{2}-\frac{1}{q_j})}\|\delta_{\bar{s}}(f_0-T)\|_2\right\}_{\langle\bar{s},\bar{\gamma}\rangle=n}\right\|_{l_{\bar{\theta}^{(2)}}}\geq$$

$$\geq C\left\|\left\{\prod_{j=1}^{m}2^{s_j(\frac{1}{2}-\frac{1}{q_j})}\|\delta_{\bar{s}}(f_0-T)\|_2\right\}_{\bar{s}\in S}\right\|_{l_{\bar{\theta}^{(2)}}}\geq$$

$$\geq Cn^{-\sum\limits_{j=2}^{m}\frac{1}{\tau_j}}\left\|\left\{\prod_{j=1}^{m}2^{s_j(\frac{1}{2}-\frac{1}{q_j})}\prod_{j=1}^{m}2^{-s_j(r_j+1-\frac{1}{p_j})}\prod_{j=1}^{m}2^{\frac{s_j}{2}}\right\}_{\bar{s}\in S}\right\|_{l_{\bar{\theta}^{(2)}}}\geq$$

$$\geq Cn^{-\sum\limits_{j=2}^{m}\frac{1}{\tau_j}}\left\|\left\{2^{-(r_1+\frac{1}{q_1}-\frac{1}{p_1})\langle\bar{s},\bar{\gamma}\rangle}\right\}_{\bar{s}\in S}\right\|_{l_{\bar{\theta}^{(2)}}}=Cn^{-\sum\limits_{j=2}^{m}\frac{1}{\tau_j}}2^{-(r_1+\frac{1}{q_1}-\frac{1}{p_1})}\left\|\{1\}_{\bar{s}\in S}\right\|_{l_{\bar{\theta}^{(2)}}}.$$

Applying the Holder's inequality $(\frac{1}{\theta_j^{(2)}}+\frac{1}{\theta_j^{(2)\prime}}=1,\ \ j=1,...,m)$ and Lemma 4 , we have

$$|S|=\sum_{\bar{s}\in S}1\leq\left\|\{1\}_{\bar{s}\in S}\right\|_{l_{\bar{\theta}^{(2)}}}\left\|\{1\}_{\bar{s}\in S}\right\|_{l_{\bar{\theta}^{(2)\prime}}}\leq$$

$$\leq\left\|\{1\}_{\bar{s}\in S}\right\|_{l_{\bar{\theta}^{(2)}}}\left\|\{1\}_{\langle\bar{s},\gamma\rangle=n}\right\|_{l_{\bar{\theta}^{(2)\prime}}}\leq Cn^{\sum\limits_{j=2}^{m}\frac{1}{\theta_j^{(2)\prime}}}\left\|\{1\}_{\bar{s}\in S}\right\|_{l_{\bar{\theta}^{(2)}}}.$$

Since $|S|\asymp n^{m-1}$, we obtain

$$n^{\nu-1}\leq Cn^{\sum\limits_{j=2}^{\nu}\frac{1}{\theta_j^{(2)\prime}}}\left\|\{1\}_{\bar{s}\in S}\right\|_{l_{\bar{\theta}^{(2)}}}.$$

Hence

$$n^{\sum\limits_{j=2}^{m}\frac{1}{\theta_j^{(2)}}}\leq C\left\|\{1\}_{\bar{s}\in S}\right\|_{l_{\bar{\theta}^{(2)}}}.$$

Therefore,

$$\|f_0-T\|^*_{\bar{q},\bar{\theta}^{(2)}}\geq C2^{-(r_1+\frac{1}{q_1}-\frac{1}{p_1})}n^{\sum\limits_{j=2}^{m}(\frac{1}{\theta_j^{(2)}}-\frac{1}{\tau_j})}$$

for any polynomial $T(\bar{x})$ with a collection of harmonics from Ω_M. Hence

$$e_M\left(S^{\bar{r}}_{\bar{p},\bar{\theta}^{(1)},\bar{\tau}}B\right)_{\bar{q},\theta^{(2)}}\geq C2^{-(r_1+\frac{1}{q_1}-\frac{1}{p_1})}n^{\sum\limits_{j=2}^{m}(\frac{1}{\theta_j^{(2)}}-\frac{1}{\tau_j})}\geq$$

$$\geq CM^{-\left(r_1+\frac{1}{q_1}-\frac{1}{p_1}\right)}(\log M)^{(m-1)\left(r_1-\frac{1}{p_1}+\frac{1}{q_1}\right)+\sum\limits_{j=2}^{m}\left(\frac{1}{\theta_j^{(2)}}-\frac{1}{\tau_j}\right)}. \qquad (9)$$

Since $\nu\leq m$, then we obtain the lower bound estimation from (9). The theorem has been proved.

V.N. Temlyakov [22, 23] developed a constructive method of estimation of the best $M-$ term approximations functions from Nikol'skii-Besov's classes $S^{\bar{r}}_{p,\tau}B$ in the space $L_q(I^m)$ in the case $1<p<2\leq q<\infty$. Let us recall some notations.

Let $\bar{r}=(r_1,...,r_m)$, $r_j>0$, $j=1,...,m$, and $l\in\mathbb{N}\cap\{0\}=\mathbb{N}_0$. Let $r=r_1$ and for a function $f\in L_1(I^m)$ suppose

$$f_{l,\bar{r}}(\bar{x})=\sum_{\bar{s}:rl\leq\langle\bar{s},\bar{r}\rangle<r(l+1)}\delta_{\bar{s}}(f,\bar{x}),\ \ \bar{x}\in I^m;$$

$$\|f_{l,\bar{r}}\|_A=\sum_{\bar{s}:rl\leq\langle\bar{s},\bar{r}\rangle<r(l+1)}\sum_{\bar{k}\in\rho(\bar{s})}|a_{\bar{k}}(f)|.$$

V.N. Temlyakov [22] considered the class

$$W_A^{\bar{r},a,b}=\Big\{f\in L_1(I^m):\|f_{l,\bar{r}}\|_A\leq 2^{-la}l^{(\nu-1)b}\Big\},$$

where $a>0,b>0$.

Lemma 3 [22]. Let $2\leq p<+\infty$ and $a>0,b>0$. Then there exists a constructive method based on the greedy algorithms which leads to the following estimation

$$e_M(W_A^{\bar{r},a,b})_p\leq CM^{-a-\frac{1}{2}}(\log M)^{(\nu-1)(a+b)}.$$

Theorem 6. Let $1<p_j<2\leq q<+\infty,\quad 1<\theta_j<\infty,\quad 1\leq\tau_j\leq\infty,\quad j=1,...,m,\quad 0<r_1=...=r_\nu<$ $<r_{\nu+1}\leq...\leq r_m,\quad r_1>\frac{1}{p_1}$ and $\frac{r_1}{p_j}=\frac{r_j}{p_1},j=1,...,\nu$ and $\frac{r_1}{p_j}<\frac{r_j}{p_1},j=\nu+1,...,m$. Then the following relation holds

$$e_n(S^{\bar{r}}_{\bar{p},\bar{\theta}^{(1)},\bar{\tau}}B)_q\asymp n^{-(r+\frac{1}{2}-\frac{1}{p_1})}(\log n)^{(\nu-1)(r-\frac{1}{p_1})+\sum\limits_{j=2}^{\nu}\frac{1}{\tau_j'}},$$

where $\tau_j'=\frac{\tau_j}{\tau_j-1},\quad j=1,...,m$.

Proof. Let $f\in S^{\bar{r}}_{\bar{p},\bar{\theta}^{(1)},\bar{\tau}}B$. By the condition of the theorem $1<p_j<2,\quad j=1,...,m$, then there exists a number p_0 such that $1<p_j<p_0<2,\quad j=1,...,m$. Therefore, by applying the Holder's inequality ($\frac{1}{p_0}+\frac{1}{p_0'}=1$) and the Hausdorf-Young's Theorem [31; 211], we get

$$\|f_{l,\bar{r}}\|_A\leq\sum_{\bar{s}:rl\leq\langle\bar{s},\bar{r}\rangle<r(l+1)}\prod_{j=1}^{m}2^{(s_j-1)\frac{1}{p_0}}\Big(\sum_{\bar{k}\in\rho(\bar{s})}|a_{\bar{k}}(f)|^{p_0'}\Big)^{\frac{1}{p_0'}}\leq$$

$$\leq C\sum_{\bar{s}:rl\leq\langle\bar{s},\bar{r}\rangle<r(l+1)}\prod_{j=1}^{m}2^{s_j\frac{1}{p_0}}\|\delta_{\bar{s}}(f)\|_{p_0}. \tag{10}$$

Now, since $p_j<p_0,\quad j=1,...,m$, then by applying the inequality of different metrics for trigonometric polynomials (see [28, 29])and the Holder's inequality, we obtain from (10) the following

$$\|f_{l,\bar{r}}\|_A\leq C\sum_{\bar{s}:rl\leq\langle\bar{s},\bar{r}\rangle<r(l+1)}\prod_{j=1}^{m}2^{s_j\frac{1}{p_j}}\|\delta_{\bar{s}}(f)\|^*_{\bar{p},\bar{\theta}}\leq$$

$$\leq C\left\|\left\{2^{\langle\bar{s},\bar{r}\rangle}\|\delta_{\bar{s}}(f)\|^*_{\bar{p},\bar{\theta}}\right\}_{rl\leq\langle\bar{s},\bar{r}\rangle<r(l+1)}\right\|_{l_{\bar{\tau}}}\left\|\left\{\prod_{j=1}^{m}2^{s_j(\frac{1}{p_j}-r_j)}\right\}_{rl\leq\langle\bar{s},\bar{r}\rangle<r(l+1)}\right\|_{l_{\bar{\tau}'}}. \tag{11}$$

Suppose $\gamma_j=\frac{r_j}{r_1},\quad \gamma_j'=\frac{r_j-\frac{1}{p_j}}{r_1-\frac{1}{p_1}},\ j=1,...,m$. Then, by the assumption of the theorem, $1=\gamma_1=...=\gamma_\nu<$ $<\gamma_{\nu+1}\leq...\leq\gamma_m$ and $\gamma_j'=\gamma_j,\quad j=1,...,\nu,\ \gamma_j=\gamma_j',\quad j=\nu+1,...,m$. Then using Lemma 1 we have

$$\left\|\left\{\prod_{j=1}^{m}2^{s_j(\frac{1}{p_j}-r_j)}\right\}_{rl\leq\langle\bar{s},\bar{r}\rangle<r(l+1)}\right\|_{l_{\bar{\tau}'}}\leq$$

$$\leq\left\|\left\{2^{-(r_1-\frac{1}{p_1})\langle\bar{s},\bar{\gamma'}\rangle}\right\}_{\langle\bar{s},\bar{\gamma}\rangle\geq l}\right\|_{l_{\bar{\tau}'}}\leq C2^{-l(r_1-\frac{1}{p_1})}l^{\sum\limits_{j=2}^{\nu}\frac{1}{\tau_j'}}, \tag{12}$$

where $\frac{1}{\tau_j'}=1-\frac{1}{\tau_j},\quad j=1,...,m$.

Next, it follows from the inequalities (11) and (12) that

$$\|f_{l,\bar{r}}\|_A\leq C2^{-l(r_1-\frac{1}{p_1})}l^{\sum\limits_{j=2}^{\nu}\frac{1}{\tau_j'}}$$

for any function $f \in S^{\bar{r}}_{\bar{p},\bar{\theta}^{(1)},\bar{\tau}}B$ in the case $r_1 > \frac{1}{p_1}$. Thus, the function $f \in S^{\bar{r}}_{\bar{p},\bar{\theta}^{(1)},\bar{\tau}}B$ belongs to the class $W^{\bar{r},a,b}_{A}$ in the case $a = r_1 - \frac{1}{p_1}$ and $b = \frac{1}{\nu-1}\sum_{j=2}^{\nu}\frac{1}{\tau_j'}$. Hence, by Lemma 3, we have

$$e_n(S^{\bar{r}}_{\bar{p},\bar{\theta}^{(1)},\bar{\tau}}B)_q \le Ce_n(W^{\bar{r},a,b}_{A})_q \le Cn^{-(r+\frac{1}{2}-\frac{1}{p_1})}(\log n)^{(\nu-1)(r-\frac{1}{p_1})+\sum_{j=2}^{\nu}\frac{1}{\tau_j'}}$$

in the case $1 < p_j < 2 \le q < +\infty, \quad 1 < \theta_j < \infty, \quad 1 \le \tau_j \le \infty, \quad j = 1, ..., m, \quad r_1 > \frac{1}{p_1}$. It proves the upper bound estimation.

Now let's consider the lower bound estimation. Since $2 \le q < \infty$, then $\|f\|_2 \le C\|f\|_q, \quad f \in L_q(I^m)$. Therefore

$$e_n(S^{\bar{r}}_{\bar{p},\bar{\theta}^{(1)},\bar{\tau}}B)_q \ge Ce_n(S^{\bar{r}}_{\bar{p},\bar{\theta}^{(1)},\bar{\tau}}B)_2.$$

Now, by letting $\theta_j = q = 2$, $j = 1, ..., m$, in Theorem 5, we get

$$e_n(S^{\bar{r}}_{\bar{p},\bar{\theta}^{(1)},\bar{\tau}}B)_2 \ge Cn^{-(r+\frac{1}{2}-\frac{1}{p_1})}(\log n)^{(\nu-1)(r+\frac{1}{2}-\frac{1}{p_1})+\sum_{j=2}^{\nu}(\frac{1}{2}-\frac{1}{\tau_j})}.$$

Hence

$$e_n(S^{\bar{r}}_{\bar{p},\bar{\theta}^{(1)},\bar{\tau}}B)_q \ge Cn^{-(r+\frac{1}{2}-\frac{1}{p_1})}(\log n)^{(\nu-1)(r+\frac{1}{2}-\frac{1}{p_1})+\sum_{j=2}^{\nu}(\frac{1}{2}-\frac{1}{\tau_j})} \ge$$

$$\ge Cn^{-(r+\frac{1}{2}-\frac{1}{p_1})}(\log n)^{(\nu-1)(r-\frac{1}{p_1})+\sum_{j=2}^{\nu}\frac{1}{\tau_j'}}.$$

It proves the theorem.

Remark. Note that the upper bound estimation in the case $p_j = \theta_j = p$, $j = 1, ..., m$, in Theorem 6 was proved by V.N. Temlyakov [22] using the constructive method.

In [32] obtained the exact estimation of the best $M-$ term approximations of Nikol'skii's and Besov's classes in the Lebesgue space with the mixed norm.

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References

1 Blozinski A.P. Multivariate rearragements and Banach function spaces with mixed norms / A.P. Blozinski // Transac. Amer. math. soc. — 1981. — Vol. 263, No. 1. — P. 146–167.

2 Никольский С.М. Приближение функций многих переменных и теоремы вложения / С.М.Никольский. — М.: Наука, 1977. — 456 с.

3 Никольский С.М. Функции с доминирующей смешанной производной, удовлетворяющей кратному условию Гельдера / С.М. Никольский // Сиб. мат. журн. — 1963. — Т. 4, № 6. — С. 1342–1364.

4 Аманов Т.И. Пространства дифференцируемых функций с доминирующей смешанной производной / Т.И. Аманов. — Алма-Ата: Наука, 1976. — 224 с.

5 Лизоркин П.И. Пространства функций смешанной с декомпозиционной точки зрения / П.И. Лизоркин, С.М. Никольский // Труды Математического института им. В.А. Стеклова. — 1989. — Т. 187. — С. 143–161.

6 Стечкин С.Б. Об абсолютной сходимости ортогональных рядов / С.Б. Стечкин // Доклады Академии наук СССР. — 1955. — Т. 102, № 1. — С. 37–40.

7 Исмагилов Р.С. Поперечники множеств в линейных нормированных пространствах и приближение функций тригонометрическими многочленами / Р.С. Исмагилов // Успехи математических наук. — 1974. — Т. 29, № 3. — С. 161–173.

8 Майоров В.Е. Тригонометрические поперечники соболевских классов W^r_p в пространстве L_q / В.Е. Майоров // Математические заметки. — 1986. — Т. 40, №2. — С. 161–173.

9 Белинский Э.С. Приближение периодических функций «плавающей» системой экспонент и тригонометрические поперечники / Э.С. Белинский // Исследования по теории функций многих вещественных переменных. — Ярославль, 1984. — С. 10–24.

10 Белинский Э.С. Приближение «плавающей» системой экспонент на классах периодических функций / Э.С. Белинский // Математический сборник. — 1987. — Т. 132, 1. — С. 20–27.

11 Белинский Э.С. Приближение «плавающей» системой экспонент на классах периодических функций с ограниченной смешанной производной / Э.С. Белинский // Исследования по теории функций многих вещественных переменных. — Ярославль, 1988. — С. 16–33.

12 Темляков В.Н. Приближение функций с ограниченной смешанной производной / В.Н. Темляков // Труды Математического института им. В.А.Стеклова. — 1986. — 178. — С. 1–112.

13 Романюк А.С. Наилучшие M-членные тригонометрические приближения классов Бесова периодических функций многих переменных / А.С. Романюк // Известия Российской академии наук. Серия математическая. — 2003. — Т. 67. — Вып. 2. — С. 61–100.

14 Романюк А.С. Приближение классов периодических функций многих переменных / А.С. Романюк // Мат. замет. — 2002. — Т. 71, № 1. — С. 109–121.

15 De Vore R.A. Nonlinear approximation by trigonometric sums / R.A. DeVore, V.N. Temlyakov // Journal of Fourier Analysis and Applications. — 1995. — Vol. 2, No. 1. — P. 29–48.

16 Temlyakov V.N. Greedy algoritm and m-term trigonometric approximation / V.N. Temlyakov // Constructive approximation. — 1998. — Vol. 14, No. 4. — P. 569–587.

17 Dinh Dung. On asymptotic order of n-term approximations and non-linear-n widths / Dinh Dung // Vietnam Journal Mathematics. — 1999. — No. 27(4). — P. 363–367.

18 Кашин Б.С. Об аппроксимативных свойствах полных ортонормированных систем / Б.С. Кашин // Труды Математического института им. В.А.Стеклова. — 1985. — 172. — С. 187–201.

19 DeVore R.A. Nonlinear approximation / R.A. DeVore // Acta Numerica. — 1998. — 7. — P. 51–150.

20 Temlyakov V.N. Nonlinear methods approximation / V.N. Temlyakov // Found. Comp. Math. — 2003. — 3. — P. 33–107.

21 Dinh Dung. Hyperbolic cross approximation / Dinh Dung, V.N. Temlyakov, T. Ullrich // ArXiv: 1601. 03978v1. — 15, Januare. — 2016. — P. 1–154.

22 Темляков В.Н. Конструктивные разреженные тригонометрические приближения и другие задачи для функций смешанной гладкости / В.Н. Темляков // Математический сборник. — 2015. — 206:11. — С. 131–160.

23 Temlyakov V.N. Constructive sparse trigonometric approximation for functions with small mixed smoothness / V.N. Temlyakov // ArXiv: 1503.00282v1 [NA]. — 2015. — P. 1–30.

24 Bazarkhanov D.B. Nonlinear tensor product approximation of functions / D.B. Bazarkhanov, V.N. Temlyakov // ArXiv: 1409.1403v1 [stat ML]. — 2014. — P. 1–23.

25 Akishev G. On approximation of function classes in Lorentz spaces with anisotropic norm / G. Akishev // Analysis in theory and applications. — 2013. — 29(4). — P. 358 – 372.

26 Акишев Г. О точности оценок наилучшего M-членного приближения класса Бесова / Г. Акишев // Сиб. электр. матем. изв. — 2010. — Т. 7. — С. 255–274.

27 Акишев Г. О порядках M-членного приближения классов в пространстве Лоренца / Г. Акишев // Математический журнал. — 2011. — 11. — 1. — С. 5–29.

28 Акишев Г. Неравенства разных метрик для тригонометрических полиномов / Г. Акишев // Уалихановские чтения – 9. — Кокшетау, 2004. — С. 3–6.

29 Нурсултанов Е. Д. Неравенства разных метрик С.М. Никольского и свойства последовательности норм сумм Фурье функции из пространства Лоренца / Е. Нурсултанов // Труды Математического института РАН. — 2006. — 255. — С. 1–18.

30 Акишев Г. О порядках M-членного приближения классов периодических функций / Г. Акишев // Математический журнал. — 2006. — Т. 6, № 4. — С. 5–11.

31 Бари Н.К. Тригонометрические ряды / Н.К. Бари. — М.: Наука, 1961. — 936 с.

32 Akishev G. On M-term approximations of the Nikol'skii-Besov class / G. Akishev // Hacet. Jour. Math. and Stat. — 2016. — Т. 45, № 2. — P. 297–310.

Ғ. Ақышев

Лоренц кеңістігінде функциялардың ең жақсы M-мүшелі жуықтауларын конструктивті әдістермен бағалау

Мақалада көпайнымалы периодты функциялардың аралас нормалы Лоренц кеңістігі, Никольский-Бесовтың функционалдық класы және функцияның ең жақсы M-мүшелі жуықтауы қарастырылды. Сондай-ақ бір Лоренц кеңістігіндегі функцияның басқасында жатуының жеткілікті шарты тағайындалған. Никольский-Бесов класындағы функциялардың ең жақсы M-мүшелі жуықтауларының жоғарыдан және төменнен бағалаулары алынған. Жоғарыдан бағалауды дәлелдеу үшін В.Н. Темляков құрған жаңа конструктивтік әдіс қолданылды.

Кілт сөздер: Лоренц, Никольский-Бесов класы, М-терминдердің ең жақсы жақтары, жуықтаулар, жеткілікті шарттар, бағалау.

Г. Акишев

Оценки наилучших M-членных приближений функций в пространстве Лоренца конструктивными методами

В статье рассмотрены пространство Лоренца периодических функций многих переменных с анизотропной нормой; функциональный класс Никольского-Бесова и наилучшее M-членное приближение функции. Установлены достаточные условия принадлежности функции из одного пространства Лоренца в другое. Получены оценки сверху и снизу наилучших M-членных приближений функций из класса Никольского-Бесова. Для доказательства оценки сверху использован новый конструктивный метод, разработанный В.Н. Темляковым.

Ключевые слова: Лоренц, класс Никольского-Бесова, лучшие аппроксимации М-термов, приближение, достаточные условия, оценка.

References

1 Blozinski, A.P. (1981). Multivariate rearragements and Banach function spaces with mixed norms // Transac. Amer. Math. Soc., 263, 1, 146–167.

2 Nikol'skii, S.M. (1977). *Priblizhenie funkcii mnohikh peremennykh i teoremy vlozheniia [Approximation of classes of functions of several variables and embeding theorems].* Moscow: Nauka [in Russian].

3 Nikol'skii, S.M. (1963). Funktsii s dominiruiushchei smeshannoi proizvodnoi, udovletvoriaiushchei kratnomu usloviiu Heldera [Functions with dominant mixed derivative satisfies multi condition Hol'der]. *Sibirskii matematicheskii zhurnal – Siberian Mathematical Journal, Vol. 4, 6,* 1342–1364 [in Russian].

4 Amanov, T.I. (1976). *Prostranstva differentsiruemykh funktsii s dominiruiushchei smeshannoi proizvodnoi [Spaces of differentiable functions with dominant mixed derivative].* Alma-Ata: Nauka [in Russian].

5 Lizorkin, P.I., Nikol'skii S.M. (1989). Prostranstva funktsii smeshannoi s dekompozitsionnoi tochki zreniia [Spaces of functions of mixed smoothness from the decomposition point of view]. *Trudy Matematicheskoho instituta imeni V.A.Steklova — Proceedings of the Steklov Mathematical Institute, Vol. 187*, 163–184 [in Russian].

6 Stechkin, S.B. (1955). Ob absolyutnoi skhodimosti ortohonalnykh riadov [On the absolute convergence of orthogonal series]. *Doklady Akademii nauk SSSR — Reports of the Academy of Sciences of the USSR, Vol. 102, 2,* 37–40 [in Russian].

7 Ismagilov, R.S. (1974). Poperechniki mnozhestv v lineinykh normirovannykh prostranstvakh i priblizhenie funktsii trihonometricheskimi mnohochlenami [Widths of sets in linear normed spaces and the approximation of functions by trigonometric polynomials]. *Uspekhi matematicheskih nauk – Successes of mathematical Sciences, Vol. 29, 3,* 161–178 [in Russian].

8 Maiorov, V.E. (1986). Trihonometricheskie poperechniki sobolevskikh klassov W_p^r v prostranstve L_q [Trigonometric diametrs of the Sobolev classes W_p^r in the space L_q]. *Matematicheskie zametki — Mathematical Notes, 40, 2,* 161–173 [in Russian].

9 Belinskii, E.S. (1984). Priblizhenie periodicheskikh funktsii «plavaiushchei» sistemoi eksponent i trihonometricheskie poperechniki [Approximation of periodic functions by a 'floating' system of exponents and trigonometric diametrs]. *Issledovaniia po teorii funktsii mnohikh veshchestvennykh peremennykh – Research on the theory of functions of many real variables,* 10-24. Yaroslavl' State University [in Russian].

10 Belinskii, E.S. (1987). Priblizhenie «plavaiushchei» sistemoi eksponent na klassakh periodicheskikh funktsii [Approximation by a 'floating' system of exponents on the classes of smooth periodic functions]. *Matematicheskii sbornik — Mathematical Collection, Vol. 132, 1,* 20–27 [in Russian].

11 Belinskii, E.S.(1988). Priblizhenie «plavaiushhei» sistemoi eksponent na klassakh periodicheskikh funktsii s ohranichennoi smeshannoi proizvodnoi [Approximation by a 'floating' system of exponentials on the classes of smooth periodic functions with bounded mixed derivative]. *Issledovaniia po teorii funktsii mnohikh veshchestvennykh peremennykh, Jaroslavl — Research on the theory of functions of many real variables Yaroslavl State University,* 16–33 [in Russian].

12 Temlyakov, V.N. (1986). Priblizhenie funktsii s ohranichennoi smeshannoi proizvodnoi [Approximation of functions with bounded mixed derivative]. *Trudy Matematicheskoho instituta imeni V.A.Steklova — Proceedings of the V.A.Steklov Mathematical Institute, 178,* 1–112 [in Russian].

13 Romanyuk, A.S. (2003). Nailuchshie M-chlennye trihonometricheskie priblizheniia klassov Besova periodicheskikh funktsii mnohih peremennykh [On the best M-term trigonometric approximations for the Besov classes of periodic functions of many variables]. *Izvestiia Rossiiskoi Akademii Nauk – Proceedings of the Russian Academy of Sciences, Vol. 67, 2,* 61–100 [in Russian].

14 Romanyuk, A.S. (2002). Priblizhenie klassov periodicheskikh funktsii mnohikh peremennykh [Approximation of the classes of periodic functions of many variables]. *Matematicheskie zametki — Mathematical Notes, Vol. 71, 1,* 109–121 [in Russian].

15 DeVore, R.A., Temlyakov, V.N. (1995). Nonlinear approximation by trigonometric sums // *Journal fourier analysis applications, Vol. 2,* 1, 29–48.

16 Temlyakov, V.N. (1998). Greedy algoritm and m-term trigonometric approximation // *Constructive approximation, Vol. 14,* 4, 569–587.

17 Dinh Dung. (1999). On asymptotic order of n-term approximations and non-linear n-widths // *Vietnam Journal of Mathematics, 27(4),* 363–367.

18 Kashin, B.S. (1985). Ob approksimativnykh svoistvakh polnykh ortonormirovannykh sistem [Approximation properties of complete orthonormal systems]. *Trudy matematicheskoho instituta imeni V.A.Steklova — Proceedings of the V.A.Steklov Institute of Mathematics, 172,* 187–201 [in Russian].

19 DeVore, R.A. (1998). Nonlinear approximation // Acta Numer, 7, 51–150.

20 Temlyakov, V.N. (2003). Nonlinear methods approximation // Found. Comp. Math., 3, 33–107.

21 Dinh Dung, Temlyakov, V.N. & Ullrich, T. (2016). Hyperbolic cross approximation // ArXiv: 1601. 03978v1. – 15, Januare. 1–154.

22 Temlyakov, V.N. (2015). Konstruktivnye razrezhennye trihonometricheskie priblizheniia i druhie zadachi dlia funktsii smeshannoi hladkosti [Constructive sparse trigonometric approximation and other problems for functions with mixed smoothness]. *Matematicheskii sbornik — Mathematical Collection, 206, 11,* 131–160 [in Russian].

23 Temlyakov, V.N. (2015). Constructive sparse trigonometric approximation for functions with small mixed smoothness // ArXiv: 1503.00282v1 [NA], 1–30.

24 Bazarkhanov, D.B. & Temlyakov, V.N. (2014). Nonlinear tensor product approximation of functions // ArXiv: 1409.1403v1 [stat ML], 1–23.

25 Akishev, G. (2013). On approximation of function classes in Lorentz spaces with anisotropic norm // Analysis in theory and applications, 29, 4, 358–372.

26 Akishev, G. (2010). O tochnosti otsenok nailuchsheho M-chlennoho priblizheniia klassa Besova [On the exact estimations of the best M-term approximation of the Besov class]. *Sibirskie elektronnye matematicheskie izvestiia — Siberian Electronic Mathematical Reports, 7,* 255–274 [in Russian].

27 Akishev, G. (2011). O poriadkakh M-chlennoho priblizheniia klassov v prostranstve Lorentsa [On the order of the M-term approximation classes in Lorentz spaces]. *Matematicheskii zhurnal — Mathematical Journal, 11, 1,* 5–29 [in Russian].

28 Akishev, G. (2004). Neravenstva raznykh metrik dlia trihonometricheskikh polinomov [Inequalities of different metric for trigonometric polynomials]. *Ualikhanovskie chteniia — 9 – Ualikhanov reading — 9, 3–6* Kokshetau [in Russian].

29 Nursultanov, E.D. (2006). Neravenstvo raznykh metrik S.M.Nikolskoho i svoistva posledovatelnosti norm summ Fure funktsii iz prostranstva Lorentsa [S.M.Nikol'skii's inequality for different metric and properties of the sequence of norms of the Fourier sums of a function in the Lorentz space]. *Trudy Matematicheskoho instituta imeni V.A.Steklova — Proceedings of the V.A.Steklov Mathematical Institute, 255,* 1–18 [in Russian].

30 Akishev, G. (2006). O poriadkakh M-chlennoho priblizheniia klassov periodicheskikh funktsii [On orders M-term approximation classes of periodic functions]. *Matematicheskii zhurnal — Mathematical Journal, 6, 4,* 5–11 [in Russian].

31 Bary, N.K. (1961). *Trihonometricheskie riady [Trigonometric series].* Moscow: Nauka [in Russian].

32 Akishev, G. (2016). On M-term approximations of the Nikol'skii-Besov class // Hacettepe Journal of Mathematics and Statistics, 45, 2, 297–310.